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FIBRED RIEMANNIAN SPACE WITH TRIPLE OF KILLING VECTORS

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Dedicated to Professor Shigeo Sasaki on his sixtieth birthday

Recently, 3-structures, almost contact, K-contact or Sasakian (normal contact), have been introduced and several interesting subjects concerning these structures have been studied ([3], [4], [5], [6], [8], [9], [13]). The 3-structure, K-contact or Sasakian, is a special kind of triples of Killing vectors, which will be defined in the present paper as a set of three unit Killing vectors ξ, η and ζ being mutually orthogonal and satisfying the structure equations $[\eta, \zeta]=2\xi$, $[\zeta, \xi]=2\eta$, $[\xi, \eta]=2\zeta$. One of purposes of the present paper is to obtain, in terms of curvatures, a condition that a triple of Killing vectors is a Sasakian 3-structure.

In §1, we recall definitions and properties of structures, K-contact or Sasakian. We define also in §1 a triple of Killing vectors and give its preliminary properties. In §2, we give fundamental concepts and divices concerning fibred Riemannian spaces with triple of Killing vectors. We state, in §3, some propositions concerning triples of Killing vectors or K-contact 3-structures as consequences of formulas established in §2. The last §4 is devoted to studying properties of Nijenhuis tensor of structure tensor fields determined by a triple of Killing vectors or a K-contact 3-structure.

§1. Preliminaries.

First, we recall some properties of a K-contact structure. Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold[1] of dimension n with metric tensor $\tilde{g}$. Let there be given in $(\tilde{M}, \tilde{g})$ a unit Killing vector ξ satisfying

$$\tilde{K}(\xi, \tilde{X})\xi = -\tilde{X} + \alpha(\tilde{X})\xi, \tag{1.1}$$

where $\tilde{K}$ denotes the curvature tensor of $(\tilde{M}, \tilde{g})$ and α the 1-form associated with ξ, i.e., $\alpha(\tilde{X}) = \tilde{g}(\xi, \tilde{X})$.[2] Then ξ is said to define a *K-contact structure* (cf. [2]). If we put, for a K-contact structure ξ,

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1) Manifolds, vector fields and geometric objects we discuss are assumed to be differentiable and of class C^∞.

2) Here and in the sequel, $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ denote arbitrary vector fields in $\tilde{M}$

$$\varphi=\tilde{\nabla}\xi, \tag{1.2}$$

$\tilde{\nabla}$ being the Riemannian connection of $(\tilde{M}, \tilde{g})$, then φ is a tensor field of type (1, 1) satisfying

$$\begin{aligned} &\varphi^2=-I+\alpha\otimes\xi, \quad \alpha(\xi)=1, \\ &\alpha\circ\varphi=0, \quad \varphi\xi=0, \quad \tilde{K}(\xi, \tilde{X})=\tilde{\nabla}_{\tilde{X}}\varphi, \end{aligned} \tag{1.3}$$

where I is the unit tensor field of type (1, 1) and the 1-form $\alpha\circ\varphi$ is defined by $(\alpha\circ\varphi)(\tilde{X})=\alpha(\varphi\tilde{X})$. If we put $\Phi=\tilde{\nabla}\alpha$, then we see that Φ is a skew-symmetric tensor field of type (0, 2), i.e., a 2-form and satisfies

$$d\Phi=0, \tag{1.4}$$

d denoting the exterior differentiation. In such a case, $\tilde{M}$ is necessarily orientable and odd-dimensional. When the condition

$$\tilde{N}+2\Phi\otimes\xi=0 \tag{1.5}$$

is satisfied, where $\tilde{N}$ is the Nijenhuis tensor of φ, the K-contact structure ξ is called a *normal contact structure* or a *Sasakian structure*. A K-contact structure ξ is Sasakian if and only if it satisfies

$$\tilde{K}(\xi, \tilde{X})=\tilde{x}\otimes\xi-\alpha\otimes\tilde{X}, \tag{1.6}$$

where $\tilde{x}$ is the 1-form associated with $\tilde{X}$ (cf. [2]).

We now assume that the Riemannian manifold $(\tilde{M}, \tilde{g})$ admits three unit Killing vectors ξ, η and ζ which are mutually orthogonal and satisfy

$$\xi=\frac{1}{2}[\eta, \zeta], \qquad \eta=\frac{1}{2}[\zeta, \xi], \qquad \zeta=\frac{1}{2}[\xi, \eta]. \tag{1.7}$$

Such a set $\{\xi, \eta, \zeta\}$ is, for simplicity, called a *triple of Killing vectors* in $(\tilde{M}, \tilde{g})$. We put

$$\varphi=\tilde{\nabla}\xi, \qquad \psi=\tilde{\nabla}\eta, \qquad \theta=\tilde{\nabla}\zeta; \tag{1.8}$$

$$\Phi=\tilde{\nabla}\alpha, \qquad \Psi=\tilde{\nabla}\beta, \qquad \Theta=\tilde{\nabla}\gamma, \tag{1.9}$$

where α, β and γ are 1-forms associated with ξ, η and ζ respectively. Then Φ, Ψ and Θ are skew-symmetric tensor fields of type (0, 2), i.e., 2-forms and equations

$$\varphi\xi=0, \qquad \psi\eta=0, \qquad \theta\zeta=0; \tag{1.10}$$

$$d\Phi=0, \qquad d\Psi=0, \qquad d\Theta=0; \tag{1.11}$$

$$\tilde{\nabla}_\xi\varphi=0, \qquad \tilde{\nabla}_\eta\psi=0, \qquad \tilde{\nabla}_\zeta\theta=0 \tag{1.12}$$

are valid, because ξ, η and ζ are unit Killing vectors. From (1. 7), we have

$$2\xi=\theta\eta-\psi\zeta, \qquad 2\eta=\varphi\zeta-\theta\xi, \qquad 2\zeta=\psi\xi-\varphi\eta. \tag{1.13}$$

On the other hand, we find

$$\psi\zeta+\theta\eta=0, \qquad \theta\xi+\varphi\zeta=0, \qquad \varphi\eta+\psi\xi=0, \tag{1.14}$$

because ξ, η and ζ are mutually orthogonal. Thus, using (1.13) and (1.14), we obtain

$$\xi=\theta\eta=-\psi\zeta, \qquad \eta=\varphi\zeta=-\theta\xi, \qquad \zeta=\psi\xi=-\varphi\eta. \tag{1.15}$$

Given a triple $\{\xi, \eta, \zeta\}$ of Killing vectors, we denote by D the distribution spanned by ξ, η and ζ. Then, by means of (1.7), D is integrable. Using (1.7), (1.8), (1.10) and (1.15), we obtain

$$\begin{gathered}\tilde{\nabla}_\xi\xi=\tilde{\nabla}_\eta\eta=\tilde{\nabla}_\zeta\zeta=0,\\ \tilde{\nabla}_\eta\zeta=-\tilde{\nabla}_\zeta\eta=\xi, \qquad \tilde{\nabla}_\zeta\xi=-\tilde{\nabla}_\xi\zeta=\eta, \qquad \tilde{\nabla}_\xi\eta=-\tilde{\nabla}_\eta\xi=\zeta.\end{gathered} \tag{1.16}$$

Thus any integral manifold F of D is totally geodesic in $(\tilde{M}, \tilde{g})$ and hence, by means of (1.16),

$$\begin{aligned}\bar{K}(\xi, \eta)\xi&=\tilde{K}(\xi, \eta)\xi\\ &=\tilde{\nabla}_\xi\tilde{\nabla}_\eta\xi-\tilde{\nabla}_\eta\tilde{\nabla}_\xi\xi-\tilde{\nabla}_{[\xi,\eta]}\xi=-\eta\end{aligned}$$

holds, where $\bar{K}$ is the curvature tensor of F with induced metric. Therefore, denoting by $\bar{\sigma}(C, D)$ the sectional curvature of F with respect to the section spanned by tangent vectors C and D to F, we have $\bar{\sigma}(\xi, \eta)=1$. Similarly, we obtain $\bar{\sigma}(\eta, \zeta)=\bar{\sigma}(\zeta, \xi)=1$. Consequently, we have

LEMMA 1.1. *In $(\tilde{M}, \tilde{g})$ with triple $\{\xi, \eta, \zeta\}$ of Killing vectors, any integral manifold of the distribution D spanned by ξ, η and ζ is totally geodesic and of constant curvature* 1.

