H. SUZUKI
KODAI MATH. SEM. REP.
25 (1973), 153—162

NOTES ON (f, U, V, u, v, λ)-STRUCTURES

BY HIDEAKI SUZUKI

Sasaki and Hatakeyama [1] proved that, if a differentiable manifold M admits an almost contact structure (φ, ξ, η), then there is in the product space $M\times R$, R being a real line, an almost complex structure F which is canonically constructed from φ, ξ and η. They defined an almost contact structure (φ, ξ, η) to be normal when this almost complex structure F is integrable in $M\times R$. The normality of an almost contact structure (φ, ξ, η) is characterized by vanishing of a certain tensor field constructed from φ, ξ, and η. Recently, Yano and Okumura [5] have defined a new structure in an even-dimensional manifold called an (f, U, V, u, v, λ)-structure as a set of a tensor field f of type (1, 1), two vector fields U, V, two 1-forms u, v and a scalar field λ satisfying certain algebraic conditions. They have showed that there exists naturally an (f, U, V, u, v, λ)-structure in a submanifold of codimension 2 immersed in an almost complex manifold or in a hypersurface immersed in an almost contact manifold [5]. One of the purposes of the present paper is to show that, if an even-dimensional manifold admits an (f, U, V, u, v, λ)-structure, then there is in the product space $M\times R^2$, R^2 being a plane, an almost complex structure F constructed from f, U, V, u, v and λ and to obtain a necessary and sufficient condition for the almost complex structure F to be integrable. Another purpose is to show that a hypersurface M immersed in a unit sphere $S^{2n+1}(1)$ is isometric to the hypersurface $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ if M satisfies certain conditions.

In §1, we recall the definition of an (f, U, V, u, v, λ)-structure and that of an (f, g, u, v, λ)-structure. In §2, we define an almost complex structure F in the product space $M\times R^2$, when an (f, U, V, u, v, λ)-structure is given in M and, by using local components of the Nijenhuis tensor of F, we define in M a tensor field T of type (1, 2), tensor fields P_1 and P_2 of type (0, 2), tensor fields Q_1 and Q_2 of type (1, 1), a vector field S, 1-forms w_1, w_2, w_3, w_4 and functions k_1, k_2. We study some properties of these tensor fields and obtain a necessary and sufficient condition for F to be integrable. In §3, we study the Riemannian connection of a Riemannian metric G defined naturally in $M\times R^2$ in terms of g, when an (f, g, u, v, λ)-structure is given in M. In the last §4, we prove a proposition stating that a hypersurface immersed in a unit sphere of odd dimension is isometric to the product of two spheres of the same dimension and of the same radius.

Received April 28, 1972.

§ 1. Preliminaries.

Let M be an m-dimensional differentiable manifold of class C^∞.[1] If there exist in M a (1, 1)-tensor field f, two vector fields U and V, 1-forms u and v and a function λ satisfying the following conditions (1. 1)~(1. 5), then we say that M has an (f, U, V, u, v, λ)-structure (f, U, V, u, v, λ).

$$f^2=-I+u\otimes U+v\otimes V, \tag{1. 1}$$

I being the unit tensor field of type (1, 1),

$$u\circ f=\lambda v, \qquad fU=-\lambda V, \tag{1. 2}$$

$$v\circ f=-\lambda u, \qquad fV=\lambda U, \tag{1. 3}$$

where 1-forms $u\circ f$ and $v\circ f$ are respectively defined by $(u\circ f)(X)=u(fX)$ and $(v\circ f)(X)=v(fX)$ for any vector field X, and

$$u(U)=1-\lambda^2, \qquad u(V)=0, \tag{1. 4}$$

$$v(U)=0, \qquad v(V)=1-\lambda^2. \tag{1. 5}$$

It is well-known that a differentiable manifold with (f, U, V, u, v, λ)-structure is necessarily of even dimension and that any submanifold of codimension 2 immersed in an almost complex manifold and any hypersurface immersed in an almost contact manifold admit an (f, U, V, u, v, λ)-structure [5]. If a manifold with (f, U, V, u, v, λ)-structure (f, U, V, u, v, λ) has a positive definite Riemannian metric g satisfying the conditions:

$$g(U, X)=u(X), \tag{1. 6}$$

$$g(V, X)=v(X), \tag{1. 7}$$

$$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y) \tag{1. 8}$$

for any vector fields X and Y, then we say that M has an (f, g, u, v, λ)-structure (f, g, u, v, λ). Any submanifold of condimension 2 immersed in an almost Hermitian manifold and any hypersurface immersed in an almost contact metric manifold admit an (f, g, u, v, λ)-structure [5].

