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SUBMANIFOLDS SATISFYING THE CONDITION $K(X, Y)\cdot K=0$

BY KUNIO SAKAMOTO

Introduction.

In 1968, Simons [7] obtained a formula giving the Laplacian of the square of length of the second fundamental tensor and applied it to the study of minimal hypersurfaces of a sphere. Nomizu and Smyth [6] applied a formula of Simons' type to the study of hypersurfaces with constant mean curvature and with non-negative sectional curvature in a Euclidean space or in a sphere. Chern, Do Carmo and Kobayashi [2] also applied Simons' formula to the study of minimal submanifolds of a sphere (see also Chern [1]). Recently, Yano and Ishihara [10] have applied a formula of Simons' type to the study of submanifolds of higher codimension with parallel mean curvature vector and with locally trivial normal bundle in a Euclidean space or in a sphere. On the other hand, Nomizu [5] studied hypresurfaces of a Euclidean space, which satisfy the condition $K(X, Y)\cdot K=0$ for all tangent vectors X and Y, K being the curvature tensor. Tanno [8], Tanno and Takahashi [9] studied hypersurfaces of a Euclidean space or of a sphere, which satisfy the condition $K(X, Y)\cdot S=0$ for all tangent vectors X and Y, S being the Ricci tensor (see also Kenmotsu [4]).

In the present paper, we shall, applying a formula of Simons' type, study submanifolds satisfying the condition $K(X, Y)\cdot K=0$ and having parallel mean curvature vector, non-negative Ricci curvature and locally trivial normal bundle in a space of constant curvature. We shall also study submanifolds with parallel second fundamental tensor and with locally trivial normal bundle in a Euclidean space or in a sphere. The main results are stated in Theorems 3. 3, 3. 4, 3. 5 and 3. 6.

§ 1. Preliminaries.

Let M^m be an m-dimensional Riemannian manifold of class C^∞ with metric tensor G, whose components are G_{ji} with respect to local coordinates $\{\xi^h\}$. Let M^n be an n-dimensional connected submanifold of class C^∞ differentiably immersed in M^m $(1<n<m)$ and suppose that the local expression of the submanifold M^n is

$$\xi^h=\xi^h(\eta^a), \tag{1. 1}$$

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where $\{\eta^a\}$ are local coordinates in the submanifold M^n. The indices $h, i, \cdots, l$ run over the range $\{1, \cdots, m\}$ and the indices $a, b, \cdots, g$ over the range $\{1, \cdots, n\}$. If we put

$$B_b{}^h = \partial_b \xi^h, \qquad \partial_b = \partial/\partial\eta^b, \tag{1.2}$$

then the Riemannian metric g of M^n induced from that of M^m is given by

$$g_{cb} = G_{ji} B_c{}^j B_b{}^i. \tag{1.3}$$

For each index b, $B_b{}^h$ denotes a local vector field tangent to M^n and the n local vector fields $B_b{}^h$ span the tangent space of the submanifold M^n at each point. We denote by $C_x{}^h$ $m-n$ mutually orthogonal local unit vector fields normal to M^n, where here and in the sequel the indices x, y, z run over the range $\{n+1, \cdots, m\}$,

If we denote by $\{{}_j{}^h{}_i\}$ and $\{{}_c{}^a{}_b\}$ the Christoffel symbols formed with G_{ji} and g_{cb} respectively, then the van der Waerden-Bortolotti covariant derivative of $B_b{}^h$ is, by definition, given by

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j B_b{}^i - \left\{ {a \atop c\ b} \right\} B_a{}^h. \tag{1.4}$$

Since $\nabla_c B_b{}^h$ is, for any fixed indices c and b, a local vector field normal to M^n, we can write

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h. \tag{1.5}$$

The local tensor field $h_{cb}{}^x$ is called the second fundamental tensor of the submanifold M^n relative to the unit normals $C_x{}^h$. Equations (1.5) are equations of Gauss for the submanifold M^n.