A triple $\{\xi, \eta, \zeta\}$ of Killing vectors is called a *K-contact 3-structure* if each of ξ, η and ζ defines a K-contact structure and the equations

$$\begin{gathered}\theta\psi=\varphi+\beta\otimes\zeta, \qquad \varphi\theta=\psi+\gamma\otimes\xi, \qquad \psi\varphi=\theta+\alpha\otimes\eta,\\ \psi\theta=-\varphi+\gamma\otimes\eta, \quad \theta\varphi=-\psi+\alpha\otimes\zeta, \quad \varphi\psi=-\theta+\beta\otimes\xi\end{gathered} \tag{1.17}$$

hold (cf. [5], [9]). In such a case, $\tilde{M}$ is necessarily orientable and of dimension $n=4m+3$. In the sequel, we assume that $\dim \tilde{M}=n=4m+3\geqq 7$ (i.e., $m\geqq 1$) for any $(\tilde{M}, \tilde{g})$ with K-contact 3-structure. A K-contact 3-structure $\{\xi, \eta, \zeta\}$ is called a *normal contact 3-structure* or *a Sasakian 3-structure* if all of ξ, η and ζ are normal contact structures. If any two of ξ, η and ζ are normal contact structures, then the triple $\{\xi, \eta, \zeta\}$ of Killing vectors is Sasakian 3-structure (cf. [5], [6], [8]).

Let $(\tilde{M}, \tilde{g})$ be an n-dimensional Riemannian manifold with triple $\{\xi, \eta, \zeta\}$ of

Killing vectors (resp. with K-contact 3-structures or with Sasakian 3-structure) and M a manifold of dimension $n-3$. Assume that there is a differentiable mapping π: $\tilde{M}\rightarrow M$, which is onto and of the maximum rank, and for any point P of M the complete inverse image $\pi^{-1}(\mathrm{P})$ is a maximal integral manifold of the distribution D spanned by ξ, η and ζ. In such a case, $(\tilde{M}, \tilde{g})$ is called a *fibred Riemannian* space with triple $\{\xi, \eta, \zeta\}$ of Killing vectors (resp. with K-contact 3-structure or with Sasakian 3-structure), each of complete inverse images $\pi^{-1}(\mathrm{P})$ the *fibre* over P and M the *base manifold*. Any Riemannian manifold with triple of Killing vectors admits locally such a structure of a fibred Riemannian space with triple of Killing vectors (i.e., for any point P of the manifold, there is a suitable neighborhood containing P and admitting a structure of a fibred Riemannian space). Thus the arguments developed for a fibred Riemannian space with triple of Killing vectors will be locally established for any Riemannian manifold with triple of Killing vectors.

§2. Fibred Riemannian space with triple of Killing vectors.

In this section, we assume that $(\tilde{M}, \tilde{g})$ is a fibred Riemannian space with triple $\{\xi, \eta, \zeta\}$ of Killing vectors and denote by π: $\tilde{M}\rightarrow M$ the projection, where M is the base manifold. Since ξ, η and ζ are Killing vectors spanning the tangent space of each fibre, $(\tilde{M}, M, \pi, \tilde{g})$ forms a fibred space with projectable Riemannian metric $\tilde{g}$ in the sense of [1], [3], [7], [11] and [12].

We take coordinate neighborhoods $\{\tilde{U}, x^h\}$ of $\tilde{M}$ and $\{U, v^a\}$ of M such that $\pi(\tilde{U})=U$. Then the projection π: $\tilde{M}\rightarrow M$ may be expressed, with respect to $\tilde{U}$ and U, by certain equations of the form[3)]

$$v^a=v^a(x^h), \tag{2.1}$$

where $v^a(x^h)$ denote the coordinates of the projection $\mathrm{P}=\pi(\tilde{\mathrm{P}})$ of a point $\tilde{\mathrm{P}}$ with coordinates x^h in $\tilde{U}$ and are differentiable functions of the variables x^h with Jacobian $(\partial v^a/\partial x^h)$ of the maximum rank $4m(=n-3)$. Take a fibre F such that $F\cap\tilde{U}\neq\phi$. We may assume that $F\cap\tilde{U}$ is connected and that we can introduce local coordinates (u^α) in $F\cap\tilde{U}$ in such a way that (v^a, u^α) is a system of coordinates in $\tilde{U}$, (v^a) being coordinates of the point $\pi(F)$ of U. Differentiating (2.1) by x^h, we put

$$E_i{}^a=\partial_i v^a, \tag{2.2}$$

where $\partial_i=\partial/\partial x^i$. Then, for each fixed index a, $E_i{}^a$ are components of a local covector field E^a in $\tilde{U}$. On the other hand, if we put $C_\alpha=\partial/\partial u^\alpha$ which is a local vector field in $\tilde{U}$ for each fixed index α, then C_α form the natural frame of each fibre in $F\cap\tilde{U}$. We denote by $C_\alpha{}^h$ components of C_α in $\tilde{U}$.

3) The indices h, i, j, k, l run over the range $\{1, 2, \cdots, n\}$ and the indices a, b, c, d, e over the range $\{1, 2, \cdots, n-3\}$. The summation convention will be used with respect to these systems of indices.

Let $\tilde{g}_{ji}$ be components of $\tilde{g}$ in $\tilde{U}$. Then the induced metric $\bar{g}$ of a fibre F has components of the form $g_{\gamma\beta}=\tilde{g}_{ji}C^j{}_\gamma C^i{}_\beta$ in $F\cap\tilde{U}$. If we put $C_i{}^\alpha=\tilde{g}_{ih}g^{\alpha\beta}C^h{}_\beta$, where $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$, and denote by C^α the local vector field with components $C_i{}^\alpha$ in $\tilde{U}$ for each fixed index α, then E^a and C^α form a coframe in $\tilde{U}$.[4] Denoting by $(E^h{}_b, C^h{}_\beta)$ the inverse matrix of $(E_i{}^a, C_i{}^\alpha)$, we obtain

$$(2.3)\qquad \begin{aligned} &E_i{}^a E^i{}_b=\delta_b^a, && E_i{}^a C^i{}_\beta=0,\\ &C_i{}^\alpha E^i{}_b=0, && C_i{}^\alpha C^i{}_\beta=\delta_\beta^\alpha. \end{aligned}$$

Then we have, in $\tilde{U}$, $n-3$ local vector fields E_b with components $E^h{}_b$ and 3 local vector fields C_β with components $C^h{}_\beta$, which form in $\tilde{U}$ a frame dual to the coframe $\{E^a, C^\alpha\}$.