§ 2. An almost complex structure in $M\times R^2$.

Suppose that a $2n$-dimensional manifold M has an (f, U, V, u, v, λ)-structure (f, U, V, u, v, λ).

1) Manifolds and geometric objects we discuss are assumed to be differentiable and of class C^∞.

We define in $M\times R^2$, R^2 being a plane, a tensor field F of type (1, 1) with local components $F_\mu{}^\lambda$ given by[2)]

$$(2.1)\qquad (F_\mu{}^\lambda)=\begin{pmatrix} f_i{}^h & U^h & V^h \\ -u_i & 0 & -\lambda \\ -v_i & \lambda & 0 \end{pmatrix}$$

in $\{W\times R^2;\ x^\lambda\}$, $\{W;\ x^h\}$ being a coordinate neighborhood of M and x^{1^*}, x^{2^*} being cartesian coordinates in R^2, where $f_i{}^h$, U^h, V^h, u_i and v_i are respectively local components of f, U, V, u and v in $\{W;\ x^h\}$. Then, taking account of (1. 1)$\sim$(1. 5), we can easily verify that $F^2=-I$ holds in $M\times R^2$. Thus we have

PROPOSITION 1. *If there is given an (f, U, V, u, v, λ)-structure in M, then the tensor field F defined by* (2. 1) *is an almost complex structure in* $M\times R^2$.

The Nijenhuis tensor N of F has local components

$$(2.2)\qquad N_{\nu\mu}{}^\lambda=F_\nu{}^\alpha\partial_\alpha F_\mu{}^\lambda-F_\mu{}^\alpha\partial_\alpha F_\nu{}^\lambda-(\partial_\nu F_\mu{}^\alpha-\partial_\mu F_\nu{}^\alpha)F_\alpha{}^\lambda$$

in $M\times R^2$ (cf. Yano [2]).[3)] Thus, using (2. 1), we can write down $N_{\nu\mu}{}^\lambda$ as follows:

$$(2.3)_1\qquad N_{ji}{}^h=\bar{N}_{ji}{}^h+(\partial_j u_i-\partial_i u_j)U^h+(\partial_j v_i-\partial_i v_j)V^h,$$

where $\bar{N}$ is the Nijenhuis tensor of f, and

$$(2.3)_2\qquad N_{ji}{}^{1^*}=-f_j{}^m\partial_m u_i+f_i{}^m\partial_m u_j+u_m(\partial_j f_i{}^m-\partial_i f_j{}^m)-\lambda(\partial_j v_i-\partial_i v_j),$$

$$(2.3)_3\qquad N_{ji}{}^{2^*}=-f_j{}^m\partial_m v_i+f_i{}^m\partial_m v_j+v_m(\partial_j f_i{}^m-\partial_i f_j{}^m)+\lambda(\partial_j u_i-\partial_i u_j),$$

$$(2.3)_4\qquad N_{1^*i}{}^h=(\mathcal{L}_U f)_i{}^h+V^h\partial_i\lambda,$$

$$(2.3)_5\qquad N_{2^*i}{}^h=(\mathcal{L}_V f)_i{}^h-U^h\partial_i\lambda,$$

$$(2.3)_6\qquad N_{1^*i}{}^{1^*}=-(\mathcal{L}_U u)_i-\lambda\partial_i\lambda,$$

$$(2.3)_7\qquad N_{1^*i}{}^{2^*}=-(\mathcal{L}_U v)_i-f_i{}^m\partial_m\lambda,$$

$$(2.3)_8\qquad N_{2^*i}{}^{1^*}=-(\mathcal{L}_V u)_i+f_i{}^m\partial_m\lambda,$$

$$(2.3)_9\qquad N_{2^*i}{}^{2^*}=-(\mathcal{L}_V v)_i-\lambda\partial_i\lambda,$$

$$(2.3)_{10}\qquad N_{1^*2^*}{}^h=[U, V]^h,$$

$$(2.3)_{11}\qquad N_{1^*2^*}{}^{1^*}=-\mathcal{L}_U\lambda,$$

$$(2.3)_{12}\qquad N_{1^*2^*}{}^{2^*}=-\mathcal{L}_V\lambda,$$

2) The indices $\alpha, \beta, \gamma, \cdots, \lambda, \mu, \nu$ run over the range $\{1, \cdots, 2n+2\}$ and $a, b, c, \cdots, i, j, k$ the range $\{1, \cdots, 2n\}$. We denote $n+1$ and $n+2$ by 1^* and 2^* respectively. The Einstein's summation convention will be used with respect to these two systems of indices.