If we denote by g^* the metric tensor induced on the normal bundle $\mathfrak{N}(M^n)$ of the submanifold M^n from the metric tensor G of M^m, then we have, for the components of g^* relative to the frame $\{C_x{}^h\}$,

$$g^*_{yx} = G_{ji} C_y{}^j C_x{}^i = \delta_{yx}. \tag{1.6}$$

If we denote by $\Gamma_c{}^x{}_y$ components of the connection ∇^* induced on $\mathfrak{N}(M^n)$ from the Riemannian connection ∇ of the ambient manifold M^m, the van der Waerden-Bortolotti covariant derivative of $C_y{}^h$ is, by definition, given by

$$\nabla_c C_y{}^h = \partial_c C_y{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j C_y{}^i - \Gamma_c{}^x{}_y C_x{}^h. \tag{1.7}$$

Since $\nabla_c C_y{}^h$ is, for any fixed c and y, a local vector field tangent to M^n, we have from $G_{ji} B_b{}^j C_y{}^i = 0$ and (1.5)

$$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h \qquad (h_c{}^a{}_y = h_{cb}{}^x g^{ba} \delta_{xy}). \tag{1.8}$$

Equations (1.8) are equations of Weingarten for the submanifold M^n. We extend the van der Waerden-Bortolotti covariant differentiation ∇_c to tensor fields of mixed type on M^n in such a way that for any tensor fields, say $T_b{}^a{}_y{}^x$ and $T_{by}{}^h$, of mixed type, the covariant derivatives are defined to be

$$\nabla_c T_b{}^a{}_y{}^x=\partial_c T_b{}^a{}_y{}^x+\left\{ {a \atop c\ e} \right\} T_b{}^e{}_y{}^x-\left\{ {e \atop c\ b} \right\} T_e{}^a{}_y{}^x+\Gamma_c{}^x{}_z T_b{}^a{}_y{}^z-\Gamma_c{}^z{}_y T_b{}^a{}_z{}^x,$$

(1. 9)

$$\nabla_c T_{by}{}^h=\partial_c T_{by}{}^h+\left\{ {h \atop j\ i} \right\} B_c{}^j T_{by}{}^i-\left\{ {a \atop c\ b} \right\} T_{ay}{}^h-\Gamma_c{}^x{}_y T_{bx}{}^h.$$

For tensor fields of mixed type, we have, from (1. 9), the Ricci formula

(1. 10) $$\nabla_d\nabla_c T_b{}^a{}_y{}^x-\nabla_c\nabla_d T_b{}^a{}_y{}^x=K_{dce}{}^a T_b{}^e{}_y{}^x-K_{dcb}{}^e T_e{}^a{}_y{}^x+K_{dcz}{}^x T_b{}^a{}_y{}^z-K_{dcy}{}^z T_b{}^a{}_z{}^x,$$

where $K_{dcb}{}^a$ and $K_{dcy}{}^x$ are curvature tensors of g of M^n and ∇^* of $\mathfrak{N}(M^n)$ respectively.

We now assume that the ambient manifold M^m is of constant curvature c, ie., that

(1. 11) $$R_{kjih}=c(G_{kh}G_{ji}-G_{jh}G_{ki}),$$

where R_{kjih} are covariant components of the curvature tensor of G of M^m. Substituting (1. 5) and (1. 8) in the Ricci formulas for $B_b{}^h$ and $C_y{}^i$ respectively, we have the structure equations of the submanifold M^n, i.e.,

(1. 12) $$K_{dcba}=c(g_{da}g_{cb}-g_{ca}g_{db})+h_{da}{}^x h_{cbx}-h_{ca}{}^x h_{dbx},$$

(1. 13) $$\nabla_d h_{cb}{}^x=\nabla_c h_{db}{}^x,$$

(1. 14) $$K_{dcy}{}^x=h_{de}{}^x h_c{}^e{}_y-h_{ce}{}^x h_d{}^e{}_y.$$

Transvecting (1. 12) with g^{da}, we find

(1. 15) $$K_{cb}=c(n-1)g_{cb}+nh^x h_{cbx}-h_{ce}{}^x h_b{}^e{}_x,$$

where $K_{cb}=K_{ecb}{}^e$ is the Ricci tensor and $h^x=(1/n)h_c{}^{cx}$ is the mean curvature vector of the submanifold M^n.

