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INVARIANT SUBMANIFOLDS OF AN f-MANIFOLD WITH COMPLEMENTED FRAMES

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To Shigeru Ishihara on his fiftieth birthday

§ 0. Introduction.

In an m-dimensional differentiable manifold M of class C^∞, a tensor field f of type (1, 1), which satisfies $f^3+f=0$ and is of constant rank r at each point of M, is called an *f-structure* of rank r and M with an f-structure an *f-manifold* ([9], [10]).

The tensor fields $-f^2$ and f^2+1, 1 being the unit tensor field, are complementary projection operators which define two complementary distributions in M corresponding to the projection operators $-f^2$ and f^2+1 respectively. The distribution corresponding to $-f^2$ is r-dimensional and that corresponding to f^2+1 $(m-r)$-dimensional.

The f-manifolds have been studied by Ishihara and the present author [3], [11].

If there exist $m-r$ vector fields U_a $(a=1, 2, \cdots, m-r)$ spanning the distribution corresponding to f^2+1 and $m-r$ 1-forms u^a satisfying

$$f^2=-1+\sum_{a=1}^{m-r} u^a \otimes U_a,$$

$$fU_a=0, \qquad u^a \circ f=0, \qquad u^a(U_b)=\delta_b^a,$$

$(a, b=1, 2, \cdots, m-r)$, where δ_b^a is the Kronecker delta, then the set (f, U_a, u^a) is called an *f-structure with complemented frames* and M an *f-manifold with complemented frames*.

The f-manifolds with complemented frames have been studied by Goldberg [1], [2], Nakagawa [5] and the present author [1], [2].

Now suppose that an n-dimensional differentiable manifold M' is immersed in an f-manifold M by the immersion i: $M' \to M$. If the tangent space of $i(M')$ is invariant by the action of f, $i(M')$ is called an *invariant submanifold* of M.

In the present paper, we consider an f-structure with complemented frames such that $r=m-2$. In § 1, we define and study the normality of such a structure.

In § 2, we show that such a structure induces an almost complex structure in

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the manifold and in §3, we study the relation between the integrability of this almost complex structure and the normality of the f-structure with complemented frames.

In §§4 and 5, we study invariant submanifolds, especially those of a normal f-manifold with complemented frames.

In §§6 and 7, we study metric f-structures with complemented frames and almost product structures related to the f-structures.

§1. f-structure with complemented frames.

Let M be an m-dimensional differentiable manifold and let there be given a tensor field f of type (1, 1) and of rank $m-2$, two vector fields U, V and two 1-forms u, v. If the set (f, U, V, u, v) satisfies

$$\begin{gathered} f^2=-1+u\otimes U+v\otimes V, \\ fU=0, \qquad fV=0, \qquad u\circ f=0, \qquad v\circ f=0, \\ u(U)=1, \qquad v(U)=0, \qquad u(V)=0, \qquad v(V)=1, \end{gathered} \tag{1.1}$$

then (f, U, V, u, v) is called an *f-structure with complemented frames* and M an *f-manifold with complemented frames* (cf. [1], [2], [5]).

We define a tensor field S of type (1, 2) by

$$S(X, Y)=[f, f](X, Y)+(du)(X, Y)U+(dv)(X, Y)V, \tag{1.2}$$

where $[f, f]$ is the Nijenhuis tensor formed with f, that is,

$$[f, f](X, Y)=[fX, fY]-f[fX, Y]-f[X, fY]+f^2[X, Y]. \tag{1.3}$$

If the tensor field S vanishes identically, then the structure is said to be *normal.*

From the definition (1.2) of S, we have, by using (1.1),

$$S(X, U)=-f[fX, U]+f^2[X, U]+(du)(X, U)U+(dv)(X, U)V.$$

But we have

$$-f[fX, U]+f^2[X, U]=f\{[U, fX]-f[U, X]\}=f(\mathcal{L}_U f)X,$$

where $\mathcal{L}_U$ denotes the Lie differentiation with respect to U, and

$$\begin{aligned} (du)(X, U)&=X(u(U))-U(u(X))-u([X, U]) \\ &=-(\mathcal{L}_U u)(X), \\ (dv)(X, U)&=X(v(U))-U(v(X))-v([X, U]) \\ &=-(\mathcal{L}_U v)(X), \end{aligned}$$

and consequently

$$S(X, U)=f(\mathcal{L}_U f)X-(\mathcal{L}_U u)(X)U-(\mathcal{L}_U v)(X)V. \tag{1.4}$$

