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ALMOST QUATERNION STRUCTURES OF THE SECOND KIND AND ALMOST TANGENT STRUCTURES

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Dedicated to Professor S. Ishihara on his fiftieth birthday

§ 0. Introduction.

A set of three tensor fields F, G and H of type $(1,1)$ in a differentiable manifold which satisfy

$$F^2=-1, \quad G^2=-1, \quad H^2=-1,$$

$$F=GH=-HG, \quad G=HF=-FH, \quad H=FG=-GF$$

is called an *almost quaternion structure* and a differentiable manifold with an almost quaternion structure an *almost quaternion manifold.*

If there exists, in an almost quaternion manifold, a system of coordinate neighborhoods with respect to which components of F, G and H are all constant, then the almost quaternion structure is said to be *integrable* and the almost quaternion manifold is called a *quaternion manifold.*

In a previous paper [8], the present authors studied integrability conditions for almost quaternion structures.

A set of three tensor fields F, G and H of type $(1,1)$ in a differentiable manifold which satisfy

$$F^2=-1, \quad G^2=1, \quad H^2=1,$$

$$F=-GH=HG, \quad G=HF=-FH, \quad H=FG=-GF$$

is called an *almost quaternion structure of the second kind* and a differentiable manifold with an almost quaternion structure of the second kind an *almost quaternion manifold of the second kind.*

The main purpose of the present paper is first of all to study integrability conditions for an almost quaternion structure of the second kind and then to apply the results to the study of almost tangent structures and tangent structures.

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§ 1. Preliminaries.

Let P and Q be two tensor fields of type (1, 1) in a differentiable manifold of class C^∞. The expression

$$
\begin{aligned}
[P, Q](X, Y) = &[PX, QY] - P[QX, Y] - Q[X, PY] \\
&+ [QX, PY] - Q[PX, Y] - P[X, QY] \\
&+ (PQ + QP)[X, Y],
\end{aligned} \tag{1.1}
$$

X and Y being arbitrary vector fields, defines a tensor field of type (1, 2) called the Nijenhuis tensor or the torsion tensor of P and Q. We note that $[P, Q](X, Y) = -[P, Q](Y, X)$, $[P, Q](X, Y) = [Q, P](X, Y)$ and if $Q = +1$ or -1, 1 denoting the unit tensor, then $[P, Q]$ vanishes identically.

As in the previous paper [8], we need following definitions and formulas. If S is a tensor field of type (1, 2) and N is a tensor field of type (1, 1), then $S \barwedge N$ is defined to be

$$
(S \barwedge N)(X, Y) = S(NX, Y) + S(X, NY) \tag{1.2}
$$

and $N \barwedge S$ to be

$$
(N \barwedge S)(X, Y) = N(S(X, Y)) \tag{1.3}
$$

for arbitrary vector fields X and Y. $S \barwedge N$ and $N \barwedge S$ are both of type (1, 2).

Then for three tensor fields, L, M and N of type (1, 1), we have [2]

$$
[L, MN] + [M, LN] = [L, M] \barwedge N + L \barwedge [M, N] + M \barwedge [L, N] \tag{1.4}
$$

and for a tensor field S of type (1, 2) and tensor fields M, N of type (1, 1),

$$
(S \barwedge M) \barwedge N - (S \barwedge N) \barwedge M = S \barwedge MN - S \barwedge NM \tag{1.5}
$$

and

$$
(M \barwedge S) \barwedge N = M \barwedge (S \barwedge N). \tag{1.6}
$$

§ 2. The almost quaternion structure of the second kind.

We consider in this section an almost quaternion manifold of the second kind, that is, a differentiable manifold in which there exist three tensor fields F, G and H of type (1, 1) satisfying

$$
\begin{aligned}
&F^2 = -1, \quad G^2 = 1, \quad H^2 = 1, \\
&F = -GH = HG, \quad G = HF = -FH, \quad H = FG = -GF.
\end{aligned} \tag{2.1}
$$

We would like to study relations which exist between Nijenhuis tensors formed with F, G and H.

The following argument is quite similar to that used in the previous paper [8], but since the difference from the previous one arising from the fact that we have here $G^2=1$, $H^2=1$ and $F=-GH=HG$ instead of $G^2=-1$, $H^2=-1$ and $F=GH=-HG$ in the previous case is so delicate that we repeat briefly the argument, similar to that used in [8].

We first put $L=M=F$, $N=G$ in (1.4) and find

$$[H, F]=F\barwedge[F, G]+\frac{1}{2}[F, F]\barwedge G. \tag{2.2}$$

We then put $L=G$, $M=N=F$ in (1.4) and find

$$[H, F]=-[F, G]\barwedge F-F\barwedge[F, G]-G\barwedge[F, F]. \tag{2.3}$$

From (2.2) and (2.3), we find

$$[H, F]=-\frac{1}{2}[F, G]\barwedge F-\frac{1}{2}G\barwedge[F, F]+\frac{1}{4}[F, F]\barwedge G \tag{2.4}$$

and

$$[F, G]\barwedge F+2F\barwedge[F, G]+G\barwedge[F, F]+\frac{1}{2}[F, F]\barwedge G=0. \tag{2.5}$$

We next put $L=M=G$, $N=F$ in (1.4) and find

$$[G, H]=-G\barwedge[F, G]-\frac{1}{2}[G, G]\barwedge F. \tag{2.6}$$

We then put $L=F$, $M=N=G$ in (1.4) and find

$$[G, H]=[F, G]\barwedge G+G\barwedge[F, G]+F\barwedge[G, G]. \tag{2.7}$$

From (2.6) and (2.7), we find

$$[G, H]=\frac{1}{2}[F, G]\barwedge G+\frac{1}{2}F\barwedge[G, G]-\frac{1}{4}[G, G]\barwedge F \tag{2.8}$$

and

$$[F, G]\barwedge G+2G\barwedge[F, G]+F\barwedge[G, G]+\frac{1}{2}[G, G]\barwedge F=0. \tag{2.9}$$

We put $L=FG$, $M=F$ and $N=G$ in (1.4) and find

$$[H, H]=-[F, F]+[H, F]\barwedge G+H\barwedge[F, G]+F\barwedge[G, H]. \tag{2.10}$$

Equations (2.2)$\sim$(2.9) are all same as in the previous paper [8], but equation (2.10) is a little different from (2.10) in the previous paper. We have the minus sign in front of $[F, F]$. The difference comes from the fact that $G^2=+1$ instead of $G^2=-1$.

We then put $L=FG$, $M=G$ and $N=F$ in (1.4) and find

$$[H, H]=[G, G]-[G, H]\barwedge F-H\barwedge [F, G]-G\barwedge [H, F]. \tag{2.11}$$

Thus, from (2.10) and (2.11), we find

$$\begin{aligned}[H, H]=\frac{1}{2}\{&-[F, F]+[G, G]+[H, F]\barwedge G\\ &-G\barwedge [H, F]-[G, H]\barwedge F+F\barwedge [G, H]\}\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&-[F, F]-[G, G]+[H, F]\barwedge G+G\barwedge [H, F]\\ &+[G, H]\barwedge F+F\barwedge [G, H]+2H\barwedge [F, G]=0.\end{aligned} \tag{2.13}$$

Equations (2.12) and (2.13) are also different from those in the previous paper [8].

Finally putting $L=M=N=F$, G or H in (1.4), we find

$$[F, F]\barwedge F=-2F\barwedge [F, F], \tag{2.14}$$

$$[G, G]\barwedge G=-2G\barwedge [G, G], \tag{2.15}$$

$$[H, H]\barwedge H=-2H\barwedge [H, H]. \tag{2.16}$$

In a previous paper, we proved a series of theorems which hold for an almost quaternion structure of the first kind. The formulas used to prove these theorems differ from (2.3)~(2.16) above only by the sign of the term $[F, F]$ in (2.10), (2.12) and (2.13) and consequently the theorems in the previous paper [8] having $[F, F]=0$ in the assumptions are also valid for the almost quaternion structure of the second kind. Thus Theorems 3.1~3.4 in the previous paper [8] are all valid also for an almost quaternion structure of the second kind.