When the ambient manifold M^m is of constant curvature c, we compute the Laplacian ΔF of the function $F=h_{cb}{}^x h^{cb}{}_x$, where $\Delta=g^{cb}\nabla_c\nabla_b$. We thus have

(1. 16) $$\frac{1}{2}\Delta F=g^{ed}(\nabla_e\nabla_d h_{cb}{}^x)h^{cb}{}_x+(\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x).$$

From the Ricci identity for $h_{cb}{}^x$ and (1. 13), we have

(1. 17) $$\frac{1}{2}\Delta F=n(\nabla_c\nabla_b h^x)h^{cb}{}_x+K_c{}^a h_{ba}{}^x h^{cb}{}_x-K_{ecba}h^{eax}h^{cb}{}_x+K_{ecy}{}^x h_b{}^{ey}h^{cb}{}_x+(\nabla_c h_{ba}{}^x)(\nabla^c h^{ba}{}_x).$$

If we substitute (1. 12), (1. 14) and (1. 15) in (1. 17), then we have (cf. [10])

$$\frac{1}{2}\Delta F=n(\nabla_c\nabla_b h^x)h^{cb}{}_x+cnF-cn^2h^xh_x-h_{ea}{}^yh_{cby}h^{ea}{}_xh^{cbx}$$

(1. 18)

$$+nh^yh_{cay}h_b{}^a{}_xh^{cbx}-K_{ecy}{}^xK^{ecy}{}_x+(\nabla_ch_{ba}{}^x)(\nabla^ch^{ba}{}_x).$$

When the normal bundle $\mathfrak{N}(M^n)$ is locally trivial, i.e., $K_{dcy}{}^x=0$, the above equation (1. 18) becomes

(1. 19)
$$\frac{1}{2}\Delta F=n(\nabla_c\nabla_b h^x)h^{cb}{}_x+cnF-cn^2h^xh_x-h_{ea}{}^yh_{cby}h^{ea}{}_xh^{cbx}$$

$$+nh^yh_{cay}h_b{}^a{}_xh^{cbx}+(\nabla_ch_{ba}{}^x)(\nabla^ch^{ba}{}_x).$$

§ 2. Submanifolds satisfying the condition $K(X, Y)\cdot K=0$.

Let M^n be a submanifold in a space M^m of constant curvature c, and suppose that the normal bundle $\mathfrak{N}(M^n)$ of M^n is locally trivial, i.e., that $K_{dcy}{}^x=0$ holds. We now consider the condition

(*)
$$K(X, Y)\cdot K=0$$

for any tangent vector X and Y of M^n, where $K(X, Y)$ operates on the tensor algebra at each point as a derivation. The condition (*) is equivalent to

(2. 1)
$$\nabla_f\nabla_eK_{dcba}-\nabla_e\nabla_fK_{dcba}=-(K_{fed}{}^gK_{gcba}+K_{fec}{}^gK_{dgba}+K_{feb}{}^gK_{dcga}+K_{fea}{}^gK_{dcbg})=0.$$

On the other hand, differentiating (1. 12) covariantly, we have

(2. 2)
$$\nabla_eK_{dcba}=(\nabla_eh_{da}{}^x)h_{cbx}+h_{da}{}^x(\nabla_eh_{cbx})-(\nabla_eh_{ca}{}^x)h_{dbx}-h_{ca}{}^x(\nabla_eh_{dbx}),$$

and hence

$$\begin{aligned}&\nabla_f\nabla_eK_{dcba}-\nabla_e\nabla_fK_{dcba}\\&=(\nabla_f\nabla_eh_{da}{}^x-\nabla_e\nabla_fh_{da}{}^x)h_{cbx}+(\nabla_f\nabla_eh_{cb}{}^x-\nabla_e\nabla_fh_{cb}{}^x)h_{dax}\\&\quad-(\nabla_f\nabla_eh_{ca}{}^x-\nabla_e\nabla_fh_{ca}{}^x)h_{dbx}-(\nabla_f\nabla_eh_{db}{}^x-\nabla_e\nabla_fh_{db}{}^x)h_{cax}.\end{aligned}$$

Applying the Ricci identity (1. 10) to $h_{cb}{}^x$ with vanishing $K_{dcy}{}^x$, we see that the equations above reduce to

$$\nabla_f\nabla_eK_{dcba}-\nabla_e\nabla_fK_{dcba}$$

(2. 3)
$$=-(K_{fed}{}^gh_{ga}{}^x+K_{fea}{}^gh_{dg}{}^x)h_{cbx}-(K_{fec}{}^gh_{gb}{}^x+K_{feb}{}^gh_{cg}{}^x)h_{dax}$$

$$+(K_{fec}{}^gh_{ga}{}^x+K_{fea}{}^gh_{cg}{}^x)h_{dbx}+(K_{fed}{}^gh_{gb}{}^x+K_{feb}{}^gh_{dg}{}^x)h_{cax}.$$

Since the normal bundle $\mathfrak{N}(M^n)$ of M^n is locally trivial, we see from (1. 14) that, for any indices x and y, $h_b{}^{ax}$ and $h_b{}^{ay}$ are commutative, i.e., $h_e{}^{ax}h_b{}^{ey}=h_e{}^{ay}h_b{}^{ex}$.