3) We denote $\partial/\partial x^\lambda$ by ∂_λ.

where $\mathcal{L}_U$ and $\mathcal{L}_V$ denote the operators of Lie derivation with respect to U and V, respectively.

We can casily verify that there are in M a tensor field T of type (1, 2) with components $N_{ji}{}^h$, two tensor fields P_1 and P_2 of type (0, 2) with components $N_{ji}{}^{1^*}$ and $N_{ji}{}^{2^*}$ respectively, two tensor fields Q_1 and Q_2 of type (1, 1) with components $N_{1^*i}{}^h$ and $N_{2^*i}{}^h$ respectively, a vector field S with components $N_{1^*2^*}{}^h$, four 1-forms w_1, w_2, w_3, w_4 with components $N_{1^*i}{}^{1^*}$, $N_{1^*i}{}^{2^*}$, $N_{2^*i}{}^{1^*}$ and $N_{2^*i}{}^{2^*}$ respectively and two functions $k_1=N_{1^*2^*}{}^{1^*}$ and $k_2=N_{1^*2^*}{}^{2^*}$ (cf. Lemma 4). The Nijenhuis tensor N of an almost complex structure F satisfies identically the conditions

(2. 4) $$N_{\nu\alpha}{}^{\lambda}F_{\mu}{}^{\alpha}+N_{\nu\mu}{}^{\alpha}F_{\alpha}{}^{\lambda}=0$$

and

(2. 5) $$N_{\nu\alpha}{}^{\lambda}F_{\mu}{}^{\alpha}-N_{\alpha\mu}{}^{\lambda}F_{\nu}{}^{\alpha}=0$$

(cf. Yano [2]). Substituting (2. 1) into (2. 4), we have

$(2.6)_1$ $$N_{jm}{}^hf_i{}^m+N_{ji}{}^mf_m{}^h+N_{ji}{}^{1^*}U^h+N_{ji}{}^{2^*}V^h+N_{1^*j}{}^hu_i+N_{2^*j}{}^hv_i=0,$$

$(2.6)_2$ $$-N_{ji}{}^mu_m+N_{jm}{}^{1^*}f_i{}^m-\lambda N_{ji}{}^{2^*}+N_{1^*j}{}^{1^*}u_i+N_{2^*j}{}^{1^*}v_i=0,$$

$(2.6)_3$ $$-N_{ji}{}^mv_m+\lambda N_{ji}{}^{1^*}+N_{jm}{}^{2^*}f_i{}^m+N_{1^*j}{}^{2^*}u_i+N_{2^*j}{}^{2^*}v_i=0,$$

$(2.6)_4$ $$N_{jm}{}^hU^m-N_{1^*j}{}^mf_m{}^h-\lambda N_{2^*j}{}^h-N_{1^*j}{}^{1^*}U^h-N_{1^*j}{}^{2^*}V^h=0,$$

$(2.6)_5$ $$N_{jm}{}^hV^m+\lambda N_{1^*j}{}^h-N_{2^*j}{}^mf_m{}^h-N_{2^*j}{}^{1^*}U^h-N_{2^*j}{}^{2^*}V^h=0,$$

$(2.6)_6$ $$N_{1^*m}{}^hf_i{}^m+N_{1^*i}{}^mf_m{}^h+N_{1^*i}{}^{1^*}U^h+N_{1^*i}{}^{2^*}V^h-N_{1^*2^*}{}^hv_i=0,$$

$(2.6)_7$ $$N_{2^*m}{}^hf_i{}^m+N_{2^*i}{}^mf_m{}^h+N_{2^*i}{}^{1^*}U^h+N_{2^*i}{}^{2^*}V^h+N_{1^*2^*}{}^hu_i=0,$$

$(2.6)_8$ $$N_{jm}{}^{1^*}U^m+N_{1^*j}{}^mu_m+\lambda N_{1^*j}{}^{2^*}-\lambda N_{2^*j}{}^{1^*}=0,$$

$(2.6)_9$ $$-N_{1^*i}{}^mu_m+N_{1^*m}{}^{1^*}f_i{}^m-\lambda N_{1^*i}{}^{2^*}-N_{1^*2^*}{}^{1^*}v_i=0,$$

$(2.6)_{10}$ $$N_{jm}{}^{1^*}V^m-N_{2^*j}{}^mu_m+\lambda N_{1^*j}{}^{1^*}+\lambda N_{2^*j}{}^{2^*}=0,$$