Any tensor field, say $\tilde{T}$ of type (1, 2), in $\tilde{M}$ is represented locally in $\tilde{U}$ as follows:

$$\begin{aligned}\tilde{T}=&\,T_{cb}{}^a E^c\otimes E^b\otimes E_a+T_{cb}{}^\alpha E^c\otimes E^b\otimes C_\alpha+\cdots\\ &+T_{\gamma\beta}{}^a C^\gamma\otimes C^\beta\otimes E_a+T_{\gamma\beta}{}^\alpha C^\gamma\otimes C^\beta\otimes C_\alpha,\end{aligned}$$

where the coefficients $T_{cb}{}^a, \cdots, T_{\gamma\beta}{}^\alpha$ are local functions in $\tilde{U}$. In the right-hand side, the first term $T_{cb}{}^a E^c\otimes E^b\otimes E_a$ determines a global tensor field in $\tilde{M}$, which is called the *horizontal part* of $\tilde{T}$ and denoted by $\tilde{T}^H$. The last term $T_{\gamma\beta}{}^\alpha C^\gamma\otimes C^\beta\otimes C_\alpha$ determines also a global tensor field in $\tilde{M}$, which is called the *vertical part* of $\tilde{T}$ and denoted by $\tilde{T}^V$. If we have $\tilde{T}=\tilde{T}^H$ (resp. $\tilde{T}=\tilde{T}^V$), then we say that $\tilde{T}$ is *horizontal* (resp. *vertical*). For a function $\tilde{f}$ in $\tilde{M}$, we define its horizontal part $\tilde{f}^H$ by $\tilde{f}^H=\tilde{f}$. For any two tensor fields $\tilde{T}$ and $\tilde{S}$ in $\tilde{M}$, we have $(\tilde{T}\otimes\tilde{S})^H=\tilde{T}^H\otimes\tilde{S}^H$.

When a horizontal tensor field, say $\tilde{T}$ of type (1, 2), has the form $\tilde{T}=T_{cb}{}^a E^c\otimes E^b\otimes E_a$ in $\tilde{U}$, we say sometimes that $\tilde{T}$ has components $T_{cb}{}^a$ in $\tilde{U}$.

A tensor field $\tilde{T}$ in $\tilde{M}$ is said to be projectable if it satisfies $(\mathcal{L}_{\tilde{X}}\tilde{T}^H)^H=0$ for any vertical vector field $\tilde{X}$ (See [1], [3], [12]), where $\mathcal{L}_{\tilde{X}}$ denote the Lie derivation with respect to $\tilde{X}$. Then a tensor field $\tilde{T}$ is projectable if $\mathcal{L}_\xi\tilde{T}=0$, $\mathcal{L}_\eta\tilde{T}=0$ and $\mathcal{L}_\zeta\tilde{T}=0$.

A function $\tilde{f}$ in $\tilde{M}$ is projectable if and only if $\mathcal{L}_{\tilde{X}}\tilde{f}=0$ for any vertical vector field $\tilde{X}$. If $\tilde{f}$ is projectable, then $\tilde{f}$ is constant along each fibre because every fibre is connected. Given a projectable function $\tilde{f}$ in $\tilde{M}$ in such a way that, for any point P of M, $f(\mathrm{P})=\tilde{f}(\tilde{\mathrm{P}})$, where $\tilde{\mathrm{P}}$ is a point of $\tilde{M}$ such that $\pi(\tilde{\mathrm{P}})=\mathrm{P}$. We call f the *projection* of $\tilde{f}$ and denote it by $p\tilde{f}$. If $\tilde{f}$ is a projectable function, then its gradient grad $\tilde{f}$ is so also. For a projectable function $\tilde{f}$, $(\mathrm{grad}\,\tilde{f})^H$ has components of the form $\partial_b\tilde{f}$ in $\tilde{U}$, where $\partial_b=E^i{}_b\partial_i$ and $\partial_i=\partial/\partial x^i$ in $\tilde{U}$, and $\mathrm{grad}\, f=p\ (\mathrm{grad}\,\tilde{f})$ $(f=p\tilde{f})$ has components of the form $\partial_b f$ in U, where $\partial_b=\partial/\partial v^b$ in U.

4) The indices $\alpha, \beta, \gamma, \delta$ run over the range $\{1, 2, 3\}$ and the summation convention will be used also with respect to this system of indices.

If we denote by π_U and $\tilde{g}_U$ respectively the restrictions of π and $\tilde{g}$ to $\tilde{U}$, then we have a fibred space $\{\tilde{U}, U, \pi_{\tilde{U}}, \tilde{g}_{\tilde{U}}\}$ with projectable Riemannian metric $\tilde{g}_U$, which is called the local fibering of $\tilde{U}$ in $\{\tilde{M}, M, \pi, \tilde{g}\}$. In the sequel, we use, concerning local geometric objects, the terminologies such as to be horizontal, to be vertical, to be projectable and etc. with respect to the local fibering in the above sense.

A tensor field, say $\tilde{T}$ of type (1, 2), in $\tilde{M}$ is projectable if and only if the local function $T_{cb}{}^a$ is projectable, where $\tilde{T}^H = T_{cb}{}^a E^c \otimes E^b \otimes E_a$. Then, for a projectable tensor field $\tilde{T}$ of this type, we can define a local tensor field T_U with components $p(T_{cb}{}^a)$ in U. The local tensor field T_U determines a global tensor field T in M which is called the *projection* of $\tilde{T}$ and denoted by $p\tilde{T}$.

Given a tensor field T in M, there is uniquely a horizontal and projectable tensor field $\tilde{T}$ in $\tilde{M}$ such that $p\tilde{T}=T$. This $\tilde{T}$ is called the *lift* of T and denoted by T^L.

The Riemannian metric $\tilde{g}$ is projectable because $\mathcal{L}_\xi \tilde{g}=0$, $\mathcal{L}_\eta \tilde{g}=0$, $\mathcal{L}_\zeta \tilde{g}=0$. If we put $g=p\tilde{g}$, then g is a Riemannian metric in M, which is called the *induced metric* of M. The Riemannian manifold (M, g) thus introduced is called the *base space*. The induced metric g of M has in U components of the form $g_{cb}=\tilde{g}_{ji}E^j{}_c E^i{}_b$, where the both sides are identified with their projections respectively. We denote by (g^{cb}) the inverse matrix (g_{cb}) in M. In the sequel, we shall identify any projectable (local) function with its projection.