Hence we see that there exist certain n mutually orthogonal unit vectors $v_1{}^a, \cdots, v_n{}^a$ such that

$$h_b{}^{ax}v_\alpha{}^b=\lambda_\alpha{}^x v_\alpha{}^a \qquad (\alpha;\ \text{not summed}) \tag{2.4}$$

at each point of M^n, where here and in the sequel indices $\alpha, \beta, \gamma, \varepsilon$ run over the range $\{1, \cdots, n\}$. We shall now compute

$$(\nabla_f\nabla_e K_{dcba}-\nabla_e\nabla_f K_{dcba})v_\beta{}^f v_\alpha{}^e v_\gamma{}^d v_\varepsilon{}^c.$$

First we find from (1.12)

$$K_{feba}v_\beta{}^f v_\alpha{}^e=(c+\sum_x\lambda_\alpha{}^x\lambda_\beta{}^x)(v_{\beta a}v_{\alpha b}-v_{\alpha a}v_{\beta b}) \qquad (\alpha\neq\beta).$$

Since we see, from (1.12), that the sectional curvature $\sigma_{\beta,\alpha}$ of M^n with respect to the plane section determined by eigenvectors v_α and v_β of $h_b{}^{ax}$'s is given by

$$\sigma_{\beta,\alpha}=c+\sum_x\lambda_\beta{}^x\lambda_\alpha{}^x \qquad (\alpha\neq\beta), \tag{2.5}$$

we have

$$K_{feb}{}^a v_\beta{}^f v_\alpha{}^e=\sigma_{\beta,\alpha}(v_\beta{}^a v_{\alpha b}-v_\alpha{}^a v_{\beta b}). \tag{2.6}$$

If we transvect (2.3) with $v_\beta{}^f v_\alpha{}^e$ and use (2.4) and (2.6), then we find

$$\begin{aligned}
&(\nabla_f\nabla_e K_{dcba}-\nabla_e\nabla_f K_{dcba})v_\beta{}^f v_\alpha{}^e\\
&=-\sigma_{\beta,\alpha}[\lambda_\beta{}^x(v_{\beta a}v_{\alpha d}+v_{\beta d}v_{\alpha a})-\lambda_\alpha{}^x(v_{\alpha a}v_{\beta d}+v_{\alpha d}v_{\beta a})]h_{cbx}\\
&\quad-\sigma_{\beta,\alpha}[\lambda_\beta{}^x(v_{\beta b}v_{\alpha c}+v_{\beta c}v_{\alpha b})-\lambda_\alpha{}^x(v_{\alpha b}v_{\beta c}+v_{\alpha c}v_{\beta b})]h_{dax}\\
&\quad+\sigma_{\beta,\alpha}[\lambda_\beta{}^x(v_{\beta a}v_{\alpha c}+v_{\beta c}v_{\alpha a})-\lambda_\alpha{}^x(v_{\alpha a}v_{\beta c}+v_{\alpha c}v_{\beta a})]h_{dbx}\\
&\quad+\sigma_{\beta,\alpha}[\lambda_\beta{}^x(v_{\beta b}v_{\alpha d}+v_{\beta d}v_{\alpha b})-\lambda_\alpha{}^x(v_{\alpha b}v_{\beta d}+v_{\alpha d}v_{\beta b})]h_{cax}.
\end{aligned} \tag{2.7}$$

Thus transvecting (2.7) with $v_\gamma{}^d v_\varepsilon{}^c$, we have from (2.4)

$$\begin{aligned}
&(\nabla_f\nabla_e K_{dcba}-\nabla_e\nabla_f K_{dcba})v_\beta{}^f v_\alpha{}^e v_\gamma{}^d v_\varepsilon{}^c\\
&=\sigma_{\beta,\alpha}\sum_x[(\lambda_\beta{}^x-\lambda_\alpha{}^x)\{-\lambda_\varepsilon{}^x(\delta_{\alpha\gamma}v_{\beta a}+\delta_{\beta\gamma}v_{\alpha a})v_{\varepsilon b}\\
&\quad-\lambda_\gamma{}^x(\delta_{\alpha\varepsilon}v_{\beta b}+\delta_{\beta\varepsilon}v_{\alpha b})v_{\gamma a}+\lambda_\gamma{}^x(\delta_{\alpha\varepsilon}v_{\beta a}+\delta_{\beta\varepsilon}v_{\alpha a})v_{\gamma b}+\lambda_\varepsilon{}^x(\delta_{\alpha\gamma}v_{\beta b}+\delta_{\beta\gamma}v_{\alpha b})v_{\varepsilon a}\}].
\end{aligned} \tag{2.8}$$