$(2.6)_{11}$ $$-N_{2^*i}{}^mu_m+N_{2^*m}{}^{1^*}f_i{}^m-\lambda N_{2^*i}{}^{2^*}+N_{1^*2^*}{}^{1^*}u_i=0,$$

$(2.6)_{12}$ $$N_{1^*m}{}^{1^*}U^m+\lambda N_{1^*2^*}{}^{1^*}=0,$$

$(2.6)_{13}$ $$N_{1^*m}{}^{1^*}V^m-N_{1^*2^*}{}^mu_m-\lambda N_{1^*2^*}{}^{2^*}=0,$$

$(2.6)_{14}$ $$N_{2^*m}{}^{1^*}U^m+N_{1^*2^*}{}^mu_m+\lambda N_{1^*2^*}{}^{2^*}=0,$$

$(2.6)_{15}$ $$N_{2^*m}{}^{1^*}V^m+\lambda N_{1^*2^*}{}^{1^*}=0,$$

$(2.6)_{16}$ $$N_{jm}{}^{2^*}U^m+N_{1^*j}{}^mv_m-\lambda N_{1^*j}{}^{1^*}-\lambda N_{2^*j}{}^{2^*}=0,$$

$(2.6)_{17}$ $$-N_{1^*i}{}^mv_m+\lambda N_{1^*i}{}^{1^*}+N_{1^*m}{}^{2^*}f_i{}^m-N_{1^*2^*}{}^{2^*}v_i=0,$$

(2. 6)$_{18}$ $$N_{jm}{}^{2^*}V^m+N_{2^*j}{}^m v_m+\lambda N_{1^*j}{}^{2^*}-\lambda N_{2^*j}{}^{1^*}=0,$$

(2. 6)$_{19}$ $$N_{1^*m}{}^{2^*}U^m+\lambda N_{1^*2^*}{}^{2^*}=0,$$

(2. 6)$_{20}$ $$N_{1^*m}{}^{2^*}V^m-N_{1^*2^*}{}^m v_m+\lambda N_{1^*2^*}{}^{1^*}=0,$$

(2. 6)$_{21}$ $$N_{2^*m}{}^{2^*}U^m+N_{1^*2^*}{}^m v_m-\lambda N_{1^*2^*}{}^{1^*}=0,$$

(2. 6)$_{22}$ $$N_{2^*m}{}^{2^*}V^m+\lambda N_{1^*2^*}{}^{2^*}=0.$$

Substituting (2. 1) into (2. 5), we have

(2. 7)$_1$ $$N_{jm}{}^h f_i{}^m-N_{mi}{}^h f_j{}^m+N_{1^*j}{}^h u_i+N_{1^*i}{}^h u_j+N_{2^*j}{}^h v_i+N_{2^*i}{}^h v_j=0,$$

(2. 7)$_2$ $$N_{jm}{}^{1^*}f_i{}^m-N_{mi}{}^{1^*}f_j{}^m+N_{1^*j}{}^{1^*}u_i+N_{1^*i}{}^{1^*}u_j+N_{2^*j}{}^{1^*}v_i+N_{2^*i}{}^{1^*}v_j=0,$$

(2. 7)$_3$ $$N_{jm}{}^{2^*}f_i{}^m-N_{mi}{}^{2^*}f_j{}^m+N_{1^*j}{}^{2^*}u_i+N_{1^*i}{}^{2^*}u_j+N_{2^*j}{}^{2^*}v_i+N_{2^*i}{}^{2^*}v_j=0,$$

(2. 7)$_4$ $$N_{jm}{}^h U^m+N_{1^*m}{}^h f_j{}^m-\lambda N_{2^*j}{}^h-N_{1^*2^*}{}^h v_j=0,$$

(2. 7)$_5$ $$N_{jm}{}^h V^m+\lambda N_{1^*j}{}^h+N_{2^*m}{}^h f_j{}^m+N_{1^*2^*}{}^h u_j=0,$$

(2. 7)$_6$ $$N_{jm}{}^{1^*}U^m+N_{1^*m}{}^{1^*}f_j{}^m-\lambda N_{2^*j}{}^{1^*}-N_{1^*2^*}{}^{1^*}v_j=0,$$

(2. 7)$_7$ $$N_{jm}{}^{1^*}V^m+\lambda N_{1^*j}{}^{1^*}+N_{2^*m}{}^{1^*}f_j{}^m+N_{1^*2^*}{}^{1^*}u_j=0,$$

(2. 7)$_8$ $$N_{1^*m}{}^{1^*}V^m+N_{2^*m}{}^{1^*}U^m=0,$$

(2. 7)$_9$ $$N_{jm}{}^{2^*}U^m+N_{1^*m}{}^{2^*}f_j{}^m-\lambda N_{2^*j}{}^{2^*}-N_{1^*2^*}{}^{2^*}v_j=0,$$