Theorem 3.5 in the previous paper [8] which says that if $[F, G]=0$, $[F, H]=0$, then $[F, F]=0$ is also valid. Because to prove this, we have only to use (2.3). From the assumptions and (2.3), we have $G\barwedge [F, F]=0$, from which, using $G^2=+1$, we have $[F, F]=0$.

Since Theorems 3.6~3.8 in the previous paper [8] are also valid for an almost quaternion structure of the second kind, we have

THEOREM 2.1. *For an almost quaternion structure* (F, G, H) *of the second kind, if two of six Nijenhuis tensors*

$$[F, F],\ [G, G],\ [H, H],\ [G, H],\ [H, F],\ [F, G]$$

vanish, then the others vanish too.

§ 3. Affine connections in an almost quaternion manifold of the second kind.

As in the previous paper [8], we discuss here affine connections in an almost quaternion manifold of the second kind.

It is well known that, when a tensor field G of type (1,1) satisfying $G^2=1$ is given, there exists an affine connection $\overset{1}{\nabla}$ such that $\overset{1}{\nabla}G=0$ and its torsion tensor is proportional to the Nijenhuis tensor $[G, G]$ formed with G (see [7]). Indeed, if we introduce first of all a symmetric affine connection $\overset{\circ}{\nabla}$ in the manifold and define an affine connection $\overset{1}{\nabla}$ by its coefficients of connection

$$\overset{1}{\Gamma}{}^h_{ji}=\overset{\circ}{\Gamma}{}^h_{ji}+\frac{1}{4}(\overset{\circ}{\nabla}_j G_i{}^t+\overset{\circ}{\nabla}_i G_j{}^t)G_t{}^h-\frac{1}{4}(\overset{\circ}{\nabla}_j G_t{}^h-\overset{\circ}{\nabla}_t G_j{}^h)G_i{}^t, \tag{3.1}$$

then we have

$$\begin{gathered}\overset{1}{\nabla}_j G_i{}^h=0,\\ \overset{1}{\Gamma}{}^h_{ji}-\overset{1}{\Gamma}{}^h_{ij}=-\frac{1}{8}[G, G]_{ji}{}^h,\end{gathered} \tag{3.2}$$

$[G, G]_{ji}{}^h$ being components of the Nijenhuis tensor $[G, G]$, where here and in the sequel Roman indices $a, b, c, \cdots, h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m\}$, $2m$ being the dimension of the manifold. The second equation of (3.2) shows that the connection $\overset{1}{\nabla}$ is symmetric if and only if $[G, G]=0$.

We now prove

THEOREM 3.1. *In order that there exists, in an almost quaternion manifold of the second kind with structure tensors F, G and H, a symmetric affine connection ∇ such that*

$$\nabla F=0, \qquad \nabla G=0, \qquad \nabla H=0,$$

it is necessary and sufficient that

$$[F, F]=0, \qquad [G, G]=0.$$

Proof. We put

$$\Gamma^h_{ji}=\overset{1}{\Gamma}{}^h_{ji}+T_{ji}{}^h, \tag{3.3}$$

where $\overset{1}{\Gamma}{}^h_{ji}$ are given by (3.1) and $T_{ji}{}^h$ are defined by

$$\begin{aligned}T_{ji}{}^h=&-\frac{1}{4}\{F_i{}^t\overset{1}{\nabla}_t F_j{}^h+(\overset{1}{\nabla}_j F_i{}^t)F_t{}^h-H_i{}^t\overset{1}{\nabla}_t H_j{}^h+(\overset{1}{\nabla}_i H_j{}^t)H_t{}^h\}\\ &+\frac{1}{4}(G_i{}^s\overset{1}{\nabla}_s F_j{}^t-G_s{}^t\overset{1}{\nabla}_i F_j{}^s)H_t{}^h-\frac{1}{4}(\overset{1}{\nabla}_j F_i{}^t+\overset{1}{\nabla}_i F_j{}^t)F_t{}^h,\end{aligned} \tag{3.4}$$

which also can be written as

$$T_{ji}{}^h = -\frac{1}{2}(\overset{1}{\nabla}_j F_i{}^t)F_t{}^h - \frac{1}{4}\{(\overset{1}{\nabla}_i F_j{}^t)F_t{}^h + F_i{}^t \overset{1}{\nabla}_t F_j{}^h - (G_i{}^t H_s{}^h \overset{1}{\nabla}_t F_j{}^s - H_i{}^t \overset{1}{\nabla}_t H_j{}^h)\}.$$

Now, denoting by ∇_j the operator of covariant differentiation with respect to Γ^h_{ji}, we see that

$$\nabla_j G_i{}^h = \overset{1}{\nabla}_j G_i{}^h + T_{jt}{}^h G_i{}^t - T_{ji}{}^t G_t{}^h,$$

that is,

$$\nabla_j G_i{}^h = T_{jt}{}^h G_i{}^t - T_{ji}{}^t G_t{}^h$$

by virtue of $\overset{1}{\nabla}_j G_i{}^h = 0$.

By a computation similar to that used in [8], we have

$$\nabla_j G_i{}^h = 0$$

using $FG=H$, $GH=-F$, $HF=G$ instead of $FG=H$, $GH=F$, $HF=G$ in [8].

On the other hand, we have

$$\nabla_j F_i{}^h = \overset{1}{\nabla}_j F_i{}^h + T_{jt}{}^h F_i{}^t - T_{ji}{}^t F_t{}^h. \tag{3.5}$$

We substitute (3.4) into (3.5), then by a straightforward computation, we find

$$\nabla_j F_i{}^h = 0.$$

We know that if we assume that $[G, G]=0$, then $\overset{1}{\nabla}$ is a symmetric affine connection and consequently from (3.3) we have in this case

$$\Gamma^h_{ji} - \Gamma^h_{ij} = T_{ji}{}^h - T_{ij}{}^h.$$

On the other hand, we have from (3.4)

$$T_{ji}{}^h - T_{ij}{}^h = \frac{1}{8}\{[F, F]_{ji}{}^h - [H, H]_{ji}{}^h - 2[F, G]_{ji}{}^t H_t{}^h\}.$$

Thus, if we assume that $[F, F]=0$ in addition to $[G, G]=0$, then by Theorem 2.1 we have

$$T_{ji}{}^h - T_{ij}{}^h = 0$$

and consequently ∇ is a symmetric affine connection such that

$$\nabla F = 0, \quad \nabla G = 0$$

and hence

$$\nabla H = \nabla(FG) = 0.$$

The converse being evident, the proof of the theorem is completed.

Combining Theorems 2.1 and 3.1, we have

THEOREM 3.2. *In order that there exists, in an almost quaternion manifold of the second kind with structure tensors F, G and H, a symmetric affine connection ∇ such that*

$$\nabla F=0, \quad \nabla G=0, \quad \nabla H=0,$$

it is necessary and sufficient that two of the Nijenhuis tensors

$$[F, F], \ [G, G], \ [H, H], \ [G, H], \ [H, F], \ [F, G]$$

vanish.

In the proof of Theorem 3.1, we have put

$$\Gamma^h_{ji}=\overset{1}{\Gamma}{}^h_{ji}+T_{ji}{}^h,$$

where $T_{ji}{}^h$ are given by (3.4). We shall now compute the torsion tensor of ∇ assuming not necessarily $[F, F]=0$, $[G, G]=0$. Denoting by

$$S_{ji}{}^h=\Gamma^h_{ji}-\Gamma^h_{ij} \tag{3.6}$$

the torsion tensor of ∇, we have from (3.2) and (3.3)

$$S_{ji}{}^h=-\frac{1}{8}[G, G]_{ji}{}^h+T_{ji}{}^h-T_{ij}{}^h. \tag{3.7}$$

On the other hand, a straightforward computation shows that

$$\begin{aligned} T_{ji}{}^h-T_{ij}{}^h=\frac{1}{8}\Big\{&[F, F]_{ji}{}^h-[G, G]_{ji}{}^h-[H, H]_{ji}{}^h \\ &-2(H\barwedge[F, G])_{ji}{}^h-\frac{1}{2}(F\barwedge[G, G]\barwedge F)_{ji}{}^h\Big\}. \end{aligned}$$

Thus we have

$$S=\frac{1}{8}\Big\{[F, F]-2[G, G]-[H, H]-2H\barwedge[F, G]-\frac{1}{2}F\barwedge[G, G]\barwedge F\Big\}, \tag{3.8}$$

where S is a tensor field of type (1, 2) with components $S_{ji}{}^h$.