We can easily verify that the right-hand side of (2.8) vanishes identically except in the following four cases: Case I $\gamma=\alpha$, $\gamma\neq\beta$, $\varepsilon\neq\alpha$, $\varepsilon\neq\beta$ $(\alpha\neq\beta)$, Case II $\gamma\neq\alpha$, $\gamma=\beta$, $\varepsilon\neq\alpha$, $\varepsilon\neq\beta$ $(\alpha\neq\beta)$, Case III $\gamma\neq\alpha$, $\gamma\neq\beta$, $\varepsilon=\alpha$, $\varepsilon\neq\beta$ $(\alpha\neq\beta)$ and Case IV $\gamma\neq\alpha$, $\gamma\neq\beta$, $\varepsilon\neq\alpha$, $\varepsilon=\beta$ $(\alpha\neq\beta)$. For these four cases, (2.8) reduces to

$$(\nabla_f\nabla_e K_{dcba}-\nabla_e\nabla_f K_{dcba})v_\beta{}^f v_\alpha{}^e v_\gamma{}^d v_\alpha{}^c=\sigma_{\beta,\alpha}\sum_x(\lambda_\beta{}^x-\lambda_\alpha{}^x)\lambda_\gamma{}^x(v_{\gamma b}v_{\beta a}-v_{\beta b}v_{\gamma a}). \tag{2.9}$$

We moreover assume that the submanifold satisfies the condition (*), which is equivalent to the condition

$$\sigma_{\beta,\alpha}\sum_{x}(\lambda_{\beta}{}^{x}-\lambda_{\alpha}{}^{x})\lambda_{\gamma}{}^{x}=0 \qquad \gamma\neq\alpha,\ \beta\ (\alpha\neq\beta) \tag{2.10}$$

because of (2. 9). Using (2. 5), we see easily that (2. 10) is equivalent to

$$\sigma_{\beta,\alpha}(\sigma_{\gamma,\beta}-\sigma_{\gamma,\alpha})=0 \qquad \gamma\neq\alpha,\ \beta\ (\alpha\neq\beta). \tag{2.11}$$

We here assume that there is at least one non-zero $\sigma_{\beta,\alpha}$. Then we may suppose that $\sigma_{1,2,\dots}$, $\sigma_{1,p}$ are non-zero and $\sigma_{1,p+1}=\cdots=\sigma_{1,n}=0$. We find from (2. 11)

$$\sigma_{\gamma,\beta}=\sigma_{\gamma,\alpha} \qquad (\beta<\alpha;\ 1,\cdots,p,\ \gamma=1,\cdots,n).$$

Thus we have

$$\sigma_{\beta,\alpha}=\sigma_{1,2} \qquad (\beta<\alpha;\ 1,\cdots,p),$$

$$\sigma_{\beta,\alpha}=0 \qquad (\beta=1,\cdots,p,\ \alpha=p+1,\cdots,n).$$

Similarly, if we suppose that $\sigma_{p+1,\ p+2},\cdots,\sigma_{p+1,q}$ are non-zero and $\sigma_{p+1,\ q+1}=\cdots=\sigma_{p+1,n}=0$, then we find

$$\sigma_{\beta,\alpha}=\sigma_{p+1,p+2} \qquad (\beta<\alpha;\ p+1,\cdots,q),$$

$$\sigma_{\beta,\alpha}=0 \qquad (\beta=p+1,\cdots,q,\ \alpha=q+1,\cdots,n).$$

In this way, we have

$$\sigma_{\beta,\alpha}=\sigma_{q+1,q+2} \qquad (\beta<\alpha;\ q+1,\cdots,r),$$

$$\sigma_{\beta,\alpha}=0 \qquad (\beta=q+1,\cdots,r,\ \alpha=r+1,\cdots,n),$$

....

as far as there is a non-zero $\sigma_{\beta,\alpha}$.