Since each fibre is totally geodesic, we have in $\tilde{U}$

$$\tilde{\nabla}_j E^h{}_b = \left\{ {a \atop c\ b} \right\} E_j{}^c E^h{}_a + h_{cb}{}^\alpha E_j{}^c C^h{}_\alpha - h_b{}^a{}_\beta C_j{}^\beta E^h{}_a, \tag{2.4}$$

$$\tilde{\nabla}_j C^h{}_\beta = -h_c{}^a{}_\beta E_j{}^c E^h{}_a + P_{c\beta}{}^\alpha E_j{}^c C^h{}_\alpha + \left\{ {\alpha \atop \gamma\ \beta} \right\} C_j{}^\gamma C^h{}_\alpha \tag{2.5}$$

and equivalently

$$\tilde{\nabla}_j E_i{}^a = -\left\{ {a \atop c\ b} \right\} E_j{}^c E_i{}^b + h_b{}^a{}_\beta (E_j{}^b C_i{}^\beta + E_i{}^b C_j{}^\beta), \tag{2.6}$$

$$\tilde{\nabla}_j C_i{}^\alpha = -h_{cb}{}^\alpha E_j{}^c E_i{}^b - P_{c\beta}{}^\alpha E_j{}^c C_i{}^\beta - \left\{ {\alpha \atop \gamma\ \beta} \right\} C_j{}^\gamma C_i{}^\beta, \tag{2.7}$$

$\{{}_c{}^a{}_b\}$ and $\{{}_\gamma{}^\alpha{}_\beta\}$ being the Christoffel's symbols constructed by g_{cb} and by $g_{\gamma\beta}$ respectively, where $h_{cb}{}^\alpha$ are local functions in $\tilde{U}$ such that $h_{cb}{}^\alpha + h_{bc}{}^\alpha = 0$, $h_b{}^a{}_\beta = g^{ac} g_{\beta\alpha} h_{bc}{}^\alpha$ and $P_{b\beta}{}^\alpha$ the functions appearing in

$$\begin{gathered} [C_\beta, E_b] = -P_{b\beta}{}^\alpha C_\alpha, \quad [C_\beta, C_\alpha] = 0, \\ \mathcal{L}_{C_\beta} E^a = 0, \qquad \mathcal{L}_{C_\beta} C^\alpha = P_{b\beta}{}^\alpha E^b. \end{gathered} \tag{2.8}$$

From (2.4)~(2.7), we have

$$[E_c, E_b] = 2h_{cb}{}^\alpha C_\alpha, \qquad [E_b, C_\beta] = P_{b\beta}{}^\alpha C_\alpha, \tag{2.9}$$

$$\mathcal{L}_{E_c}E^a=0, \qquad \mathcal{L}_{E_c}C^\alpha=2h_{cb}{}^\alpha E^b-P_{c\beta}{}^\alpha C^\beta.$$

We can verify that, for any projectable tensor field $\tilde{T}$ in $\tilde{M}$, $\tilde{\nabla}_{\tilde{X}^H}\tilde{T}^H$ is projectable and

$$p(\tilde{\nabla}_{\tilde{X}^H}\tilde{T}^H)=\nabla_X T, \qquad T=p\tilde{T}, \qquad X=p\tilde{X}, \tag{2.10}$$

where $\tilde{X}$ is a projectable vector field in $\tilde{M}$ (cf. [1], [3], [11], [12]).

We now need the well known Ricci identities

$$\tilde{\nabla}_k\tilde{\nabla}_j E^h{}_b-\tilde{\nabla}_j\tilde{\nabla}_k E^h{}_b=\tilde{K}_{kji}{}^h E^i{}_b,$$

$$\tilde{\nabla}_k\tilde{\nabla}_j C^h{}_\beta-\tilde{\nabla}_j\tilde{\nabla}_k C^h{}_\beta=\tilde{K}_{kji}{}^h C^i{}_\beta,$$

where $\tilde{K}_{kji}{}^h$ are components in $\tilde{U}$ of the curvature tensor $\tilde{K}$ of $(\tilde{M}, \tilde{g})$. If we take account of (2. 4)～(2. 7) and use the Ricci identities given above, then we find the following structure equations (cf. [1], [3], [7], [11], [12], [14]):

$$\tilde{K}_{\delta\gamma\beta}{}^\alpha=\bar{K}_{\delta\gamma\beta}{}^\alpha, \tag{2.11}$$

$$\tilde{K}_{\delta\gamma\beta}{}^a=0, \tag{2.12}$$

which are respectively the equation of Gauss and that of Codazzi for the immersion of each fibre in $(\tilde{M}, \tilde{g})$, where

$$\tilde{K}_{\delta\gamma\beta}{}^\alpha=\tilde{K}_{kji}{}^h C^k{}_\delta C^j{}_\gamma C^i{}_\beta C_h{}^\alpha, \ \tilde{K}_{\delta\gamma\beta}{}^a=\tilde{K}_{kji}{}^h C^k{}_\delta C^j{}_\gamma C^i{}_\beta E_h{}^a$$

and $\bar{K}_{\delta\gamma\beta}{}^\alpha$ are components of the curvature tensor $\bar{K}$ of the fibre. The equation of Ricci for the fibre is equivalent to (2. 15) or to (2. 19) which will be given later;

$$\tilde{K}_{dcb}{}^a-K_{dcb}{}^a=-h_d{}^a{}_\alpha h_{cb}{}^\alpha+h_c{}^a{}_\alpha h_{db}{}^\alpha+2h_{dc}{}^\alpha h_b{}^a{}_\alpha, \tag{2.13}$$

$$\tilde{K}_{dcb}{}^\alpha=\nabla_d h_{cb}{}^\alpha-\nabla_c h_{db}{}^\alpha+P_{d\gamma}{}^\alpha h_{cb}{}^\gamma-P_{c\gamma}{}^\alpha h_{db}{}^\gamma, \tag{2.14}$$

$$\begin{aligned}\tilde{K}_{dc\beta}{}^\alpha=&\,\partial_d P_{c\beta}{}^\alpha-\partial_c P_{d\beta}{}^\alpha-P_{d\beta}{}^\gamma P_{c\gamma}{}^\alpha+P_{c\beta}{}^\gamma P_{d\gamma}{}^\alpha\\&+h_d{}^b{}_\beta h_{cb}{}^\alpha-h_c{}^b{}_\beta h_{db}{}^\alpha-2h_{dc}{}^\gamma\left\{\begin{matrix}\alpha\\ \gamma\ \beta\end{matrix}\right\},\end{aligned} \tag{2.15}$$

which are respectively the equation of co-Gauss, that of co-Codazzi and that of co-Ricci for the fibering of $(\tilde{M}, \tilde{g})$, where

$$\tilde{K}_{dcb}{}^a=\tilde{K}_{kji}{}^h E^k{}_d E^j{}_c E^i{}_b E_h{}^a, \ \tilde{K}_{dcb}{}^\alpha=\tilde{K}_{kji}{}^h E^k{}_d E^j{}_c E^i{}_b C_h{}^\alpha,$$

$$\tilde{K}_{dc\beta}{}^\alpha=\tilde{K}_{kji}{}^h E^k{}_d E^j{}_c C^i{}_\beta C_h{}^\alpha,$$

$$\nabla_d h_{cb}{}^\alpha=\partial_d h_{cb}{}^\alpha-\left\{\begin{matrix}a\\ d\ c\end{matrix}\right\}h_{ab}{}^\alpha-\left\{\begin{matrix}a\\ d\ b\end{matrix}\right\}h_{ca}{}^\alpha$$

and $K_{dcb}{}^a$ are components in $\tilde{U}$ of the lift K^L of the curvature tensor K of the base space (M, g);

$$\tilde{K}_{\delta cb}{}^{\alpha} = \nabla_\delta h_{cb}{}^{\alpha} + h_b{}^{\alpha}{}_\delta h_{ca}{}^{\alpha}, \tag{2.16}$$

where

$$\tilde{K}_{\delta cb}{}^{\alpha} = \tilde{K}_{kji}{}^h C^k{}_\delta E^j{}_c E^i{}_b C_h{}^{\alpha}, \quad \nabla_\delta h_{cb}{}^{\alpha} = \partial_\delta h_{cb}{}^{\alpha} + \left\{ \begin{matrix} \alpha \\ \delta\ \beta \end{matrix} \right\} h_{cb}{}^{\beta},$$

∂_β being defined by $\partial_\beta = C^i{}_\beta \partial_i$ in $\tilde{U}$.