On the other hand, we have from (2.6)

$$-\frac{1}{2}[G, G]\barwedge F=[G, H]+G\barwedge[F, G]$$

and consequently

$$-\frac{1}{2}F\barwedge[G, G]\barwedge F=F\barwedge[G, H]+H\barwedge[F, G].$$

Thus substituting this into (3.8), we find

$$S=\frac{1}{8}\{[F,F]-2[G,G]-[H,H]+F\barwedge[G,H]-H\barwedge[F,G]\}. \tag{3.9}$$

Now we let F, $-H$, G play the roles of F, G, H respectively in the discussion above, then we obtain an affine connection $'\nabla$ such that

$$'\nabla F=0, \quad '\nabla H=0, \quad '\nabla G=0$$

and that the torsion tensor $'S$ of $'\nabla$ is given by

$$'S=\frac{1}{8}\{[F,F]-[G,G]-2[H,H]-F\barwedge[G,H]+G\barwedge[F,H]\}. \tag{3.10}$$

Next, we let iG, iF, H ($i^2=-1$) play the roles of F, G, H respectively in the same discussion above. By the definition of ∇, its coefficients Γ_{ji}^h are still real even if iG, iF, H play the roles of F, G, H in (3.1), (3.3) and (3.4). We thus obtain an affine connection $''\nabla$ such that

$$''\nabla G=0, \quad ''\nabla F=0, \quad ''\nabla H=0$$

and that

$$''S=\frac{1}{8}\{2[F,F]-[G,G]-[H,H]-G\barwedge[F,H]+H\barwedge[F,G]\}. \tag{3.11}$$

Thus if we define an affine connection by

$$\frac{1}{3}(\Gamma_{ji}^h+'\Gamma_{ji}^h+''\Gamma_{ji}^h),$$

$'\Gamma_{ji}^h$ and $''\Gamma_{ji}^h$ being respectively components of the affine connections $'\nabla$ and $''\nabla$, the covariant derivatives of F, G and H with respect to this connection vanish and the torsion tensor of this affine connection is given by

$$\frac{1}{6}\{[F,F]-[G,G]-[H,H]\}.$$

§ 4. Integrability conditions.

We say that an almost quaternion structure (F, G, H) of the second kind is *integrable* if the manifold M admitting (F, G, H) can be covered by a system of coordinate neighborhoods in which components of F, G and H are all constant.

If we put

$$A=\frac{1}{2}(I-G), \qquad B=\frac{1}{2}(I+G)$$

then from $FG+GF=0$, we have $FA=BF$ which shows that $r(A)=r(B)$, r denoting the rank. On the other hand, since $A+B=I$, $AB=0$ we have $r(A)+r(B)=2m$. Thus we see that

$$r(A)=r(B)=m.$$

Assume that $[G, G]=0$, then there exists a coordinate system with respect to which G has components of the form

$$G=\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \tag{4.1}$$

where E is the $m\times m$ unit matrix (see [3]).

We represent components of the tensor field F with respect to this coordinate system by

$$F=\begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

where F_1, F_2, F_3 and F_4 are $m\times m$ matrices. Then from $FG+GF=0$, we find

$$F_1=0, \qquad F_4=0$$

and from $F^2=-1$

$$F_2F_3=F_3F_2=-E.$$

Thus F has the form

$$F=\begin{pmatrix} 0 & F' \\ F'' & 0 \end{pmatrix}, \tag{4.2}$$

where

$$F'F''=F''F'=-E. \tag{4.3}$$

We now consider the condition

$$[F, F]=0. \tag{4.4}$$

If we represent components of F by

$$F=\begin{pmatrix} 0 & F_{\bar{\lambda}}{}^{\kappa} \\ F_{\lambda}{}^{\bar{\kappa}} & 0 \end{pmatrix}, \tag{4.5}$$

then (4.3) can be written as

$$F_{\lambda}{}^{\bar{\alpha}}F_{\bar{\alpha}}{}^{\kappa}=-\delta_{\lambda}^{\kappa}, \qquad F_{\bar{\lambda}}{}^{\alpha}F_{\alpha}{}^{\bar{\kappa}}=-\delta_{\bar{\lambda}}^{\bar{\kappa}} \tag{4.6}$$

and (4.4) as

$$\begin{aligned} &\partial_\mu F_\lambda{}^{\bar\kappa} - \partial_\lambda F_\mu{}^{\bar\kappa} = 0, \\ (4.7) \qquad &\partial_{\bar\mu} F_{\bar\lambda}{}^{\kappa} - \partial_{\bar\lambda} F_{\bar\mu}{}^{\kappa} = 0, \end{aligned}$$

∂_μ and $\partial_{\bar\mu}$ denoting $\partial/\partial x^\mu$ and $\partial/\partial x^{\bar\mu}$ respectively, where here and in the sequel Greek indices $\kappa, \lambda, \mu, \cdots$ run over the range $\{1, 2, \cdots, m\}$ and $\bar\kappa, \bar\lambda, \bar\mu, \cdots$ the range $\{m+1, m+2, \cdots, 2m\}$.

We have

PROPOSITION 4.1. *The symmetric affine connection appearing in Theorem* 3.2 *has the following components with respect to a coordinate system in which G has components* (4.1):

$$\begin{aligned} &\Gamma_{\mu\lambda}^{\kappa} = -(\partial_\mu F_\lambda{}^{\bar\alpha}) F_{\bar\alpha}{}^{\kappa} \\ (4.8) \qquad &\Gamma_{\bar\mu\bar\lambda}^{\bar\kappa} = -(\partial_{\bar\mu} F_{\bar\lambda}{}^{\alpha}) F_\alpha{}^{\bar\kappa}, \end{aligned}$$

all the others being zero.

Proof. From (4.1) and

$$\nabla_j G_i{}^h = \partial_j G_i{}^h + \Gamma_{jt}^h G_i{}^t - \Gamma_{ji}^t G_t{}^h = 0,$$

we find that Γ_{ji}^h are all zero except $\Gamma_{\mu\lambda}^{\kappa}$ and $\Gamma_{\bar\mu\bar\lambda}^{\bar\kappa}$. On the other hand, from (4.5) and

$$\nabla_j F_i{}^h = \partial_j F_i{}^h + \Gamma_{jt}^h F_i{}^t - \Gamma_{ji}^t F_t{}^h = 0,$$

we find

$$\partial_{\bar\mu} F_{\bar\lambda}{}^{\kappa} - \Gamma_{\bar\mu\bar\lambda}^{\bar\alpha} F_{\bar\alpha}{}^{\kappa} = 0,$$

$$\partial_\mu F_\lambda{}^{\bar\kappa} - \Gamma_{\mu\lambda}^{\alpha} F_\alpha{}^{\bar\kappa} = 0,$$

from which, using (4.3),

$$\Gamma_{\mu\lambda}^{\kappa} = -(\partial_\mu F_\lambda{}^{\bar\alpha}) F_{\bar\alpha}{}^{\kappa}, \qquad \Gamma_{\bar\mu\bar\lambda}^{\bar\kappa} = -(\partial_{\bar\mu} F_{\bar\lambda}{}^{\alpha}) F_\alpha{}^{\bar\kappa}.$$

Thus the proposition is proved.

We denote by

$$R_{kji}{}^h = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \Gamma_{kt}^h \Gamma_{ji}^t - \Gamma_{jt}^h \Gamma_{ki}^t$$

the curvature tensor of Γ_{ji}^h, then we have

PROPOSITION 4.2. *The curvature tensor R of ∇ appearing in Theorem* 3.2 *has the following components with respect to a coordinate system in which G has components* (4.1):

$$\begin{aligned} &R_{\bar\nu\mu\lambda}{}^{\kappa} = -R_{\mu\bar\nu\lambda}{}^{\kappa} = \partial_{\bar\nu} \Gamma_{\mu\lambda}^{\kappa}, \\ (4.9) \qquad &R_{\nu\bar\mu\bar\lambda}{}^{\bar\kappa} = -R_{\bar\mu\nu\bar\lambda}{}^{\bar\kappa} = \partial_\nu \Gamma_{\bar\mu\bar\lambda}^{\bar\kappa}, \end{aligned}$$

all the others being zero.