If we denote by S the Ricci tensor, we easily find

$$S(v_{\alpha},v_{\alpha})=K_{cb}v_{\alpha}{}^{c}v_{\alpha}{}^{b}=\sum_{\beta\neq\alpha}\sigma_{\beta,\alpha} \qquad (\alpha;\ \text{fixed}). \tag{2.12}$$

Hence, when we assume that the Ricci tensor S is non-negative, taking account of the behavior of the sectional curvatures $\sigma_{\beta,\alpha}$, explained above, we see that the sectional curvature $\sigma_{\beta,\alpha}$ is non-netative for all β and α. Using (2. 4) and (2. 5), we find from (1. 19) (cf. [10])

$$\frac{1}{2}\varDelta F=n(\nabla_{c}\nabla_{b}h^{x})h^{cb}{}_{x}+(\nabla_{c}h_{ba}{}^{x})(\nabla^{c}h^{ba}{}_{x})+\sum_{\alpha<\beta}\sum_{x}(\lambda_{\beta}{}^{x}-\lambda_{\alpha}{}^{x})^{2}\sigma_{\beta,\alpha}. \tag{2.13}$$

Therefore we have

PROPOSITION 2.1. *Let M^n $(n\geqq 3)$ be a submanifold immersed in a space of constant curvature and satisfy the conditions*:

(A) *The normal bundle $\mathfrak{N}(M^n)$ is locally trivial*;

(B) *The mean curvature vector is parallel in $\mathfrak{N}(M^n)$, i.e., $\nabla_c h^x=0$*;

(C) *$K(X, Y)\cdot K=0$ for any tangent vectors X and Y of M^n*;

(D) *The Ricci tensor is non-negative.*

If M^n is compact, then we have

$$\nabla_c h_{ba}{}^x=0 \qquad \text{for any indices } c, b \text{ and } a, \tag{2.14}$$

$$(\lambda_\beta{}^x-\lambda_\alpha{}^x)^2\sigma_{\beta,\alpha}=0 \qquad \text{for any indices } \alpha, \beta\ (\alpha\neq\beta) \text{ and } x. \tag{2.15}$$

PROPOSITION 2.2. *Let M^n $(n\geqq 3)$ be a submanifold immersed in a space of constant curvature and satisfy the conditions* (A), (B), (C) *and* (D) *in Proposition* 2.1. *If $F=h_{cb}{}^x h^{cb}{}_x$ is constant, we have* (2.14) *and* (2.15).

§3. Submanifolds with parallel second fundamental tensor.

Let M^n be a connected submanifold with parallel second fundamental tensor, i.e., $\nabla_c h_{ba}{}^x=0$, in a space M^m of constant curvature c and suppose that the normal bundle $\mathfrak{N}(M^n)$ is locally trivial. Then we easily see that all of the eigenvalues $\lambda_\alpha{}^x$ of the second fundamental tensor are constant and that each of eigenspaces of the second fundamental tensor is of constant dimension. If we denote by λ_α the normal vector fields with components $\lambda_\alpha{}^h=\lambda_\alpha{}^x C_x{}^h$, then they are globally defined. When we fix the normals $C_x{}^h$, we can identify λ_α with a vector of R^{m-n} with components $(\lambda_\alpha{}^{n+1}, \cdots, \lambda_\alpha{}^m)$ and the inner product of λ_α and λ_β with the usual inner product $(\lambda_\alpha, \lambda_\beta)$ in R^{m-n}. If all of the eigenvector fields corresponding to λ_α form a p_α-dimensional distribution, then we say that the multiplicity of λ_α is p_α.

Let $\mu_1, \cdots, \mu_N$ be distinct vectors of eigenvalues and let $p_1, \cdots, p_N$ be the multiplicity of $\mu_1, \cdots, \mu_N$. We denote by D_A the distribution formed by all eigenvector fields corresponding to μ_A of multiplicity p_A, where the index A runs over the range $\{1, \cdots, N\}$. Taking a vector field X^a belonging to D_A, we have

$$h_b{}^{ax}X^b=\mu_A{}^x X^a \tag{3.1}$$

and hence

$$h_b{}^{ax}\nabla_c X^b=\mu_A{}^x\nabla_c X^a, \tag{3.2}$$

since $\nabla_c h_b{}^{ax}=0$ and $\mu_A{}^x$ are constant. If a vector field Y^a belongs to D_A, then we find from (3.2)

$$h_b{}^{ax}(Y^c\nabla_c X^b-X^c\nabla_c Y^b)=\mu_\alpha{}^x(Y^c\nabla_c X^a-X^c\nabla_c Y^a). \tag{3.3}$$