(2. 7)$_{10}$ $$N_{jm}{}^{2^*}V^m+\lambda N_{1^*j}{}^{2^*}+N_{2^*m}{}^{2^*}f_j{}^m+N_{1^*2^*}{}^{2^*}u_j=0,$$

(2. 7)$_{11}$ $$N_{1^*m}{}^{2^*}V^m-N_{m2^*}{}^{2^*}U^m=0.$$

Now, we assume that the function $1-\lambda^2$ is non-zero almost everywhere in M. Transvecting (2. 6)$_1$ with U^i and with V^i, we get respectively

(2. 8)$_1$ $$N_{1^*j}{}^h=\frac{1}{1-\lambda^2}(\lambda N_{jm}{}^h V^m-N_{ji}{}^m f_m{}^h U^i-N_{ji}{}^{1^*}U^i U^h-N_{ji}{}^{2^*}U^i V^h)$$

and

(2. 8)$_2$ $$N_{2^*j}{}^h=\frac{-1}{1-\lambda^2}(\lambda N_{jm}{}^h U^m+N_{ji}{}^m f_m{}^h V^i+N_{ji}{}^{1^*}V^i U^h+N_{ji}{}^{2^*}V^i V^h).$$

Transvecting (2. 6)$_2$ with U^i and with V^i, we have respectively

(2. 8)$_3$ $$N_{1^*j}{}^{1^*}=\frac{1}{1-\lambda^2}(N_{ji}{}^m U^i u_m+\lambda N_{jm}{}^{1^*}V^m+\lambda N_{ji}{}^{2^*}U^i)$$

and

(2. 8)$_4$ $$N_{2^*j}{}^{1^*}=\frac{1}{1-\lambda^2}(N_{ji}{}^m V^i u_m-\lambda N_{jm}{}^{1^*}U^m+\lambda N_{ji}{}^{2^*}V^i).$$

Similarly, transvecting $(2.6)_3$ with U^i and with V^i, we have respectively

$$(2.8)_5 \qquad N_{1^*j}{}^{2^*}=\frac{1}{1-\lambda^2}(N_{ji}{}^m U^i v_m-\lambda N_{ji}{}^{1^*}U^i+\lambda N_{jm}{}^{2^*}V^m)$$

and

$$(2.8)_6 \qquad N_{2^*j}{}^{2^*}=\frac{1}{1-\lambda^2}(N_{ji}{}^m V^i v_m-\lambda N_{ji}{}^{1^*}V^i-\lambda N_{jm}{}^{2^*}U^m).$$

By the same devices as above, we have from $(2.6)_6$

$$(2.8)_7 \qquad N_{1^*2^*}{}^h=\frac{1}{1-\lambda^2}(N_{1^*m}{}^h U^m+N_{1^*i}{}^m V^i f_m{}^h+N_{1^*i}{}^{1^*}U^h V^i+N_{1^*i}{}^{2^*}V^i V^h).$$

Transvecting $(2.6)_9$ with V^i, we obtain

$$(2.8)_8 \qquad N_{1^*2^*}{}^{1^*}=\frac{1}{1-\lambda^2}(-N_{1^*i}{}^m V^i u_m+\lambda N_{1^*m}{}^{1^*}U^m-\lambda N_{1^*i}{}^{2^*}V^i).$$

Transvecting $(2.7)_9$ with V^j, we have

$$(2.8)_9 \qquad N_{1^*2^*}{}^{2^*}=\frac{1}{1-\lambda^2}(N_{jm}{}^{2^*}V^j U^m+\lambda N_{1^*m}{}^{2^*}U^m-\lambda N_{2^*j}{}^{2^*}V^j).$$

By means of (2.8), we have

LEMMA 2. *If the function* $1-\lambda^2$ *is non-zero almost every-where in M, then the nine sets of components of the Nijenhuis tensor* $N_{1^*j}{}^h$, $N_{2^*j}{}^h$, $N_{1^*j}{}^{1^*}$, $N_{2^*j}{}^{1^*}$, $N_{1^*j}{}^{2^*}$, $N_{2^*j}{}^{2^*}$, $N_{1^*2^*}{}^h$, $N_{1^*2^*}{}^{1^*}$ *and* $N_{1^*2^*}{}^{2^*}$ *are expressed as linear combinations of the other three sets of components* $N_{ji}{}^h$, $N_{ji}{}^{1^*}$ *and* $N_{ji}{}^{2^*}$ *almost everywhere in M.*