In the left hand side of (2.13), $\tilde{K}_{dcb}{}^a$ denote components in $\tilde{U}$ of $\tilde{K}^H$, which is projectable since $\mathcal{L}_\xi \tilde{K} = \mathcal{L}_\eta \tilde{K} = \mathcal{L}_\zeta \tilde{K} = 0$. Therefore the right hand side of (2.13) denotes a horizontal and projectable tensor $\tilde{P}$ of type (1, 3), whose components in $\tilde{U}$ are given by

$$\tilde{P}_{dcb}{}^a = -h_d{}^a{}_\alpha h_{cb}{}^{\alpha} + h_c{}^a{}_\alpha h_{db}{}^{\alpha} + 2h_{dc}{}^{\alpha} h_b{}^a{}_\alpha. \tag{2.17}$$

Since each fibre is of constant curvature 1, we have $\bar{K}_{\delta\gamma\beta}{}^{\alpha} = \delta^\alpha_\delta g_{\gamma\beta} - \delta^\alpha_\gamma g_{\delta\beta}$. Thus (2.11) reduces to

$$\tilde{K}_{\delta\gamma\beta}{}^{\alpha} = \delta^\alpha_\delta g_{\gamma\beta} - \delta^\alpha_\gamma g_{\delta\beta}. \tag{2.18}$$

The equation (2.12) shows that *the connection induced in the normal bundle of each fibre has zero curvature tensor.*

The Jacobi identity $[[E_d, E_c], C_\beta] + [[E_c, C_\beta], E_d] + [[C_\beta, E_d], E_c] = 0$ is equivalent to the identity

$$\partial_d P_{c\beta}{}^{\alpha} - \partial_c P_{d\beta}{}^{\alpha} - P_{d\beta}{}^{\gamma} P_{c\gamma}{}^{\alpha} + P_{c\beta}{}^{\gamma} P_{d\gamma}{}^{\alpha} + 2\partial_\beta h_{dc}{}^{\alpha} = 0.$$

Using the identity above, we see that the equation (2.15) of co-Ricci is equivalent to

$$\tilde{K}_{dc\beta}{}^{\alpha} = h_d{}^b{}_\beta h_{cb}{}^{\alpha} - h_c{}^b{}_\beta h_{db}{}^{\alpha} - 2\nabla_\beta h_{dc}{}^{\alpha}. \tag{2.19}$$

If we put in $\tilde{U}$

$$\begin{aligned} &\varphi^H = \varphi_c{}^b E^c \otimes E_b, \quad \psi^H = \psi_c{}^b E^c \otimes E_b, \quad \theta^H = \theta_c{}^b E^c \otimes E_b; \\ &\Phi^H = \varphi_{cb} E^c \otimes E^b, \quad \Psi^H = \psi_{cb} E^c \otimes E^b, \quad \Theta^H = \theta_{cb} E^c \otimes E^b, \end{aligned} \tag{2.20}$$

then we obtain $\varphi_{cb} = \varphi_c{}^a g_{ab}$, $\psi_{cb} = \psi_c{}^a g_{ab}$ and $\theta_{cb} = \theta_c{}^a g_{ab}$.

If we put in $\tilde{U}$

$$\xi = a^\alpha C_\alpha, \quad \eta = b^\alpha C_\alpha, \quad \zeta = c^\alpha C_\alpha \tag{2.21}$$

and $a_\beta = a^\alpha g_{\alpha\beta}$, $b_\beta = b^\alpha g_{\alpha\beta}$, $c_\beta = c^\alpha g_{\alpha\beta}$, then we find

$$\begin{aligned} &C_\alpha = a_\alpha \xi + b_\alpha \eta + c_\alpha \zeta, \\ &a_\beta a^\alpha + b_\beta b^\alpha + c_\beta c^\alpha = \delta^\alpha_\beta \end{aligned} \tag{2.22}$$

and, using (1.16),

$$\nabla_\gamma a_\beta = -(b_\gamma c_\beta - c_\gamma b_\beta), \quad \nabla_\gamma b_\beta = -(c_\gamma a_\beta - a_\gamma c_\beta), \quad \nabla_\gamma c_\beta = -(a_\gamma b_\beta - b_\gamma a_\beta), \tag{2.23}$$

where $\nabla_\gamma a_\beta=\partial_\gamma a_\beta-\{_\gamma{}^\alpha{}_\beta\}a_\alpha$ and etc. Since ξ, η and ζ are Killing vectors, each of the operators $\mathcal{L}_\xi$, $\mathcal{L}_\eta$ and $\mathcal{L}_\zeta$ commutes with the covariant differentiation $\tilde{\nabla}$. Thus, using (1. 7), we find

(2. 24) $$\mathcal{L}_D\varphi=2(r\psi-q\theta),\ \mathcal{L}_D\psi=2(p\theta-r\varphi),\ \mathcal{L}_D\theta=2(q\varphi-p\psi)$$

for any linear combination $D=p\xi+q\eta+r\zeta$ with constant coefficients p, q and r, or equivalently

(2. 25) $$\partial_\beta\varphi_{cb}=2(c_\beta\psi_{cb}-b_\beta\theta_{cb}),\ \partial_\beta\psi_{cb}=2(a_\beta\theta_{cb}-c_\beta\varphi_{cb}),\ \partial_\beta\theta_{cb}=2(b_\beta\varphi_{cb}-a_\beta\psi_{cb}).$$

Next, using (2. 24), we have

$$\mathcal{L}_D(\Phi\otimes\varphi+\Psi\otimes\psi+\Theta\otimes\theta)=0,\quad \mathcal{L}_D(\Phi\wedge\Phi+\Psi\wedge\Psi+\Theta\wedge\Theta)=0$$

for any linear combination D of ξ, η and ζ with constant coefficients. Therefore two tensor fields

(2. 26) $$\Lambda=\Phi\otimes\varphi+\Psi\otimes\psi+\Theta\otimes\theta,\quad \tilde{S}=\Phi\wedge\Phi+\Psi\wedge\Psi+\Theta\wedge\Theta$$

are projectable. Thus, if we put

(2. 27) $$\Lambda=p\tilde{\Lambda},\qquad S=p\tilde{S}.$$

then Λ and S are tensor fields in the base space (M, g).

We obtained, in (2. 8), $[C_\beta, E_b]=-P_{b\beta}{}^\alpha C_\alpha$, which is equivalent to

$$C^j{}_\beta\tilde{\nabla}_jE^h{}_b-E^j{}_b\tilde{\nabla}_jC^h{}_\beta=-P_{b\beta}{}^\alpha C^h{}_\alpha.$$

Substituting (2. 4) and (2. 22) into the equation above and taking account of (2. 25), we obtain

(2. 28) $$h_{cb}{}^\alpha=-(a^\alpha\varphi_{cb}+b^\alpha\psi_{cb}+c^\alpha\theta_{cb}),$$

(2. 29) $$\begin{aligned}P_{b\beta}{}^\alpha&=(\partial_b a_\beta)a^\alpha+(\partial_b b_\beta)b^\alpha+(\partial_b c_\beta)c^\alpha\\&=-a_\beta(\partial_b a^\alpha)-b_\beta(\partial_b b^\alpha)-c_\beta(\partial_b c^\alpha).\end{aligned}$$

If we substitute (2. 28) and (2. 29) into (2. 13), (2. 14) and (2. 19), then we have, using (2. 23) and (2. 25), respectively

(2. 30) $$\tilde{K}_{dcb}{}^a-K_{dcb}{}^a=-\Lambda_{cbd}{}^a+\Lambda_{dbc}{}^a+2\Lambda_{dcb}{}^a,$$