Thus we see that the distribution D_A and the orthogonal complement $\bar{D}_A$ of D_A are both integrable and parallel. Therefore, if we denote by M_A and $\bar{M}_A$ some integral manifolds of D_A and $\bar{D}_A$ respectively, they are totally geodesic submanifolds in M^n and M^n is locally a pythagorean product $M_A \times \bar{M}_A$. Since, for any vector fields X^a and Y^a tangent to M_A, we have

$$X^c \nabla_c (Y^a B_a{}^h) = (X^c \nabla_c Y^a) B_a{}^h + \mu_A{}^x g_{ca} X^c Y^a C_x{}^h,$$

we see that M_A is totally umbilical in the ambient manifold M^m if $\mu_A \neq 0$ and that M_A is totally geodesic in the ambient manifold M^m if $\mu_A = 0$. Thus we have (cf. [10])

LEMMA 3. 1. *Let M^n be a submanifold with parallel second fundamental tensor immersed in a space M^m of constant curvature and assume that the normal bundle $\mathfrak{N}(M^n)$ of M^n is locally trivial. If distinct vectors of eigenvalues of the second fundamental tensor are given by $\mu_1, \cdots, \mu_N$, then M^n is locally a pythagorean product $M_1 \times \cdots \times M_N$, where M_A $(A=1, \cdots, N)$ is a totally umbilical submanifold in M^m with mean curvature vector μ_A if $\mu_A \neq 0$ and M_A is a totally geodesic submanifold in M^m if $\mu_A = 0$. In particular the normal bundle $\mathfrak{N}(M_A)$ of M_A in M^m is locally trivial.*

Let M^n be an n-dimensional submanifold with parallel second fundamental tensor immersed in a space M^m of constant curvature c and suppose that the normal bundle $\mathfrak{N}(M^n)$ is locally trivial. If u^a and v^a are unit vector belonging to D_A and D_B respectively, then we have

$$K_{dcba} v^d u^c u^b v^a = 0$$

and hence, from (1. 12),

$$K_{dcba} v^d u^c u^b v^a = c + \sum \mu_A{}^x \mu_B{}^x = c + (u_A, \mu_B) = 0.$$

We note that we have this result under the assumptions in Propositions 2. 1 and 2. 2. We have known the following lemma (cf. [10]).

LEMMA 3. 2. *Let $\mu_1, \cdots, \mu_N$ be distinct vectors belonging to R^{m-n} such that $(\mu_A, \mu_B) = k$ $(A \neq B;\ A, B = 1, \cdots, N)$. If $\mu_1, \cdots, \mu_N$ span an r-dimensional subspace, $(m-n \geqq r > 0)$, then $N = r$ or $N = r+1$. When $N = r+1$, and when $\mu_1, \cdots, \mu_N$ span an r-dimensional subspace,*

$$\begin{vmatrix} (\mu_1, \mu_1) & k & \cdots\cdots & k \\ k & (\mu_2, \mu_2) & \cdots & k \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots} \\ k & k & \cdots\cdots & (\mu_N, \mu_N) \end{vmatrix} = 0.$$

If $k=0$, then one of $\mu_1, \cdots, \mu_N$ is necessarily zero.

In general, a submanifold M^n immersed in an m-dimensional space M^m is said to be of essential codimension r $(0 \leqq r \leqq m-n)$, if there exists in the ambient manifold M^m an $(n+r)$-dimensional totally geodesic submanifold containing M^n as a submanifold and no such a totally geodesic submanifold of dimension less than $n+r$. The subspace in the normal space at a point P of M^n spanned by normal vectors $v^c u^b h_{cb}{}^x C_x{}^h$, u^a and v^a being any tangent vectors of M^n at P, is called the first normal space at P.

We now assume that the ambient manifold M^m is an m-dimensional Euclidean space R^m. Then, from the above Lemma 3. 2, we see that the first normal space is of constant dimension r and $N=r$ or $N=r+1$, if $\mu_1, \cdots, \mu_N$ span an r-dimensional subspace of R^{m-n}, and that one of $\mu_1, \cdots, \mu_N$ is necessarily zero if $N=r+1$. If X^a, Y^a and Z^a are vector fields tangent to M^n, then we have

$$Z^e\nabla_e(X^cY^bh_{cb}{}^x)C_x{}^h=(Z^e\nabla_eX^c)Y^bh_{cb}{}^xC_x{}^h+X^c(Z^e\nabla_eY^b)h_{cb}{}^xC_x{}^h,$$