On the other hand, transvecting $(2.6)_1$ with u_h and with v_h, we have respectively

$$(2.9)_1 \qquad N_{ji}{}^{1^*}=-\frac{1}{1-\lambda^2}(N_{jm}{}^h f_i{}^m u_h+\lambda N_{ji}{}^m v_m+N_{1^*j}{}^h u_h u_i+N_{2^*j}{}^h u_h v_i)$$

and

$$(2.9)_2 \qquad N_{ji}{}^{2^*}=-\frac{1}{1-\lambda^2}(N_{jm}{}^h f_i{}^m v_h-\lambda N_{ji}{}^m u_m+N_{1^*j}{}^h v_h u_i+N_{2^*j}{}^h v_h v_i),$$

which show that $N_{ji}{}^{1^*}$ and $N_{ji}{}^{2^*}$ can be expressed as linear combinations of $N_{ji}{}^h$, $N_{1^*j}{}^h$ and $N_{2^*j}{}^h$. Thus, taking account of $(2.9)_1$ and $(2.9)_2$, we have from Lemma 2.

LEMMA 3. *If the function* $1-\lambda^2$ *is non-zero almost everywhere in M, then the nine sets of components of the Nijenhuis tensor* $N_{ji}{}^{1^*}$, $N_{ji}{}^{2^*}$, $N_{1^*j}{}^{1^*}$, $N_{2^*j}{}^{1^*}$, $N_{1^*j}{}^{2^*}$, $N_{2^*j}{}^{2^*}$, $N_{1^*2^*}{}^h$, $N_{1^*2^*}{}^{1^*}$ *and* $N_{1^*2^*}{}^{2^*}$ *are expressed as linear combinations of the other three* $N_{ji}{}^h$, $N_{1^*j}{}^h$ *and* $N_{2^*j}{}^h$ *almost everywhere in M.*

If a symmetric affine connection ∇ is given in M, then we can easily see that the components $N_{ji}{}^{h}$, $N_{ji}{}^{1^*}$ and $N_{ji}{}^{2^*}$ can be written as follows:

$$(2.10)_1 \quad \begin{aligned} N_{ji}{}^{h} = & f_j{}^m \nabla_m f_i{}^h - f_i{}^m \nabla_m f_j{}^h - f_m{}^h(\nabla_j f_i{}^m - \nabla_i f_j{}^m) \\ & + U^h(\nabla_j u_i - \nabla_i u_j) + V^h(\nabla_j v_i - \nabla_i v_j), \end{aligned}$$

$$(2.10)_2 \quad N_{ji}{}^{1^*} = -f_j{}^m \nabla_m u_i + f_i{}^m \nabla_m u_j + u_m(\nabla_j f_i{}^m - \nabla_i f_j{}^m) - \lambda(\nabla_j v_i - \nabla_i v_j),$$

$$(2.10)_3 \quad N_{ji}{}^{2^*} = -f_j{}^m \nabla_m v_i + f_i{}^m \nabla_m v_j + v_m(\nabla_j f_i{}^m - \nabla_i f_j{}^m) + \lambda(\nabla_j u_i - \nabla_i u_j),$$

that is, we find that all the partial differentiations ∂_i involved in $N_{ji}{}^{h}$, $N_{ji}{}^{1^*}$ and $N_{ji}{}^{2^*}$ can be replaced by the covariant differentiations ∇_i. Thus we have

LEMMA 4. *If M is a differentiable manifold with (f, U, V, u, v, λ)-structure, then the sets of components $N_{ji}{}^{h}$, $N_{ji}{}^{1^*}$, $N_{ji}{}^{2^*}$, $N_{1^*j}{}^{h}$, $N_{2^*j}{}^{h}$, $N_{1^*j}{}^{1^*}$, $N_{2^*j}{}^{1^*}$, $N_{1^*j}{}^{2^*}$, $N_{2^*j}{}^{2^*}$, $N_{1^*2^*}{}^{h}$, $N_{1^*2^*}{}^{1^*}$ and $N_{1^*2^*}{}^{2^*}$ of the Nijenhuis tensor of the almost complex structure F in $M\times R^2$ define twelve tensor fields in the manifold M, which are determined by the given (f, U, V, u, v, λ)-structure.*

We can get directly from Lemmas 1 and 2.