Proof. From Proposition 4.1 and the definition of $R_{kji}{}^{h}$, we have

$$R_{\nu\mu\lambda}{}^{\kappa}=\partial_{\nu}\Gamma^{\kappa}_{\mu\lambda}-\partial_{\mu}\Gamma^{\kappa}_{\nu\lambda}+\Gamma^{\kappa}_{\nu\omega}\Gamma^{\omega}_{\mu\lambda}-\Gamma^{\kappa}_{\mu\omega}\Gamma^{\omega}_{\nu\lambda},$$

from which, substituting (4.7),

$$\begin{aligned}R_{\nu\mu\lambda}{}^{\kappa}=&-(\partial_{\nu}\partial_{\mu}F_{\lambda}{}^{\bar{\alpha}})F_{\bar{\alpha}}{}^{\kappa}-(\partial_{\mu}F_{\lambda}{}^{\bar{\alpha}})(\partial_{\nu}F_{\bar{\alpha}}{}^{\kappa})\\&+(\partial_{\mu}\partial_{\nu}F_{\lambda}{}^{\bar{\alpha}})F_{\bar{\alpha}}{}^{\kappa}+(\partial_{\nu}F_{\lambda}{}^{\bar{\alpha}})(\partial_{\mu}F_{\bar{\alpha}}{}^{\kappa})+(\partial_{\nu}F_{\omega}{}^{\bar{\alpha}})F_{\bar{\alpha}}{}^{\kappa}(\partial_{\mu}F_{\lambda}{}^{\bar{\beta}})F_{\bar{\beta}}{}^{\omega}\\&-(\partial_{\mu}F_{\omega}{}^{\bar{\alpha}})F_{\bar{\alpha}}{}^{\kappa}(\partial_{\nu}F_{\lambda}{}^{\bar{\beta}})F_{\bar{\beta}}{}^{\omega},\end{aligned}$$

from which, using (4.3),

$$\begin{aligned}R_{\nu\mu\lambda}{}^{\kappa}=&-(\partial_{\mu}F_{\lambda}{}^{\bar{\alpha}})(\partial_{\nu}F_{\bar{\alpha}}{}^{\kappa})+(\partial_{\nu}F_{\lambda}{}^{\bar{\alpha}})(\partial_{\mu}F_{\bar{\alpha}}{}^{\kappa})\\&-F_{\omega}{}^{\bar{\alpha}}(\partial_{\nu}F_{\bar{\alpha}}{}^{\kappa})(\partial_{\mu}F_{\lambda}{}^{\bar{\beta}})F_{\bar{\beta}}{}^{\omega}+F_{\omega}{}^{\bar{\alpha}}(\partial_{\mu}F_{\bar{\alpha}}{}^{\kappa})(\partial_{\nu}F_{\lambda}{}^{\bar{\beta}})F_{\bar{\beta}}{}^{\omega},\end{aligned}$$

or again using (4.3),

$$R_{\nu\mu\lambda}{}^{\kappa}=0.$$

Similarly we have

$$R_{\bar{\nu}\bar{\mu}\bar{\lambda}}{}^{\bar{\kappa}}=0.$$

Also from Proposition 4.1 and the definition of $R_{kji}{}^{h}$, we can easily see that

$$R_{\bar{\nu}\mu\lambda}{}^{\kappa}=-R_{\mu\bar{\nu}\lambda}{}^{\kappa}=\partial_{\bar{\nu}}\Gamma^{\kappa}_{\mu\lambda},$$

$$R_{\nu\bar{\mu}\bar{\lambda}}{}^{\bar{\kappa}}=-R_{\bar{\mu}\nu\bar{\lambda}}{}^{\bar{\kappa}}=\partial_{\nu}\Gamma^{\bar{\kappa}}_{\bar{\mu}\bar{\lambda}},$$

all the other $R_{kji}{}^{h}$ being zero. Thus the Proposition is proved.

We now prove

THEOREM 4.1. *A necessary and sufficient condition that an almost quaternion structure (F, G, H) of the second kind be integrable is that*

$$[F, F]=0, \quad [G, G]=0, \quad R=0,$$

where R is the curvature tensor of the symmetric affine connection appearing in Theorem 3.2.

Proof. The necessity being evident, we shall prove the sufficiency. Following the assumption $[F, F]=0$, $[G, G]=0$, we can choose a coordinate system in which the tensor field G has components (4.1) and consequently the tensor field F has components (4.2).

We are now going to find a coordinate system $\{y^{h}\}$ in which the components $\tilde{G}_{i}{}^{h}$ of G in $\{y^{h}\}$ are still given by

(4.10) $$\tilde{G}_i{}^h=\begin{pmatrix} \delta_\lambda^\kappa & 0 \\ 0 & -\delta_{\bar{\lambda}}^{\bar{\kappa}} \end{pmatrix}$$

and the components $\tilde{F}_i{}^h$ of F in $\{y^h\}$ are given by

(4.11) $$\tilde{F}_i{}^h=\begin{pmatrix} 0 & C_{\bar{\lambda}}{}^\kappa \\ C_\lambda{}^{\bar{\kappa}} & 0 \end{pmatrix},$$

where C's are constants satisfying

(4.12) $$C_\lambda{}^{\bar{\alpha}}C_{\bar{\alpha}}{}^\kappa=-\delta_\lambda^\kappa, \qquad C_{\bar{\lambda}}{}^\alpha C_\alpha{}^{\bar{\kappa}}=-\delta_{\bar{\lambda}}^{\bar{\kappa}}.$$

If we put

(4.13) $$y^h=y^h(x^i),$$

then we have from (4.1) and (4.10)

$$\frac{\partial y^k}{\partial x^i}\tilde{G}_k{}^h=\frac{\partial y^h}{\partial x^j}G_i{}^j,$$

which are equivalent to

$$\frac{\partial y^\kappa}{\partial x^{\bar{\lambda}}}=0, \qquad \frac{\partial y^{\bar{\kappa}}}{\partial x^\lambda}=0$$

and consequently the coordinate transformation (4.13) must be of the form

(4.14) $$y^\kappa=y^\kappa(x^\lambda), \qquad y^{\bar{\kappa}}=y^{\bar{\kappa}}(x^{\bar{\lambda}}).$$

On the other hand, we have from (4.5) and (4.11)

$$\frac{\partial y^k}{\partial x^i}C_k{}^h=\frac{\partial y^h}{\partial x^j}F_i{}^j,$$

which are equivalent to

(4.15) $$\frac{\partial y^{\bar{\alpha}}}{\partial x^{\bar{\lambda}}}C_{\bar{\alpha}}{}^\kappa=\frac{\partial y^\kappa}{\partial x^\alpha}F_{\bar{\lambda}}{}^\alpha, \qquad \frac{\partial y^\alpha}{\partial x^\lambda}C_\alpha{}^{\bar{\kappa}}=\frac{\partial y^{\bar{\kappa}}}{\partial x^{\bar{\alpha}}}F_\lambda{}^{\bar{\alpha}},$$

or to

(4.16) $$\frac{\partial y^\alpha}{\partial x^\lambda}C_\alpha{}^{\bar{\kappa}}=\frac{\partial y^{\bar{\kappa}}}{\partial x^{\bar{\alpha}}}F_\lambda{}^{\bar{\alpha}}$$

by virtue of (4.6) and (4.12).