Similarly, we can prove

$$S(X, V)=f(\mathcal{L}_V f)X-(\mathcal{L}_V u)(X)U-(\mathcal{L}_V v)(X)V. \tag{1.5}$$

We also have, from the definition (1.2) of S,

$$u(S(X, Y))=u([fX, fY])+(du)(X, Y).$$

But, from

$$(du)(fX, fY)=(fX)u(fY)-(fY)u(fX)-u([fX, fY]),$$

we have

$$u([fX, fY])=-(du)(fX, fY)$$

and consequently

$$u(S(X, Y))=(du)(X, Y)-(du)(fX, fY).$$

Replacing X by fX in this equation, we find, using (1.1),

$$\begin{aligned} &u(S(fX, Y)) \\ &=(du)(fX, Y)-(du)(-X+u(X)U+v(X)V, fY) \\ &=(du)(fX, Y)+(du)(X, fY)-u(X)(du)(U, fY) \\ &\quad -v(X)(du)(V, fY). \end{aligned}$$

But, we get

$$\begin{aligned} (du)(U, fY)&=Uu(fY)-(fY)u(U)-u([U, fY]) \\ &=-u((\mathcal{L}_U f)Y)-u(f\mathcal{L}_U Y)=(\mathcal{L}_U u)(fY) \end{aligned}$$

because of $u\circ f=0$, and similarly

$$(du)(V, fY)=(\mathcal{L}_V u)(fY),$$

and consequently

$$\begin{aligned} u(S(fX, Y))&=(du)(fX, Y)+(du)(X, fY) \\ &\quad -u(X)(\mathcal{L}_U u)(fY)-v(X)(\mathcal{L}_V u)(fY). \end{aligned} \tag{1.6}$$

We can also prove

$$v(S(fX, Y))=(dv)(fX, Y)+(dv)(X, fY) \tag{1.7}$$
$$-u(X)(\mathcal{L}_U v)(fY)-v(X)(\mathcal{L}_V v)(fY).$$

We now assume that the f-structure with complemented frames (f, U, V, u, v) is normal, then, from (1.4), we have

$$f(\mathcal{L}_U f)X-(\mathcal{L}_U u)(X)U-(\mathcal{L}_U v)(X)V=0,$$

from which, using (1.1),

$$\mathcal{L}_U u=0, \qquad \mathcal{L}_U v=0, \qquad f(\mathcal{L}_U f)=0.$$

Applying f to the last equation, we have

$$-\mathcal{L}_U f+u\circ(\mathcal{L}_U f)\otimes U+v\circ(\mathcal{L}_U f)\otimes V=0,$$

$$-\mathcal{L}_U f-\{(\mathcal{L}_U u)\circ f\}\otimes U-\{(\mathcal{L}_U v)\circ f\}\otimes V=0,$$

and consequently

$$\mathcal{L}_U f=0.$$

Similarly, from (1.5), we have

$$\mathcal{L}_V u=0, \qquad \mathcal{L}_V v=0, \qquad \mathcal{L}_V f=0.$$

From (1.6) and (1.7), we have

$$du \barwedge f=0 \quad \text{and} \quad dv \barwedge f=0$$

respectively, where we have put

$$(\omega \barwedge f)(X, Y)=\omega(fX, Y)+\omega(X, fY)$$

for a 2-form ω.

On the other hand, computing $\mathcal{L}_U(fV)=0$, we find

$$f\mathcal{L}_U V=0,$$

from which, applying f,

$$-\mathcal{L}_U V+u(\mathcal{L}_U V)U+v(\mathcal{L}_U V)V=0,$$

or

$$\mathcal{L}_U V=0, \quad \text{that is,} \quad [U, V]=0.$$

Thus we have

THEOREM 1.1. *If an f-structure with complemented frames (f, U, V, u, v) is normal, then we have*

$$\mathcal{L}_U f=0, \quad \mathcal{L}_U u=0, \quad \mathcal{L}_U v=0,$$
$$\mathcal{L}_V f=0, \quad \mathcal{L}_V u=0, \quad \mathcal{L}_V v=0, \tag{1.8}$$
$$du \barwedge f=0, \quad dv \barwedge f=0, \quad [U, V]=0.$$

§ 2. Almost complex structure F.