(2. 31) $$\tilde{K}_{dcb}{}^\alpha=a^\alpha\nabla_b\varphi_{cd}+b^\alpha\nabla_b\psi_{cd}+c^\alpha\nabla_b\theta_{cd},$$

(2. 32) $$\begin{aligned}\tilde{K}_{dc\beta}{}^\alpha=&(L_{cd}+{}'L_{cd})(c_\beta b^\alpha-b_\beta c^\alpha)+(M_{cd}+{}'M_{cd})(a_\beta c^\alpha-c_\beta a^\alpha)\\&+(N_{cd}+{}'N_{cd})(b_\beta a^\alpha-a_\beta b^\alpha),\end{aligned}$$

where we have put

$$L_b{}^a=\varphi_b{}^a+\psi_c{}^a\theta_b{}^c,\quad M_b{}^a=\psi_b{}^a+\theta_c{}^a\varphi_b{}^c,\quad N_b{}^a=\theta_b{}^a+\varphi_c{}^a\psi_b{}^c,$$

$$'L_b{}^a=\varphi_b{}^a-\theta_c{}^a\psi_b{}^c,\quad 'M_b{}^a=\psi_b{}^a-\varphi_c{}^a\theta_b{}^c,\quad 'N_b{}^a=\theta_b{}^a-\psi_c{}^a\varphi_b{}^c,$$

$$L_{cb}=L_c{}^a g_{ab},\qquad M_{cb}=M_c{}^a g_{ab},\quad \cdots,\quad \text{etc.},$$

$$\nabla_d\varphi_{cb}=\partial_d\varphi_{cb}-\left\{ {a \atop d\ c} \right\}\varphi_{ab}-\left\{ {a \atop d\ b} \right\}\varphi_{ca},\ \cdots,\ \text{etc.},$$

$\Lambda_{dcb}{}^a$ denoting components in $\tilde{U}$ of Λ^H, that is,

$$\Lambda_{dcb}{}^a=\varphi_{dc}\varphi_b{}^a+\psi_{dc}\psi_b{}^a+\theta_{dc}\theta_b{}^a. \tag{2.33}$$

By a similar way, we have, from (2.16),

$$\begin{aligned}\tilde{K}_{\delta cb}{}^\alpha=&-(\varphi_{ab}\varphi_c{}^a a_\delta a^\alpha+\psi_{ab}\psi_c{}^a b_\delta b^\alpha+\theta_{ab}\theta_c{}^a c_\delta c^\alpha)\\ &-a_\delta(N_{cb}b^\alpha-'M_{cb}c^\alpha)-b_\delta(L_{cb}c^\alpha-'N_{cb}a^\alpha)-c_\delta(M_{cb}a^\alpha-'L_{cb}b^\alpha).\end{aligned} \tag{2.34}$$

The Ricci tensor $\tilde{R}$ of $(\tilde{M}, \tilde{g})$ has the form

$$\tilde{R}=\tilde{K}_{cb}E^c\otimes E^b+\tilde{K}_{\gamma b}C^\gamma\otimes E^b+\tilde{K}_{c\beta}E^c\otimes C^\beta+\tilde{K}_{\gamma\beta}C^\gamma\otimes C^\beta \tag{2.35}$$

in $\tilde{U}$. Then, using (2.12), (2.18), (2.30), (2.31), (2.32), (2.33) and (2.34), we find

$$\begin{aligned}&\tilde{K}_{cb}=K_{cb}+2\Lambda_{acb}{}^a,\quad \tilde{K}_{\gamma b}=a_\gamma\nabla_a\varphi_b{}^a+b_\gamma\nabla_a\psi_b{}^a+c_\gamma\nabla_a\theta_b{}^a,\\ &\tilde{K}_{\gamma\beta}=(\varphi_{cb}\varphi^{cb})a_\gamma a_\beta+(\psi_{cb}\psi^{cb})b_\gamma b_\beta+(\theta_{cb}\theta^{cb})c_\gamma c_\beta+2g_{\gamma\beta},\end{aligned} \tag{2.36}$$

where $\varphi^{cb}=g^{ce}\varphi_e{}^b$, $\cdots$, etc., $\nabla_c\varphi_b{}^a=\partial_c\varphi_b{}^a+\{{}_c{}^a{}_e\}\varphi_b{}^e-\{{}_c{}^e{}_b\}\varphi_e{}^a$, $\cdots$, etc. and $K_{cb}=K_{acb}{}^a$ are components in $\tilde{U}$ of the lift R^L of the Ricci tensor R of the base space (M, g).

3. Some propositions.

The curvature tensor $\tilde{K}$ of $(\tilde{M}, \tilde{g})$ with triple of Killing vectors is projectable. Then, from (2.30), we have

PROPOSITION 3.1. *The curvature tensor K of the base space (M, g) of a fibred Riemonnian space with triple of Killing vectors is given by $K(X, Y)Z=(p\tilde{K})(X, Y)Z+\Lambda(Y, Z, X)-\Lambda(X, Z, Y)-2\Lambda(X, Y, Z)$, X, Y and Z being arbitrary vector fields in M, where $\Lambda=p\tilde{\Lambda}$ is a tensor field of type* (1.3) *defined in M by* (2.27) (cf. [3], [7]).

If we take account of (1.10) and (1.15), we find

$$\varphi=\varphi^H+\beta\otimes\zeta-\gamma\otimes\eta,\quad \psi=\psi^H+\gamma\otimes\xi-\alpha\otimes\zeta,\quad \theta=\theta^H+\alpha\otimes\eta-\beta\otimes\xi \tag{3.1}$$

and hence

$$\varphi^2=(\varphi^H)^2-(\beta\otimes\eta+\gamma\otimes\zeta),\ \psi^2=(\psi^H)^2-(\alpha\otimes\xi+\gamma\otimes\zeta),\ \theta^2=(\theta^H)^2-(\alpha\otimes\xi+\beta\otimes\eta); \tag{3.2}$$

$$\theta\psi=\theta^H\psi^H+\gamma\otimes\eta,\quad \varphi\theta=\varphi^H\theta^H+\alpha\otimes\zeta,\quad \psi\varphi=\psi^H\varphi^H+\beta\otimes\xi,$$
$$\psi\theta=\psi^H\theta^H+\beta\otimes\zeta,\quad \theta\varphi=\theta^H\varphi^H+\gamma\otimes\xi,\quad \varphi\psi=\varphi^H\psi^H+\alpha\otimes\eta. \tag{3. 3}$$

We now suppose that $p\tilde{K}=K$ (resp. $p\tilde{R}=R$) holds. Then, by (2. 30) (resp. by (2. 36)), we obtain $-\Lambda_{cbd}{}^a+\Lambda_{dbc}{}^a+2\Lambda_{dcb}{}^a=0$ (resp. $\Lambda_{acb}{}^a=0$), from which, by contraction, $\varphi_b{}^a\varphi_a{}^b+\psi_b{}^a\psi_a{}^b+\theta_b{}^a\theta_a{}^b=0$ and hence $\varphi_b{}^a=\theta_b{}^a=\psi_b{}^a=0$, i.e., $\varphi^H=\psi^H=\theta^H=0$. Thus, using (3. 1), we find

$$\tilde{\nabla}\xi=\beta\otimes\zeta-\gamma\otimes\eta,\quad \tilde{\nabla}\eta=\gamma\otimes\xi-\alpha\otimes\zeta,\quad \tilde{\nabla}\zeta=\alpha\otimes\eta-\beta\otimes\xi.$$