To eliminate constants C's, we differentiate (4.16) partially with respect to x^μ and take account of (4.14), then we obtain

$$\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda} C_\alpha{}^{\bar\kappa} = \frac{\partial y^{\bar\kappa}}{\partial x^{\bar\alpha}} (\partial_\mu F_\lambda{}^{\bar\alpha}),$$

from which, transvecting with $C_{\bar\kappa}{}^\omega$ and using (4.12),

$$-\frac{\partial^2 y^\omega}{\partial x^\mu \partial x^\lambda} = \left(\frac{\partial y^{\bar\kappa}}{\partial x^{\bar\alpha}} C_{\bar\kappa}{}^\omega\right)(\partial_\mu F_\lambda{}^{\bar\alpha}),$$

or, using the first equation of (4.15),

$$\frac{\partial^2 y^\omega}{\partial x^\mu \partial x^\lambda} = -\frac{\partial y^\omega}{\partial x^\nu} (\partial_\mu F_\lambda{}^{\bar\alpha}) F_{\bar\alpha}{}^\nu .$$

Thus we have obtained a system of differential equations

$$\frac{\partial^2 y^\kappa}{\partial x^\mu \partial x^\lambda} = \frac{\partial y^\kappa}{\partial x^\alpha} \Gamma^\alpha_{\mu\lambda}, \tag{4.17}$$

where $\Gamma^\alpha_{\mu\lambda}$ are given by (4.8). Similarly, we have

$$\frac{\partial^2 y^{\bar\kappa}}{\partial x^{\bar\mu} \partial x^{\bar\lambda}} = \frac{\partial y^{\bar\kappa}}{\partial x^{\bar\alpha}} \Gamma^{\bar\alpha}_{\bar\mu\bar\lambda}, \tag{4.18}$$

where $\Gamma^{\bar\alpha}_{\bar\mu\bar\lambda}$ are given by (4.8).

But we assumed that $R=0$, which means that

$$\partial_{\bar\nu} \Gamma^\kappa_{\mu\lambda} = 0, \qquad \partial_\nu \Gamma^{\bar\kappa}_{\bar\mu\bar\lambda} = 0$$

and consequently that $\Gamma^\kappa_{\mu\lambda}$ are functions of x^ν only and $\Gamma^{\bar\kappa}_{\bar\mu\bar\lambda}$ functions of $x^{\bar\nu}$ only. Moreover we have

$$R_{\nu\mu\lambda}{}^\kappa = \partial_\nu \Gamma^\kappa_{\mu\lambda} - \partial_\mu \Gamma^\kappa_{\nu\lambda} + \Gamma^\kappa_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} - \Gamma^\kappa_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} = 0,$$

$$R_{\bar\nu\bar\mu\bar\lambda}{}^{\bar\kappa} = \partial_{\bar\nu} \Gamma^{\bar\kappa}_{\bar\mu\bar\lambda} - \partial_{\bar\mu} \Gamma^{\bar\kappa}_{\bar\nu\bar\lambda} + \Gamma^{\bar\kappa}_{\bar\nu\bar\alpha}\Gamma^{\bar\alpha}_{\bar\mu\bar\lambda} - \Gamma^{\bar\kappa}_{\bar\mu\bar\alpha}\Gamma^{\bar\alpha}_{\bar\nu\bar\lambda} = 0,$$

which show that (4.17) and (4.18) are both completely integrable and admit solutions $y^\kappa = y^\kappa(x^\alpha)$, $y^{\bar\kappa} = y^{\bar\kappa}(x^{\bar\alpha})$ such that

$$\left|\frac{\partial y^\kappa}{\partial x^\lambda}\right| \neq 0, \qquad \left|\frac{\partial y^{\bar\kappa}}{\partial x^{\bar\lambda}}\right| \neq 0$$

respectively.

Using these solutions, we effect a coordinate transformation. Then it is easily seen that the tensor G has components given by (4.10). Denoting by

$$\tilde F_i{}^h = \begin{pmatrix} f_\lambda{}^\kappa & f_{\bar\lambda}{}^\kappa \\ f_\lambda{}^{\bar\kappa} & f_{\bar\lambda}{}^{\bar\kappa} \end{pmatrix}, \tag{4.19}$$

the components of the tensor field F in $\{y^h\}$, we have

$$\frac{\partial y^k}{\partial x^i} f_k{}^h = \frac{\partial y^h}{\partial x^j} F_i{}^j, \tag{4.20}$$

from which we have

$$f_\lambda{}^{\bar\kappa} = 0, \qquad f_{\bar\lambda}{}^{\kappa} = 0 \tag{4.21}$$

and

$$\frac{\partial y^{\bar\alpha}}{\partial x^{\bar\lambda}} f_{\bar\alpha}{}^{\kappa} = \frac{\partial y^\kappa}{\partial x^\alpha} F_{\bar\lambda}{}^\alpha, \tag{4.22}$$

$$\frac{\partial y^\alpha}{\partial x^\lambda} f_\alpha{}^{\bar\kappa} = \frac{\partial y^{\bar\kappa}}{\partial x^{\bar\alpha}} F_\lambda{}^{\bar\alpha}. \tag{4.23}$$

Since $F_i{}^h$ satisfies (4.6), $f_i{}^h$ satisfies

$$f_\lambda{}^{\bar\alpha} f_{\bar\alpha}{}^{\kappa} = -\delta_\lambda^\kappa, \qquad f_{\bar\lambda}{}^\alpha f_\alpha{}^{\bar\kappa} = -\delta_{\bar\lambda}^{\bar\kappa}. \tag{4.24}$$

We prove that the functions $f_{\bar\lambda}{}^\kappa$ and $f_\lambda{}^{\bar\kappa}$ are constant.

Differentiating (4.22) partially with respect to x^μ and taking account of (4.14), we have

$$\frac{\partial y^{\bar\alpha}}{\partial x^{\bar\lambda}} \partial_\mu f_{\bar\alpha}{}^{\kappa} = \frac{\partial^2 y^\kappa}{\partial x^\mu \partial x^\alpha} F_{\bar\lambda}{}^\alpha + \frac{\partial y^\kappa}{\partial x^\alpha} \partial_\mu F_{\bar\lambda}{}^\alpha,$$

from which, substituting (4.8) and (4.17),

$$\frac{\partial y^{\bar\alpha}}{\partial x^{\bar\lambda}} \partial_\mu f_{\bar\alpha}{}^{\kappa} = -\frac{\partial y^\kappa}{\partial x^\omega} (\partial_\mu F_\alpha{}^{\bar\beta}) F_{\bar\beta}{}^\omega F_{\bar\lambda}{}^\alpha + \frac{\partial y^\kappa}{\partial x^\alpha} \partial_\mu F_{\bar\lambda}{}^\alpha = 0$$

and consequently

$$\partial_\mu f_{\bar\alpha}{}^\kappa = 0.$$

Differentiating (4.22) partially with respect to $x^{\bar\mu}$ and taking account of (4.14), we find

$$\frac{\partial^2 y^{\bar\alpha}}{\partial x^{\bar\mu} \partial x^{\bar\lambda}} f_{\bar\alpha}{}^\kappa + \frac{\partial y^{\bar\alpha}}{\partial x^{\bar\lambda}} \partial_{\bar\mu} f_{\bar\alpha}{}^\kappa = \frac{\partial y^\kappa}{\partial x^\alpha} (\partial_{\bar\mu} F_{\bar\lambda}{}^\alpha),$$

from which, substituting (4.8) and (4.18),

$$-\frac{\partial y^{\bar\alpha}}{\partial x^{\bar\omega}} (\partial_{\bar\mu} F_{\bar\lambda}{}^\beta) F_\beta{}^{\bar\omega} f_{\bar\alpha}{}^\kappa + \frac{\partial y^{\bar\alpha}}{\partial x^{\bar\lambda}} \partial_{\bar\mu} f_{\bar\alpha}{}^\kappa = \frac{\partial y^\kappa}{\partial x^\alpha} (\partial_{\bar\mu} F_\lambda{}^\alpha),$$

or, using (4.23),

$$-\frac{\partial y^{\omega}}{\partial x^{\beta}} f_{\omega}{}^{\bar{\alpha}}(\partial_{\bar{\mu}} F_{\bar{\lambda}}{}^{\beta}) f_{\bar{\alpha}}{}^{\kappa} + \frac{\partial y^{\bar{\alpha}}}{\partial x^{\bar{\lambda}}} \partial_{\bar{\mu}} f_{\bar{\alpha}}{}^{\kappa} = \frac{\partial y^{\kappa}}{\partial x^{\alpha}} (\partial_{\bar{\mu}} F_{\bar{\lambda}}{}^{\alpha}),$$

and consequently

$$\frac{\partial y^{\bar{\alpha}}}{\partial x^{\bar{\lambda}}} \partial_{\bar{\mu}} f_{\bar{\alpha}}{}^{\kappa} = 0,$$

that is

$$\partial_{\bar{\mu}} f_{\bar{\alpha}}{}^{\kappa} = 0.$$

Thus we see that $f_{\bar{\alpha}}{}^{\kappa}$ are constant.