We define a tensor field F of type (1, 1) by

$$FX=fX-v(X)U+u(X)V \tag{2.1}$$

for an arbitrary vector field X. Then we have

$$\begin{aligned} F^2X &= f(fX-v(X)U+u(X)V) \\ &\quad -v(fX-v(X)U+u(X)V)U \\ &\quad +u(fX-v(X)U+u(X)V)V \\ &= f^2X-u(X)U-v(X)V, \end{aligned}$$

that is,

$$F^2=-1 \tag{2.2}$$

and consequently F is an almost complex structure (cf. [1]).

We can easily verify that

$$FU=V, \quad FV=-U, \tag{2.3}$$

$$u\circ F=-v, \quad v\circ F=u. \tag{2.4}$$

Conversely, suppose that an almost complex manifold M with structure tensor F admits a vector field U and a 1-form u such that

$$u(U)=1, \quad u(FU)=0. \tag{2.5}$$

We define a vector field V and a 1-form v by

$$V=FU, \tag{2.6}$$

$$v=-u\circ F \tag{2.7}$$

respectively, and a tensor field f by

$$f=F+v\otimes U-u\otimes V. \tag{2.8}$$

Then, from (2.5), (2.6) and (2.7), we have

$$u(V)=0, \quad v(U)=0, \quad v(V)=1,$$

and, from (2.8),

$$fU=0, \qquad fV=0, \qquad u\circ f=0, \qquad v\circ f=0.$$

We can also verify that

$$f^2=-1+u\otimes U+v\otimes V.$$

These equations show that M admits an f-structure with complemented frames (f, U, V, u, v). Thus we have

THEOREM 2.1. *In order that a manifold admits an f-structure with complemented frames* (f, U, V, u, v), *it is necessary and sufficient that the manifold admits an almost complex structure* F, *a vector field* U *and a* 1*-form* u *such that* $u(U)=1$ *and* $u(FU)=0$.

§ 3. Integrability condition of *F*.

We compute the Nijenhuis tensor

$$[F, F](X, Y)=[FX, FY]-F[FX, Y]-F[X, FY]-[X, Y]$$

formed with F. By the definition of F, we have

$$\begin{aligned}[F, F](X, Y)=&[fX-v(X)U+u(X)V, fY-v(Y)U+u(Y)V]\\ &-f[fX-v(X)U+u(X)V, Y]\\ &+v([fX-v(X)U+u(X)V, Y])U\\ &-u([fX-v(X)U+u(X)V, Y])V\\ &-f[X, fY-v(Y)U+u(Y)V]\\ &+v([X, fY-v(Y)U+u(Y)V])U\\ &-u([X, fY-v(Y)U+u(Y)V])V-[X, Y],\end{aligned}$$

from which, after some calculations,

$$\begin{aligned}[F, F](X, Y)=&[f, f](X, Y)+(du)(X, Y)U+(dv)(X, Y)V\\ &-(dv\barwedge f)(X, Y)U+(du\barwedge f)(X, Y)V\\ &-v(X)(\mathcal{L}_U f)Y+v(Y)(\mathcal{L}_U f)X-u(X)(\mathcal{L}_V f)Y+u(Y)(\mathcal{L}_V f)X\\ &+(v(X)(\mathcal{L}_U v)(Y)-v(Y)(\mathcal{L}_U v)(X)-u(X)(\mathcal{L}_V v)(Y)+u(Y)(\mathcal{L}_V v)(X))U\\ &+(u(X)(\mathcal{L}_V u)(Y)-u(Y)(\mathcal{L}_V u)(X)-v(X)(\mathcal{L}_U u)(Y)+v(Y)(\mathcal{L}_U u)(X))V\\ &+(u(X)v(Y)-u(Y)v(X))[U, V].\end{aligned} \tag{3.1}$$

Thus, we have (cf. [1], [2])

THEOREM 3.1. *If an f-structure with complemented frames (f, U, V, u, v) is normal, then the almost complex structure F defined by* (2.1) *is integrable.*

§4. Invariant submanifolds.

Let M' be an n-dimensional differentiable manifold $(1<n<m)$ and suppose that M' is immersed in M by the immersion i: $M' \to M$. We denote by B the differential di of the immersion i.

We assume that the vector field U is tangent to $i(M')$, any vector tangent to $i(M')$ annihilate the 1-form v and the tangent space to $i(M')$ is invariant by f. Then we have

$$U=BU' \tag{4.1}$$

for a vector field U' of M',

$$v(BX')=0 \tag{4.2}$$

for any vector field X' of M', and

$$f(BX')=Bf'X' \tag{4.3}$$

for a tensor field f' of M' and an arbitrary vector field X' of M'. For simplicity, we call such a submanifold an *invariant submanifold* with respect to U and v. We can also define an invariant submanifold with respect to V and u.