Taking account of these equations, we can easily verify that $(\tilde{M}, \tilde{g})$ is locally a pythagorean product of a fibre and a Riemannian manifold. Thus we have

PROPOSITION 3. 2. *Let $\tilde{K}$ and $\tilde{R}$ (resp. K and R) be respectively the curvature and the Ricci tensors of a fibred Riemannian space $(\tilde{M}, \tilde{g})$ with triple of Killing vectors (resp. of the base space). Then we have $p\tilde{K}=K$ (or $p\tilde{R}=R$) if and only if $(\tilde{M}, \tilde{g})$ is locally a pythagorean product of a fibre and a Riemannian manifold.*

Denote by $\sigma(\tilde{X}, \tilde{Y})$ the sectional curvature of $(\tilde{M}, \tilde{g})$ with respect to the section spanned by $\tilde{X}$ and $\tilde{Y}$. Thus we obtain, for unit vectors $\tilde{X}^V$ and $\tilde{Y}^H$, $\sigma(\tilde{X}^V, \tilde{Y}^H) = -\tilde{K}_{\delta c\beta a}X^\delta Y^c X^\beta Y^a$, where $\tilde{K}_{\delta c\beta a}=\tilde{K}_{\delta c\beta}{}^e g_{ea}$, $\tilde{K}_{\delta c\beta}{}^e=\tilde{K}_{kji}{}^h C^k{}_\delta E^j{}_c C^i{}_\beta E_h{}^e$ and $\tilde{X}^V = X^\beta C_\beta$, $\tilde{Y}^H=Y^b E_b$. Therefore, using (2. 34), we find for a unit vector $\tilde{Y}^H$

$$\sigma(\xi, \tilde{Y}^H)=-(\varphi_{ac}\varphi_b{}^a)Y^cY^b,\quad \sigma(\eta, \tilde{Y}^H)=-(\psi_{ac}\psi_b{}^a)Y^cY^b,$$
$$\sigma(\zeta, \tilde{Y}^H)=-(\theta_{ac}\theta_b{}^a)Y^cY^b. \tag{3. 4}$$

Thus, if and only if $\sigma(\tilde{X}^V, \tilde{Y}^H)=1$ for any $\tilde{X}$ and $\tilde{Y}$, we have

$$(\varphi^H)^2=-I^H,\quad (\psi^H)^2=-I^H,\quad (\theta^H)^2=-I^H, \tag{3. 5}$$

which are equivalent respectively to

$$\varphi^2=-I+\alpha\otimes\xi,\quad \psi^2=-I+\beta\otimes\eta,\quad \theta^2=-I+\gamma\otimes\zeta \tag{3. 6}$$

by means of (3. 2). Thus, taking account of Lemma 1, we have

PROPOSITION 3. 3. *In $(\tilde{M}, \tilde{g})$ with triple $\{\xi, \eta, \zeta\}$ of Killing vectors, each of ξ, η and ζ is a K-contact structure if and only if the sectional curvature of $(\tilde{M}, \tilde{g})$ with respect to any section containing at least one vertical vector is equal to* 1.

If ξ, η and ζ are all K-contact structures, then we have, from (2. 36) and (3. 5),

$$\tilde{K}_{cb}=K_{cb}-6g_{cb},\qquad \tilde{K}_{\gamma\beta}=(n-1)g_{\gamma\beta},\qquad K_{\gamma b}=0.$$

The last equality can be derived from Lemma 8 in [10]. Thus we have

LEMMA 3. 4. *Assume that, for a triple $\{\xi, \eta, \zeta\}$ of Killing vectors, each of ξ, η and ζ is a K-contact structure. Then, the base space (M, g) is a Einstein space if* and only if $(\tilde{M}, \tilde{g})$ *is so.*

We now consider the condition

$$(\tilde{K}(\tilde{X}^V, \tilde{Y}^H)\tilde{Z}^H)^V = -\tilde{g}(\tilde{Y}^H, \tilde{Z}^H)\tilde{X}^V, \tag{3.7}$$

which implies (3.5) and hence (3.6). Taking account of (2.32), we see that (3.7) implies

$$L_{cb} = 'L_{cb} = M_{cb} = 'M_{cb} = N_{cb} = 'N_{cb} = 0,$$

that is,

$$\theta^H\phi^H = -\phi^H\theta^H = \varphi^H, \quad \varphi^H\theta^H = -\theta^H\varphi^H = \phi^H, \quad \phi^H\varphi^H = -\varphi^H\phi^H = \theta^H, \tag{3.8}$$

which are equivalent to (1.17) by means of (3.3). Thus $\{\xi, \eta, \zeta\}$ is necessarily a K-contact 3-structure. Therefore we have

PROPOSITION 3.5. *A triple of Killing vectors is a K-contact* 3*-structure if and only if the condition* (3.7) *is satisfied* (cf. [9]).

By means of (2.31), the condition

$$(\tilde{K}(\tilde{X}^H, \tilde{Y}^H)\tilde{Z}^H)^V = 0 \tag{3.9}$$

is satisfied if and only if

$$\nabla_c\varphi_b{}^a = 0, \qquad \nabla_c\phi_b{}^a = 0, \qquad \nabla_c\theta_b{}^a = 0 \tag{3.10}$$

hold. The equations (3.10) imply $\nabla\Lambda = 0$ and $\nabla S = 0$ in (M, g). Thus we have

PROPOSITION 3.6. *If, in* $(\tilde{M}, \tilde{g})$ *with triple of Killing vectors, the condition* (3.9) *is satisfied, then* $\nabla\Lambda = 0$ *and* $\nabla S = 0$ *hold in* (M, g).

Taking account of (2.12), (2.18), (2.30), (2.31) and (2.32), we see that, for a K-contact 3-structure $\{\xi, \eta, \zeta\}$, ξ satisfies the condition (1.6) if and only if $\nabla_c\varphi_b{}^a = 0$ holds. Thus we have

LEMMA 3.7. *For a K-contact* 3*-structure* $\{\xi, \eta, \zeta\}$, ξ *is a Sasakian structure if and only if* $\nabla_c\varphi_b{}^a = 0$ *holds.*

If, for a K-contact 3-structure $\{\xi, \eta, \zeta\}$, $\nabla_c\varphi_b{}^a = 0$ and $\nabla_c\phi_b{}^a = 0$ are satisfied, then $\nabla_c\theta_b{}^a = 0$ holds because of $\theta_b{}^a = \phi_c{}^a\varphi_b{}^c$ (cf. (3.8)). Thus, as a corollary to Lemma 3.7, we have the following well known

PROPOSITION 3.8. *A K-contact* 3*-structure* $\{\xi, \eta, \zeta\}$ *is a Sasakian* 3*-structure if and only if two of* ξ, η *and* ζ *are Sasakian structures.*

Combining Propositions 3.3, 3.5, 3.6 and Lemma 3.7, we have

PROPOSITION 3.9. *A triple of Killing vectors is a Sasakian* 3*-structure if and only if the conditions* (3.7) *and* (3.9) *are satisfied.*

PROPOSITION 3. 10. *If* $(\tilde{M}, \tilde{g})$ *with triple* $\{\xi, \eta, \zeta\}$ *of Killing vectors is of constant curvature* c, *then* $c=1$ *and* $\{\xi, \eta, \zeta\}$ *is a Sasakian* 3-*structure.*

Recently, Kashiwada [4] has proved that any Riemannian manifold admitting a Sasakian 3-structure is a Einstein space. Then, by means of Lemma 3. 4, we see that, for any fibred Riemannian space with Sasakian 3-structure, the base space is a Einstein space.