We can similarly prove that $f_{\alpha}{}^{\bar{\kappa}}$ are also constant. Thus the theorem is proved.

Combining Theorems 2.1 and 4.1, we have

THEOREM 4.2. *A necessary and sufficient condition for an almost quaternion structure* (F, G, H) *of the second kind to be integrable is that two of Nijenhuis tensors*

$$[F, F],\ [G, G],\ [H, H],\ [G, H],\ [H, F],\ [F, H]$$

vanish and $R=0$.

§ 5. Concomitants of nilpotent tensors of type (1, 1).

We consider, in this section, two tensor fields P and Q of type $(1, 1)$ satisfying

$$P^2=0, \qquad Q^2=0, \qquad PQ+QP=1. \tag{5.1}$$

Putting $L=M=P$, $N=Q$ in (1.4), we have

$$[P, PQ]+[P, PQ]=[P, P]\barwedge Q+P\barwedge[P, Q]+P\barwedge[P, Q],$$

or

$$[P, PQ]=P\barwedge[P, Q]+\frac{1}{2}[P, P]\barwedge Q, \tag{5.2}$$

and putting $L=N=P$, $M=Q$ in (1.4),

$$[P, QP]+[Q, P^2]=[P, Q]\barwedge P+P\barwedge[Q, P]+Q\barwedge[P, P],$$

or

$$[P, QP]=[P, Q]\barwedge P+P\barwedge[P, Q]+Q\barwedge[P, P]. \tag{5.3}$$

Since we have, from the third equation of (5.1),

$$[P, PQ]+[P, QP]=0, \tag{5.4}$$

we have, from (5.2) and (5.3),

$$[P, PQ]=-\frac{1}{2}[P, Q]\barwedge P-\frac{1}{2}Q\barwedge[P, P]+\frac{1}{4}[P, P]\barwedge Q \tag{5.5}$$

and

$$[P, Q]\barwedge P+2P\barwedge[P, Q]+Q\barwedge[P, P]+\frac{1}{2}[P, P]\barwedge Q=0. \tag{5.6}$$

Interchanging P and Q in (5.2), (5.3), (5.4), (5.5) and (5.6) above, we have respectively

$$[Q, QP]=Q\barwedge[P, Q]+\frac{1}{2}[Q, Q]\barwedge P, \tag{5.7}$$

$$[Q, PQ]=[P, Q]\barwedge Q+Q\barwedge[P, Q]+P\barwedge[Q, Q], \tag{5.8}$$

$$[Q, PQ]+[Q, QP]=0, \tag{5.9}$$

$$[Q, PQ]=\frac{1}{2}[P, Q]\barwedge Q+\frac{1}{2}P\barwedge[Q, Q]-\frac{1}{4}[Q, Q]\barwedge P \tag{5.10}$$

and

$$[P, Q]\barwedge Q+2Q\barwedge[P, Q]+P\barwedge[Q, Q]+\frac{1}{2}[Q, Q]\barwedge P=0. \tag{5.11}$$

Now, putting $L=M=N=P$ in (1.4), we find

$$[P, P]\barwedge P=-2P\barwedge[P, P] \tag{5.12}$$

and putting $L=M=N=Q$, we find

$$[Q, Q]\barwedge Q=-2Q\barwedge[Q, Q]. \tag{5.13}$$

Also, putting $L=PQ$, $M=P$, $N=Q$ in (1.4) and using $Q^2=0$, we find

$$[PQ, PQ]=[P, PQ]\barwedge Q+P\barwedge[PQ, Q]+PQ\barwedge[P, Q] \tag{5.14}$$

and putting $L=QP$, $M=Q$, $N=P$, we find

$$[QP, QP]=[Q, QP]\barwedge P+Q\barwedge[QP, P]+QP\barwedge[P, Q]. \tag{5.15}$$

On the other hand, we have, from the third equation of (5.1),

$$[PQ, PQ]=[I-QP, I-QP]=[QP, QP]$$

and consequently, we have, from (5.14) and (5.15),

$$
\begin{aligned}
&[PQ, PQ]=[QP, QP] \\
(5.16)\qquad &=\frac{1}{2}\{[P, PQ]\barwedge Q-Q\barwedge[P, PQ]+[Q, QP]\barwedge P \\
&\qquad -P\barwedge[Q, QP]+[P, Q]\}
\end{aligned}
$$

and

$$
\begin{aligned}
(5.17)\qquad &[P, PQ]\barwedge Q+Q\barwedge[P, PQ]+P\barwedge[Q, PQ]+[Q, PQ]\barwedge P \\
&+2PQ\barwedge[P, Q]-[P, Q]=0.
\end{aligned}
$$

Now we have from (5.1)

$$(PQ)^2=PQ, \qquad (QP)^2=QP, \qquad (PQ)(QP)=(QP)(PQ)=0,$$
$$PQ+QP=1,$$

which show that PQ and QP are complementary projection operators and consequently define two complementary distributions. We call the distribution defined by the projection operator PQ the *horizontal distribution* and that defined by the projection operator QP the *vertical distribution*.

A vector X in the horizontal distribution, that is, a vector X satisfying $PQX=X$, or $QPX=0$ is said to be *horizontal* and a vector Y in the vertical distribution, that is, a vector Y satisfying $QPY=Y$, or $PQY=0$ said to be *vertical*.

Since $QP(PX)=0$ and $P(PQX)=0$ for an arbitrary vector X, the operator P transforms a vector into a horizontal vector and a horizontal vector into a zero vector.

Since $PQ(QX)=0$ and $Q(QPX)=0$ for an arbitrary vector X, the operator Q transforms a vector into a vertical vector and a vertical vector into a zero vector.

We here show that if two $2m\times 2m$ matrices P and Q satisfy (5.1), then the rank $r(P)$ of P and that $r(Q)$ of Q are both equal to m.

First of all if we put $F=P-Q$, then we have

$$PF=-PQ, \qquad FP=-QP, \qquad FQ=PQ$$

and $F^2=-1$, from which we see that F is regular. Thus we have

$$r(P)=r(PQ)=r(QP)=r(Q).$$

Denoting by V^{2m} a $2m$-dimensional vector space, we put

$$\Pi_1=\{V\in V^{2m}|PQV=V\},$$

$$\Pi_2=\{V\in V^{2m}|QPV=V\}.$$

Then, since $PQ+QP=1$, we have

$$V^{2m}=\Pi_1+\Pi_2 \qquad \text{(direct sum)}$$

and consequently we have

$$2m = \dim \Pi_1 + \dim \Pi_2,$$

that is

$$r(PQ) + r(QP) = 2m,$$

from which, using $r(PQ) = r(QP)$,

$$r(PQ) = r(QP) = m.$$

Thus

$$r(P) = r(Q) = m.$$

We assume throughout this section that $[Q, Q] = 0$. The reason why we use terminologies above and this assumption will appear in § 7.

THEOREM 5.1. *If* $[Q, Q] = 0$, *then the vector field*

$$[QP, QP](X, Y) \qquad or \qquad [PQ, PQ](X, Y)$$

is vertical for arbitrary vector fields X *and* Y.

Proof. First of all we have from the assumption $[Q, Q] = 0$ and (5.7)

$$[Q, QP] = Q \barwedge [P, Q],$$

from which

$$PQ \barwedge [Q, QP] = 0.$$

On the other hand, from (5.15) we find

$$PQ \barwedge [QP, QP] = PQ \barwedge ([Q, QP] \barwedge P) = (PQ \barwedge [Q, QP]) \barwedge P$$

by virtue of (1.6), from which

$$PQ \barwedge [QP, QP] = 0.$$

Thus $PQ[QP, QP](X, Y) = 0$, which shows that $[QP, QP](X, Y)$ is a vertical vector field. Since $[QP, QP] = [PQ, PQ]$, $[PQ, PQ](X, Y)$ is also vertical.