Now, applying f to (4.1), we find

$$0=fU=fBU'=Bf'U'$$

because of (4.3), from which,

$$f'U'=0.$$

Applying f to (4.3), we find

$$-BX'+u(BX')U+v(BX')V=Bf'^2X',$$

from which, using (4.1) and (4.2),

$$f'^2X'=-X'+u'(X')U',$$

where we have put $u'(X')=u(BX')$.

From (4.3), we find

$$0=u(f(BX'))=u(Bf'X'),$$

and consequently

$$u'(f'X')=0.$$

From (4.1), we have

$$1=u(U)=u(BU'),$$

that is,

$$u'(U')=1.$$

Thus, summing up, we have

$$(4.4)\qquad \begin{gathered} f'^2=-1+u'\otimes U', \\ f'U'=0, \qquad u'\circ f'=0, \\ u'(U')=1, \end{gathered}$$

which show that (f', U', u') defines an almost contact structure (cf. [6], [7], [8]). Thus we have

THEOREM 4.1. *An invariant submanifold with respect to U and v of a manifold with f-structure with complemented frames (f, U, V, u, v) admits an almost contact structure.*

We can prove a similar theorem on an invariant submanifold with respect to V and u.

§ 5. Invariant submanifolds of a normal *f*-manifold with complemented frames.

We now compute the expression $S(BX', BY')$ for an invariant submanifold with respect to U and v. We have

$$\begin{aligned} & S(BX', BY') \\ &=[fBX', fBY']-f[fBX', BY']-f[BX', fBY']+f^2[BX', BY'] \\ &\quad +(du)(BX', BY')U+(dv)(BX', BY')V \\ &=[Bf'X', Bf'Y']-f[Bf'X', BY']-f[BX', Bf'Y']+f^2[BX', BY'] \\ &\quad +(du)(BX', BY')U+(dv)(BX', BY')V \\ &=B[f'X', f'Y']-fB[f'X', Y']-fB[X', f'Y']+f^2B[X', Y']+(du')(X', Y')U, \end{aligned}$$

because of

$$(du)(B'X', B'Y')=(du')(X', Y'),\ (dv)(B'X', B'Y')=0,$$

and consequently

$$\begin{aligned} & S(B'X', B'Y') \\ &=B\{[f'X', f'Y']-f'[f'X', Y']-f'[X', f'Y']+f'^2[X', Y']+(du')(X', Y')U'\}. \end{aligned}$$

Thus, we have

THEOREM 5.1. *An invariant submanifold with respect to U and v of a manifold with normal f-structure with complemented frames (f, U, V, u, v) admits an almost normal contact structure.*

We can prove a similar theorem on invariant submanifold with respect to V and u.

§ 6. Metric f-structure with complemented frames.

Let M be an m-dimensional differentiable manifold with f-structure with complemented frames (f, U, V, u, v). If there exists on M a Riemannian metric g satisfying

$$(6.1)\qquad \begin{aligned} &g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y), \\ &u(X)=g(U, X),\ v(X)=g(V, X) \end{aligned}$$

for arbitrary vector fields X and Y, we call the structure (f, U, V, u, v) a metric f-structure with complemented frames and denote it by (f, g, u, v).

Suppose that M admits an (f, g, u, v)-structure. We know that the tensor field F of type (1, 1) defined by (2.1) is an almost complex structure. For this tensor field F, we have

$$\begin{aligned} &g(FX, FY) \\ &=g(fX-v(X)U+u(X)V, fY-v(Y)U+u(Y)V) \\ &=g(fX, fY)+v(X)v(Y)+u(X)u(Y) \end{aligned}$$

and consequently

$$(6.2)\qquad g(FX, FY)=g(X, Y).$$

Thus (F, g) defines an almost Hermitian structure.