because of $\nabla_c h_{ba}{}^x=0$. Thus the first normal space is parallel in the normal bundle $\mathfrak{N}(M^n)$. Therefore we see that the essential codimension is r, i.e., that M^n is immersed in an $(n+r)$-dimensional plane in R^m, if $\mu_1, \cdots, \mu_N$ span an r-dimensional subspace of R^{m-n} (cf. [3]) Since it is easily verified that the second fundamental tensor of M_A $(A=1, \cdots, N)$ in R^m is parallel and that the first normal space of M_A in R^m is of constant dimension 1 if $\mu_A\neq 0$, we see from Lemma 3. 1 that M_A is immersed in an (p_A+1)-dimensional plane in R^m as a totally umbilical hypersurface if $\mu_A\neq 0$ and that, in particular, if M_A is of dimension 1, M_A is a curve of constant curvature in a 2-dimensional plane in R^m. Therefore we have (cf. [5], [6] and [10])

THEOREM 3. 3. *Let M^n be a connected complete submanifold of dimension n with parallel second fundamental tensor immersed in a Euclidean space R^m of dimension m $(1<n<m)$ and suppose that the normal bundle is locally trivial. Then M^n is a sphere $S^n(r)$ of dimension n with radius r, an n-dimensional plane R^n, a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\ p_1+\cdots+p_N=n,\ p_1,\cdots,p_N\geqq 1,\ 1<N\leqq m-n, \tag{3. 4}$$

or a pythagorean product of the form

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)\times R^p,\ p_1+\cdots+p_N+p=n,\ p_1,\cdots,p_N,p\geqq 1,\ 1<N\leqq m-n, \tag{3. 5}$$

where $S^p(r)$ is a p-dimensional sphere with radius r and R^p is a p-dimensional plane. If M^n is a pythagorean product of the form (3. 4) *or* (3. 5), *then M^n is of essential codimension N.*

In the case where the ambient manifold M^m is an m-dimensional sphere $S^m(a)$ with radius a, we have (see [10])

THEOREM 3. 4. *Let M^n be an n-dimensional connected complete submanifold with parallel second fundamental tensor immersed in an m-dimensional sphere $S^m(a)$ with radius a $(0<a, 1<n<m)$ and suppose that the normal bundle is locally trivial. Then M^n is a small sphere, a great sphere or a pythagorean product of a certain number of spheres. If, moreover, M^n is of essential codimension $m-n$, then M^n is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\ p_1+\cdots+p_N=n,\ p_1,\cdots,p_N\geqq 1,\ r_1{}^2+\cdots+r_N{}^2=a^2,\ N=m-n+1, \tag{3. 6}$$

or a pythagorean product of the form

$$(3.7)\quad \begin{aligned}&\Sigma^{p_1}(r_1)\times\cdots\times\Sigma^{p_{N'}}(r_{N'})\subset\Sigma^{m-1}(r),\\ &p_1+\cdots+p_{N'}=n,\ p_1,\ \cdots,\ p_{N'}\geqq 1,\ r_1^2+\cdots+r_{N'}^2=r^2<a^2,\ N'=m-n,\end{aligned}$$

where $\Sigma^p(r)$ *is a* p*-dimensional small sphere with radius* r *in* $S^m(a)$.

Taking account of Proposition 2.1, we have, as a corollary to Theorems 3.3 and 3.4,

THEOREM 3.5. *Let* M^n *be a connected submanifold immersed in a Euclidean space* R^m *(resp. a sphere* $S^m(a)$*)* $(3\leqq n<m)$ *and satisfy the conditions* (A), (B), (C) *and* (D) *stated in Proposition* 2.1. *If* M^n *is compact, then* M^n *is a sphere or a pythagorean product of the form* (3.4) *(resp. a small sphere, or a pythagorean product of a certain number of spheres).*

Taking account of Proposition 2.2, we have, as a corollary to Theorems 3.3 and 3.4,

THEOREM 3.6. *Let* M^n *be a connected complete submanifold immersed in a Euclidean space* R^m *(resp. a sphere* $S^m(a)$*)* $(3\leqq n<m)$ *and satisfy the conditions* (A), (B), (C) *and* (D) *stated in Proposition* 2.1. *If* $F=h_{cb}{}^x h^{cb}{}_x$ *is constant, then we have the same conclusion as in Theorem* 3.3 *(resp. as in Theorem* 3.4*).*

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.