COROLLARY 1. *If* $[Q, Q] = 0$, *then the vertical distribution is integrable.*

Proof. Let X and Y be arbitrary vector fields, then we have, by the definition of the Nijenhuis tensor,

$$[QPX, QPY] = \frac{1}{2}[QP, QP](X, Y) + QP[QPX, Y] + QP[X, QPY] - QP[X, Y].$$

Since $[QP, QP](X, Y)$ is vertical by Theorem 5.1, the equation above shows that $[QPX, QPY]$ is vertical and consequently the vertical distribution is integrable.

COROLLARY 2. *If* $[Q, Q]=0$, *a necessary and sufficient condition for the horizontal distribution to be integrable is that*

$$[PQ, PQ]=0.$$

Proof. A necessary and sufficient condition for the horizontal distribution to be integrable is that $[PQX, PQY]$ is horizontal for arbitrary vector fields X and Y. But, we have, from the definition of the Nijenhuis tensor

$$[PQX, PQY]=\frac{1}{2}[PQ, PQ](X, Y)+PQ[PQX, Y]+PQ[X, PQY]-PQ[X, Y].$$

Since $[PQ, PQ](X, Y)$ is always vertical by Theorem 5.1, a necessary and sufficient condition for $[PQX, PQY]$ to be horizontal is the vanishing of $[PQ, PQ](X, Y)$ for arbitrary vector fields X and Y. Thus the corollary is proved.

THEOREM 5.2. *If* $[P, Q]=0$ *and* $[Q, Q]=0$, *then* $[PQ, PQ]=0$.

Proof. We have, from the assumption and (5.7),

$$[Q, QP]=0, \tag{5.18}$$

and from (5.3)

$$[P, QP]=Q\barwedge[P, P]. \tag{5.19}$$

Substituting (5.18) and (5.19) into (5.15), we find

$$\begin{aligned}[PQ, PQ]&=[QP, QP]\\&=Q\barwedge(Q\barwedge[P, P])\\&=Q^2\barwedge[P, P]\\&=0,\end{aligned}$$

which proves the theorem.

Combining Theorem 5.2 and Corollary 2 to Theorem 5.1, we have

COROLLARY 1. *If* $[P, Q]=0$ *and* $[Q, Q]=0$, *then the horizontal distribution is integrable.*

We have also

THEOREM 5.3. *If* $[P, P]=0$ *and* $[Q, Q]=0$, *then we have* $[P, Q]=0$.

Proof. From (5.6) and $[P, P]=0$, we have

$$[P, Q] \barwedge P=-2P \barwedge [P, Q] \tag{5.20}$$

and from (5.11) and $[Q, Q]=0$

$$[P, Q] \barwedge Q=-2Q \barwedge [P, Q]. \tag{5.21}$$

We find, from (5.2) and $[P, P]=0$,

$$Q \barwedge [P, PQ]=QP \barwedge [P, Q] \tag{5.22}$$

and, from (5.2) and $[P, P]=0$,

$$\begin{aligned}[P, PQ] \barwedge Q &= (P \barwedge [P, Q]) \barwedge Q \\ &= P \barwedge ([P, Q] \barwedge Q)\end{aligned}$$

and consequently using (5.21)

$$[P, PQ] \barwedge Q=-2PQ \barwedge [P, Q]. \tag{5.23}$$

Similarly, we have, from (5.7) and $[Q, Q]=0$,

$$P \barwedge [Q, QP]=PQ \barwedge [P, Q]$$

or

$$P \barwedge [Q, PQ]=-PQ \barwedge [P, Q] \tag{5.24}$$

and

$$\begin{aligned}[Q, QP] \barwedge P &= (Q \barwedge [P, Q]) \barwedge P \\ &= Q \barwedge ([P, Q] \barwedge P)\end{aligned}$$

or using (5.20)

$$\begin{aligned}&[Q, QP] \barwedge P=-2QP \barwedge [P, Q], \\ &[Q, PQ] \barwedge P=2QP \barwedge [P, Q].\end{aligned} \tag{5.25}$$

Substituting (5.22), (5.23), (5.24) and (5.25) into (5.17), we find

$$-2PQ \barwedge [P, Q]+QP \barwedge [P, Q]-PQ \barwedge [P, Q]+2QP \barwedge [P, Q]+2PQ \barwedge [P, Q]-[P, Q]=0,$$

that is, using $QP-1=-PQ$,

$$PQ \barwedge [P, Q]-QP \barwedge [P, Q]=0,$$

from which, using $(PQ)^2=PQ$, $(QP)^2=QP$, $(PQ)(QP)=(QP)(PQ)=0$, we find

$$PQ \barwedge [P, Q]=0, \qquad QP \barwedge [P, Q]=0.$$

Thus, adding these equations and remembering $PQ+QP=1$, we have $[P, Q]=0$, which proves the theorem.

Thus combining Corollary 1 to Theorem 5.2 and this theorem, we have

COROLLARY. *If* $[P, P]=0$ *and* $[Q, Q]=0$, *then the horizontal distribution is integrable.*

We also have

THEOREM 5.4. *If* $[P, Q]$ *is vertical, that is,* $[P, Q](X, Y)$ *is vertical for arbitrary vector fields* X *and* Y, *and* $[Q, Q]=0$, *then we have*

$$[PQ, PQ]=2[P, Q].$$

Proof. Since $[P, Q](X, Y)$ is vertical for arbitrary vector fields, we have $[P, Q]=(QP)\barwedge[P, Q]$. Substituting this and $[Q, Q]=0$ into (5.7), we find

$$[Q, QP]=0. \tag{5.26}$$

On the other hand, from (5.2) and $[P, Q]=(QP)\barwedge[P, Q]$, we have

$$Q\barwedge[P, PQ]=[P, Q]+\frac{1}{2}Q\barwedge([P, P]\barwedge Q), \tag{5.27}$$

and from (5.6)

$$Q\barwedge([P, Q]\barwedge P)+2QP\barwedge[P, Q]+\frac{1}{2}Q\barwedge([P, P]\barwedge Q)=0,$$

$$(Q\barwedge[P, Q])\barwedge P+2(1-PQ)\barwedge[P, Q]+\frac{1}{2}Q\barwedge[P, P]\barwedge Q=0,$$

or, using $[P, Q]=(QP)\barwedge[P, Q]$,

$$Q\barwedge[P, P]\barwedge Q=-4[P, Q]. \tag{5.28}$$

From (5.27) and (5.28), we find

$$Q\barwedge[P, PQ]=-[P, Q]. \tag{5.29}$$

From $QP\barwedge[P, Q]=[P, Q]$, (5.15) and (5.26), we have

$$\begin{aligned}[PQ, PQ]&=[QP, QP]\\ &=Q\barwedge[QP, P]+[P, Q]\end{aligned}$$

or using (5.29)

$$[PQ, PQ]=2[P, Q],$$

which proves the theorem.

§ 6. Relations between (F, G, H) and (P, Q).

We first suppose that there are given, in a differentiable manifold, two tensor fields P and Q of type (1,1) satisfying (5.1). Then we can easily verify that tensor fields F, G, H of type (1,1) defined by

$$F=P-Q, \quad G=P+Q, \quad H=FG=PQ-QP \tag{6.1}$$

satisfy (2.1) and consequently define an almost quaternion structure of the second kind.

By a straightforward computation, we have

$$[F, F]=[P, P]-2[P, Q]+[Q, Q], \tag{6.2}$$

$$[G, G]=[P, P]+2[P, Q]+[Q, Q], \tag{6.3}$$

$$[F, G]=[P, P]-[Q, Q], \tag{6.4}$$

$$[F, H]=2[P, PQ]+2[Q, QP], \tag{6.5}$$

$$[G, H]=2[P, PQ]-2[Q, QP], \tag{6.6}$$

$$[H, H]=4[PQ, PQ]. \tag{6.7}$$

From Corollary 2 to Theorem 5.1 and (6.7), we have

THEOREM 6.1. *If* $[Q,Q]=0$, *a necessary and sufficient condition for the horizontal distribution to be integrable is that* $[H,H]=0$.