PROPOSITION 5. *The complex structure F in $M\times R^2$ is integrable if and only if the three tensors $N_{ji}{}^{h}$, $N_{ji}{}^{1^*}$ and $N_{ji}{}^{2^*}$ vanish identically in M, or, if and only if the three tensors $N_{ji}{}^{h}$, $N_{1^*j}{}^{h}$ and $N_{2^*j}{}^{h}$ vanish identically in M.*

We see from Proposition 5 that if the almost complex structure F is integrable in $M\times R^2$, then (f, U, V, u, v, λ)-structure is normal in the sense of [5].

§3. A Riemannian metric in $M\times R^2$.

Let M be a differentiable manifold with (f, g, u, v, λ)-structure. If we consider a Riemannian metric G in $M\times R^2$ with components

$$(G_{\mu\lambda}) = \begin{pmatrix} g_{ji} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

g_{ji} being the components of the Riemannian metric g in M, then we see easily that (G, F) defines an almost Hermitian structure in $M\times R^2$, F being the almost complex structure defined by (2.1), that is,

$$(3.1) \quad F_\beta{}^\mu F_\alpha{}^\lambda G_{\mu\lambda} = G_{\beta\alpha},$$

where $F_\mu{}^\lambda$ are components of F. We denote the Christoffel symbols formed with G and those formed with g respectively by $\widetilde{\{{}_\nu{}^\lambda{}_\mu\}}$ and by $\{{}_j{}^h{}_i\}$. Then we find easily

$$\widetilde{\left\{ {h \atop j\ i} \right\}} = \left\{ {h \atop j\ i} \right\},$$

(3. 2)
$$\widetilde{\left\{ {1^* \atop j\ i} \right\}} = \widetilde{\left\{ {2^* \atop j\ i} \right\}} = \widetilde{\left\{ {\lambda \atop 1^*\ 1^*} \right\}} = \widetilde{\left\{ {\lambda \atop 1^*\ 2^*} \right\}} = \widetilde{\left\{ {\lambda \atop 2^*\ 2^*} \right\}}$$
$$= \widetilde{\left\{ {h \atop 1^*\ i} \right\}} = \widetilde{\left\{ {h \atop 2^*\ i} \right\}} = 0.$$

We denote by $\tilde{\nabla}_\kappa$ and ∇_k the covariant differentiation with respect $\widetilde{\{{}_\nu{}^\lambda{}_\mu\}}$ and with respect to $\{{}_j{}^h{}_i\}$, respectively. Then the covariant derivative of F with respect to $\widetilde{\{{}_\nu{}^\lambda{}_\mu\}}$ is given by

(3. 3)
$$\tilde{\nabla}_\kappa F_\mu{}^\lambda = \partial_\kappa F_\mu{}^\lambda + \widetilde{\left\{ {\lambda \atop \kappa\ \alpha} \right\}} F_\mu{}^\alpha - \widetilde{\left\{ {\alpha \atop \kappa\ \mu} \right\}} F_\alpha{}^\lambda,$$

that is, given by

(3. 4)$_1$ $\quad \tilde{\nabla}_k F_j{}^i = \nabla_k f_j{}^i,$

(3. 4)$_2$ $\quad \tilde{\nabla}_k F_j{}^{1^*} = -\nabla_k u_j,$

(3. 4)$_3$ $\quad \tilde{\nabla}_k F_j{}^{2^*} = -\nabla_k v_j,$

(3. 4)$_4$ $\quad \tilde{\nabla}_k F_{1^*}{}^i = \nabla_k u^i,$

(3. 4)$_5$ $\quad \tilde{\nabla}_k F_{2^*}{}^i = \nabla_k v^i,$

(3. 4)$_6$ $\quad \tilde{\nabla}_{1^*} F_j{}^i = \tilde{\nabla}_{2^*} F_j{}^i = \tilde{\nabla}_{1^*} F_{1^*}{}^{2^*} = \tilde{\nabla}_{1^*} F_{2^*}{}^{1^*} = \tilde{\nabla}_{2^*} F_{1^*}{}^{2^*} = \tilde{\nabla}_{2^*} F_{2^*}{}^{1^*} = 0,$

(3. 4)$_7$ $\quad \tilde{\nabla}_k F_{2^*}{}^{1^*} = -\tilde{\nabla}_k F_{1^*}{}^{2^*} = -\nabla_k \lambda.$

Hence we have

PROPOSITION 6. *Suppose that M has an (f, g, u, v, λ)-structure. Then a necessary and sufficient condition for the product Riemannian manifold $M \times R^2$ to be a Kählerian space with (G, F) is that all of f, u, v and λ are covariantly constant in M.*

§4. Hypersurfaces in a unit sphere.