§4. Nijenhuis tensors.

In $(\tilde{M}, \tilde{g})$ with triple $\{\xi, \eta, \zeta\}$ of Killing vectors, the Nijenhuis tensor $\tilde{N}(\varphi, \psi)$ of φ and ψ is, by definition, a tensor field of type (1, 2) with components

$$\begin{aligned}2\tilde{N}(\varphi, \psi)_{ji}{}^h &= \varphi_j{}^k\tilde{\nabla}_k\psi_i{}^h-\varphi_i{}^k\tilde{\nabla}_k\psi_j{}^h+\psi_j{}^k\tilde{\nabla}_k\varphi_i{}^h-\psi_i{}^k\tilde{\nabla}_k\varphi_j{}^h\\ &\quad -\varphi_k{}^h(\tilde{\nabla}_j\psi_i{}^k-\tilde{\nabla}_i\psi_j{}^k)-\psi_k{}^h(\tilde{\nabla}_j\varphi_i{}^k-\tilde{\nabla}_i\varphi_j{}^k)\end{aligned} \tag{4. 1}$$

and the Nijenhuis tensor $\tilde{N}(\varphi)$ of φ is defined by

$$\tilde{N}(\varphi)=\tilde{N}(\varphi, \varphi). \tag{4. 2}$$

As we have seen in (3. 1),

$$\varphi=\varphi^H+\gamma\otimes\eta-\beta\otimes\zeta.$$

Differentiating this covariantly and using (2. 4) and (2. 6), we have

$$\tilde{\nabla}_{E_c}\varphi=(\nabla_c\varphi_b{}^a)E^b\otimes E_a+(\varphi_b{}^a h_c{}^b{}_\beta)C^\beta\otimes E_a+(\varphi_b{}^a h_{ca}{}^\alpha)E^b\otimes C_\alpha, \tag{4. 3}$$

$$\tilde{\nabla}_{E_c}\Phi=(\nabla_c\varphi_{ba})E^b\otimes E^a+(\varphi_{ba}h_c{}^b{}_\beta)C^\beta\otimes E^a+(\varphi_{ba}h_c{}^a{}_\alpha)E^b\otimes C^\alpha. \tag{4. 4}$$

If we use (4. 3), we have, from (4. 2) and (4. 1),

$$\begin{aligned}\tilde{N}(\varphi)^H &= N(\varphi)_{cb}{}^a E^c\otimes E^b\otimes E_a,\\ \tilde{N}(\varphi, \psi)^H &= N(\varphi, \psi)_{cb}{}^a E^c\otimes E^b\otimes E_a\end{aligned} \tag{4. 5}$$

respectively, where

$$\begin{aligned}2N(\varphi, \psi)_{cb}{}^a &= \varphi_c{}^d\nabla_d\psi_b{}^a-\varphi_b{}^d\nabla_d\psi_c{}^a+\psi_c{}^d\nabla_d\psi_b{}^a-\psi_b{}^d\nabla_d\varphi_c{}^a\\ &\quad -\varphi_d{}^a(\nabla_c\psi_b{}^d-\nabla_b\psi_c{}^d)-\psi_d{}^a(\nabla_c\varphi_b{}^d-\nabla_b\varphi_c{}^d),\\ N_{cb}{}^a(\varphi) &= N(\varphi, \varphi)_{cb}{}^a,\end{aligned} \tag{4. 6}$$

On the other hand, we can verify

$$\begin{aligned}\tilde{N}(\varphi)^H &= \tilde{N}(\varphi)+2\Phi\otimes\xi,\\ \tilde{N}(\varphi, \psi)^H &= \tilde{N}(\varphi, \psi)+\Phi\otimes\eta+\Psi\otimes\xi.\end{aligned} \tag{4. 7}$$

Thus, taking account of (1. 5) and (4. 7), we see that, for a K-contact 3-structure $\{\xi, \eta, \zeta\}$, ξ is a Sasakian structure if and only if $\tilde{N}(\varphi)^H=0$. By means of Proposition 3. 8, a K-contact 3-structure $\{\xi, \eta, \zeta\}$ is a Sasakian 3-structure if and only if two of $\tilde{N}(\varphi)^H$, $\tilde{N}(\psi)^H$ and $\tilde{N}(\theta)^H$ vanish.

The condition $\tilde{N}(\varphi, \psi)^H=0$ is equivalent to

$$\varphi_c{}^d \nabla_d \psi_{ba} - \varphi_b{}^d \nabla_d \psi_{ca} + \psi_c{}^d \nabla_d \varphi_{ba} - \psi_b{}^d \nabla_d \varphi_{ca} - \varphi_a{}^d \nabla_d \psi_{cb} - \psi_a{}^d \nabla_d \varphi_{cb} = 0 \tag{4. 8}$$

by means of the identities

$$\nabla_c \varphi_{ba} + \nabla_b \varphi_{ac} + \nabla_a \varphi_{cb} = 0, \quad \nabla_c \psi_{ba} + \nabla_b \psi_{ac} + \nabla_a \psi_{cb} = 0, \quad \nabla_c \theta_{ba} + \nabla_b \theta_{ac} + \nabla_a \theta_{cb} = 0$$

which are consequences of $d\Phi=0$, $d\Psi=0$, $d\Theta=0$ and the equation (4. 4). If we add (4. 8) to the equation obtained by interchanging indices in (4. 8) in such a way that $c \to b \to a \to c$, then we have

$$\varphi_b{}^d \nabla_d \psi_c{}^a + \psi_b{}^d \nabla_d \varphi_c{}^a = 0. \tag{4. 9}$$

Conversely, (4. 9) implies (4. 8). Thus the condition $\tilde{N}(\varphi, \psi)^H=0$ is equivalent to (4. 9). Similarly, we can verify that the condition $\tilde{N}(\varphi)^H=0$ is equivalent to

$$\varphi_b{}^d \nabla_d \varphi_c{}^a = 0. \tag{4. 10}$$

For a K-contact 3-structure, the conditions (4. 9) and (4. 10) are equivalent respectively to

$$\varphi_b{}^d \nabla_d \varphi_c{}^a = \psi_b{}^d \nabla_d \psi_c{}^a, \tag{4. 11}$$

$$\nabla_d \varphi_c{}^a = 0. \tag{4. 12}$$

Thus, for a K-contact 3-structure, the conditions $\tilde{N}(\varphi, \psi)^H=0$ and $\tilde{N}(\varphi)^H=0$ are equivalent to (4. 11) and (4. 12) respectively. Thus we have

PROPOSITION 4. 1. *A K-contact 3-structure $\{\xi, \eta, \zeta\}$ is a Sasakian 3-structure if and only if two of $\tilde{N}^H(\varphi)$, $\tilde{N}^H(\psi)$ and $\tilde{N}^H(\varphi, \psi)$ vanish. In this case, $\tilde{N}^H(\theta)$, $\tilde{N}^H(\psi, \theta)$ and $\tilde{N}^H(\theta, \varphi)$ vanish.*

Recently, it has been proved in [13] that a K-contact 3-structure $\{\xi, \eta, \zeta\}$ is a Sasakian 3-structure if and only if one of $\tilde{N}^H(\psi, \theta)$, $\tilde{N}^H(\theta, \varphi)$ and $\tilde{N}^H(\varphi, \psi)$ vanishes.

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