Thus, using Theorem 3.1, we have

THEOREM 6.1. *If a metric f-structure with complemented frames (f, g, u, v) is normal, then the almost complex structure F is Hermitian.*

Let $i(M')$ be an invariant submanifold with respect to U and v of a manifold M with a metric f-structure with complemented frames. The manifold M being a Riemannian manifold with metric tensor g, $i(M')$ is also a Riemannian manifold with metric tensor

$$(6.3)\qquad g'(X', Y')=g(BX', BY').$$

In the first equation of (6.1), replacing X and Y by BX' and BY' respectively, we have

$$g(fBX', fBY')=g(BX', BY')-u(BX')u(BY')-v(BX')v(BY').$$

But, $i(M')$ being invariant, we have

$$g(Bf'X', Bf'Y')=g(BX', BY')-u(BX')u(BY'),$$

that is,

$$g'(f'X', f'Y')=g'(X', Y')-u'(X')u'(Y').$$

We also have, replacing X by BX' in the second equation of (6.1),

$$u(BX')=g(U, BX'),\ u(BX')=g(BU', BX'),$$

that is,

$$u'(X')=g'(U', X').$$

From the third equation of (6.1), we have

$$v(BX')=g(V, BX'),$$

that is,

$$0=g(V, BX'),$$

which shows that V is a unit normal to the submanifold $i(M')$. Thus we have

THEOREM 6.2. *An invariant submanifold with respect to U and v of a manifold with metric f-structure with complemented frames (f, g, u, v) admits an almost contact metric structure.*

Suppose that the (f, g, u, v)-structure satisfies

$$du(X, Y)=g(fX, Y), \tag{6.4}$$

then, for an invariant submanifold with respect to U and v of a manifold with metric f-structure with complemented frames (f, g, u, v), we have

$$du(BX', BY')=g(fBX', BY'),$$

$$du(BX', BY')=g(Bf'X', BY'),$$

$$du'(X', Y')=g'(f'X', Y')$$

and consequently the almost contact structure induced on the invariant submanifold is contact. Thus we have

THEOREM 6.3. *An invariant submanifold with respect to U and v of a manifold with normal metric f-structure with complemented frames (f, g, u, v) satisfying* (6.4) *admits a normal metric contact structure, that is, a Sasakian structure.*

We can prove a similar theorem on an invariant submanifold with respect to V and u of a manifold with normal metric f-structure with complemented frames satisfying

$$dv(X, Y)=g(fX, Y). \tag{6.5}$$

§7. Almost product structure.

Let M be a differentiable manifold with metric f-structure with complemented frames (f, g, u, v), and suppose that there are given two complementary projection tensors P and Q:

$$P^2=P, \qquad Q^2=Q, \qquad PQ=QP=0, \qquad P+Q=1. \tag{7.1}$$

If we put

$$K=P-Q,$$

then we find

$$K^2=1, \qquad P=\frac{1}{2}(1+K), \qquad Q=\frac{1}{2}(1-K).$$

We call a structure given by (P, Q) or by K an *almost product structure*. If

$$\nabla P=0, \qquad \nabla Q=0$$

or

$$\nabla K=0,$$

where ∇ denotes the covariant differentiation with respect to g, we say that the almost product structure is integrable.

If M is complete and simply connected and the almost product structure is integrable, then we have

$$M=M'\times M'',$$

the distribution defined by P being tangent to M' and that defined by Q tangent to M''.

We now suppose that a complete and simply connected manifold M with normal metric f-structure with complemented frames (f, g, u, v) satisfying (6.4) admits an almost product structure (P, Q) such that

$$PU=U, \qquad v\circ P=0, \qquad QfP=0, \tag{7.2}$$

and that the almost product structure is integrable.

Since $M=M'\times M''$ and the last equation of (7.2) can be written as

$$fP=PfP,$$

we see that M' is an invariant submanifold with respect to U and v of $M'\times M''$ with normal metric f-structure with complemented frames (f, g, u, v) satisfying (6.4). Thus, by Theorem 6.3, M' admits a Sasakian structure. Thus we have

THEOREM 7.1. *Suppose that a complete and simply connected M with normal metric f-structure with complemented frames* (f, g, u, v) *satisfying* (6.4) *admits an* *integrable almost product structure (P, Q) such that (7.2) holds, then $M=M'\times M''$ where M' is the integral submanifold of the distribution defined by P and M' is a Sasakian manifold.*

We can prove a similar theorem in which (6.4) is replaced by (6.5), (7.2) by

$$QV=V, \qquad u\circ Q=0, \qquad PfQ=0 \tag{7.3}$$

and M' and M'' are interchanged.

Thus combining Theorem 7.1 and a theorem similar to this, we have

THEOREM 7.2. *Suppose that a complete and simply connected M with normal metric f-structure with complemented frames (f, g, u, v) satisfying* (6.4) *and* (6.5) *admits an integrable almost product structure (P, Q) such that* (7.2) *and* (7.3) *hold, then $M=M'\times M''$ where M' is an integral submanifold of the distribution defined by P and M'' that defined by Q and M' and M'' are both Sasakian.* (cf. [4]).

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