Let M be a hypersurface immersed in a unit sphere $S^{2n+1}(1)$ with canonical almost contact structure. Then there is an (f, g, u, v, λ)-structure (f, g, u, v, λ) induced in M, which has the following properties:

$$\nabla_j f_i^h = -g_{ji}u^h + \delta_j^h u_i - k_{ji}v^h + k_j^h v_i, \tag{4.1}$$

$$\nabla_j u_i = f_{ji} - \lambda k_{ji}, \tag{4.2}$$

$$\nabla_j v_i = -k_{jm}f_i^m + \lambda g_{ji}, \tag{4.3}$$

$$\nabla_j \lambda = k_{ji}u^i - v_j, \tag{4.4}$$

where k_{ji} is the second fundamental tensor of the hypersurface M relative to $S^{2n+1}(1)$ [3]. We now assume that the induced (f, g, u, v, λ)-structure (f, g, u, v, λ) of M satisfies the condition that the tensor fields P_1 and w_1 defined in §2 vanish identically, i.e., that

$$N_{ji}{}^{1^*} = 0, \qquad N_{1^*j}{}^{1^*} = 0 \tag{4.5}$$

hold identically. Substituting (4. 1)~(4. 4) into (4. 5), we have

$$u_m k_j{}^m v_i - u_m k_i{}^m v_j = 0 \tag{4.6}$$

and

$$-u^m \nabla_m u_j - u_m \nabla_j u^m - \lambda \nabla_j \lambda = 0. \tag{4.7}$$

Transvecting (4. 6) with v^i, we obtain

$$u_m k_j{}^m = \alpha v_j, \tag{4.8}$$

where

$$\alpha = (u_m k_i{}^m v^i)/(1-\lambda^2). \tag{4.9}$$

Substituting (4. 8) into (4. 4), we find

$$\nabla_j \lambda = (\alpha - 1)v_j. \tag{4.10}$$

Substituting $u_m \nabla_j u^m = -\lambda \nabla_j \lambda$, which is a direct consequence of $u_m u^m = 1-\lambda^2$ (cf. (1. 4)), into (4. 7), we have

$$u^m \nabla_m u_j = 0. \tag{4.11}$$

Transvecting (4. 2) with u^j and using (4. 8), we find

$$\begin{aligned} u^j \nabla_j u_i &= f_{ji}u^j - \lambda k_{ji}u^j \\ &= -f_i{}^j u_j - \lambda k_i{}^j u_j \\ &= -\lambda v_i - \lambda \alpha v_i \\ &= -\lambda(1+\alpha)v_i. \end{aligned} \tag{4.12}$$

Thus, from (4. 11) and (4. 12), we have $\alpha = -1$. Therefore, (4. 10) reduces to

(4. 13) $$\nabla_j\lambda=-2v_j.$$

On the other hand, Yano [4] has recently proved

THEOREM. *Suppose that a complete and orientable 2n-dimensional Riemannian manifold* M^{2n} *is immersed in* $S^{2n+1}(1)$ *as a hypersurface. If the* (f, g, u, v, λ)*-structure* (f, g, u, v, λ) *induced on this hypersurface is such that* $(1-\lambda^2)$ *is non-zero almost everywhere in* M^{2n}, *and, if it satisfies* $\nabla_i\lambda=-2v_i$, *then* M^{2n} *is isometric to* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

If we now take account of this theorem and of (4. 13), we have

PROPOSITION 7. *Let M be a complete 2n-dimensional hypersurface immersed in a unit sphere* $S^{2n+1}(1)$ *with natural almost contact structure. Denote by* (f, g, u, v, λ) *the induced* (f, g, u, v, λ)*-structure of M. If* $(1-\lambda^2)$ *is non-zero almost everywhere in M, and, if the tensor field* P_1 *with components* $N_{ji}{}^{1^*}$ *and the covector field* w_1 *with components* $N_{1^*i}{}^{1^*}$ *vanish identically in M, then M is isometric to* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

BIBLIOGRAPHY

[1] SASAKI, S., AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to almost contact structure II. Tôhoku Math. J. **13** (1961), 281–294.

[2] YANO, K., Differential geometry on complex and almost complex spaces. Pergamon Press (1965).

[3] YANO, K., On (f, g, u, v, λ)-structures induced on a hypersurface of an odd-dimensional sphere. Tôhoku Math. J. **23** (1971), 671–680.

[4] YANO, K., Differential geometry of $S^n\times S^n$ To appear in J. Diff. Geom.

[5] YANO, K., AND M. OKUMURA, On (f, g, u, v, λ)-structures. Kōdai Math. Sem. Rep. **22** (1970), 401–423.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.