We now prove

THEOREM 6.2. *A necessary and sufficient condition for the almost quaternion structure* (F,G,H) *of the second kind defined by* (6.1) *to be integrable is that*

$$[P,P]=0, \qquad [P,Q]=0, \qquad [Q,Q]=0,$$

and $R=0$, R *being the curvature tensor appearing in Theorem* 4.1.

Proof. Suppose first that the almost quaternion structure (F,G,H) of the second kind defined by (6.1) is integrable, then we have

$$[F,F]=0, \qquad [F,G]=0, \qquad [G,G]=0$$

and consequently, we have, from (6.2), (6.3) and (6.4),

$$[P,P]=0, \qquad [P,Q]=0, \qquad [Q,Q]=0.$$

Conversely, if these equations are satisfied, then (6.2) and (6.3) show that $[F,F]=0$ and $[G,G]=0$. Thus by Theorem 4.1, the almost quaternion structure (F,G,H) of the second kind is integrable.

Combining Theorems 6.1 and 6.2, we have

COROLLARY. *If the almost quaternion structure* (F, G, H) *of the second kind defined by* (6.1) *is integrable, then the horizontal distribution is integrable.*

We next suppose that there is given, in a differentiable manifold, an almost quaternion structure (F, G, H) of the second kind. Then we can easily see that tensor fields P and Q of type (1,1) defined by

$$P=\frac{1}{2}(F+G), \qquad Q=-\frac{1}{2}(F-G) \tag{6.8}$$

satisfy (5.1). Thus P and Q define two complementary projection operators PQ and QP which are given by

$$PQ=\frac{1}{2}(1-H), \qquad QP=\frac{1}{2}(1+H). \tag{6.9}$$

Using (6.8) and (6.9), we find

$$[P, P]=\frac{1}{4}\{[F, F]+2[F, G]+[G, G]\}, \tag{6.10}$$

$$[P, Q]=-\frac{1}{4}\{[F, F]-[G, G]\}, \tag{6.11}$$

$$[Q, Q]=\frac{1}{4}\{[F, F]-2[F, G]+[G, G]\}, \tag{6.12}$$

$$[P, PQ]=-\frac{1}{4}\{[G, H]+[H, F]\}, \tag{6.13}$$

$$[Q, QP]=\frac{1}{4}\{[G, H]-[H, F]\}, \tag{6.14}$$

$$[PQ, PQ]=\frac{1}{4}[H, H]. \tag{6.15}$$

In the remaining part of this section we assume that $[Q, Q]=0$. We first have

THEOREM 6.3. *Under the assumption* $[Q, Q]=0$, $[F, F]=0$ *implies* $[H, H]=0$.

Proof. From (6.2), $[Q, Q]=0$ and $[F, F]=0$, we have

$$[P, P]=2[P, Q]$$

and consequently, from (6.3) and (6.4), we find

$$[G, G]=2[P, P] \quad \text{and} \quad [F, G]=[P, P]$$

respectively, from which

$$[G, G]=2[F, G]. \tag{6.16}$$

Thus from (2.5) and (6.16), we have, using $[F, F]=0$,

$$[G, G]\barwedge F=-2F\barwedge[G, G]. \tag{6.17}$$

Also, from (2.2) and (6.16), we have, using $[F, F]=0$,

$$[H, F]=\frac{1}{2}F\barwedge[G, G], \tag{6.18}$$

and, from (2.6) and (6.16),

$$[G, H]=-\frac{1}{2}G\barwedge[G, G]-\frac{1}{2}[G, G]\barwedge F,$$

or, using (6.17),

$$[G, H]=-\frac{1}{2}G\barwedge[G, G]+F\barwedge[G, G]. \tag{6.19}$$

Substituting $[F, F]=0$, (6.16), (6.18) and (6.19) into (2.10), we find

$$[H, H]=\frac{1}{2}F\barwedge[G, G]\barwedge G+\frac{1}{2}H\barwedge[G, G]$$
$$-\frac{1}{2}FG\barwedge[G, G]-[G, G],$$

or, using (2.15),

$$[H, H]=-H\barwedge[G, G]-[G, G]. \tag{6.20}$$

Since $H^2=1$, we have, from (6.20),

$$H\barwedge[H, H]=[H, H]. \tag{6.21}$$

But, from $H=PQ-QP$ and $PQ+QP=1$, we have $H=2PQ-1$ and consequently substituting this into (6.21), we find

$$PQ\barwedge[PQ, PQ]=[PQ, PQ]. \tag{6.22}$$

On the other hand, following Theorem 5.1, under the assumption $[Q, Q]=0$, the vector $[PQ, PQ](X, Y)$ is vertical for arbitrary vector fields X and Y and consequently

$$PQ\barwedge[PQ, PQ]=0,$$

thus, from (6.22) we have $[PQ, PQ]=0$, which proves the theorem.

Combining Corollary 2 to Theorem 5.1 and Theorem 6.3, we have

COROLLARY 1. *Under the assumption* $[Q, Q]=0$, *a necessary and sufficient condition for the horizontal distribution to be integrable is that* $[F, F]=0$.

We also have, from Theorems 4.1 and 6.3,

COROLLARY 2. *Under the assumption* $[Q, Q]=0$, *in order for the almost quaternion structure* (F, G, H) *of the second kind defined by* (6.1) *to be integrable, it is necessary and sufficient that* $[F, F]=0$ *and* $R=0$.

Theorem 6.2 and this corollary give

COROLLARY 3. *Under the assumption* $[Q, Q]=0$, $[F, F]=0$ *is equivalent to* $[P, P]=0$, $[P, Q]=0$.

We further prove

THEOREM 6.4. *If* $[P, Q]=0$ *and* $[Q, Q]=0$, *then* $[F, F]=0$.

Proof. From the assumptions, (6.2), (6.3) and (6.4), we have

$$[F, F]=[G, G]=[F, G]. \tag{6.23}$$

Substituting this into (2.2), we find

$$[H, F]=F\barwedge[F, F]+\frac{1}{2}[G, G]\barwedge G,$$

or using (2.15)

$$[H, F]=F\barwedge[F, F]-G\barwedge[G, G].$$

Similarly we have, from (2.6) and (2.14),

$$[G, H]=-G\barwedge[G, G]-\frac{1}{2}[F, F]\barwedge F,$$

this is,

$$[G, H]=F\barwedge[F, F]-G\barwedge[G, G],$$

and consequently we have

$$[H, F]=[G, H]=F\barwedge[F, F]-G\barwedge[G, G]. \tag{6.24}$$

Substituting (6.23) and (6.24) into (2.13), we find

$$\begin{aligned}&-[F, F]-[G, G]+F\barwedge[G, G]\barwedge G-G\barwedge[G, G]\barwedge G\\&+GF\barwedge[F, F]-[G, G]+F\barwedge[F, F]\barwedge F-G\barwedge[F, F]\barwedge F\\&-[F, F]-FG\barwedge[G, G]+2H\barwedge[F, F]=0,\end{aligned}$$

from which, using (2.14) and (2.15),

$$-[F,F]-[G,G]-2H\barwedge[G,G]+2[G,G]$$
$$-H\barwedge[F,F]-[G,G]+2[F,F]-2H\barwedge[F,F]$$
$$-[F,F]-H\barwedge[G,G]+2H\barwedge[F,F]=0,$$

from which

$$H\barwedge[F,F]=0.$$

But, since $H^2=1$, we have from this equation

$$[F,F]=0.$$

Equation (6.2) and this theorem give

COROLLARY 1. *If* $[P,Q]=0$ *and* $[Q,Q]=0$, *then* $[P,P]=0$.

From Theorem 5.3 and this corollary, we have

COROLLARY 2. *Under the assumption* $[Q,Q]=0$, $[P,P]=0$ *is equivalent to* $[P,Q]=0$.

§ 7. Tangent bundles.

Suppose now that the manifold is the tangent bundle $T(M)$ of an n-dimensional differentiable manifold M of class C^∞. It is well known that there exists a tensor field Q of type (1,1) which has components of the form

$$Q=\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}, \tag{7.1}$$

with respect to the so-called induced coordinate system [5, 6, 9, 10], E being the unit matrix and Q satisfies $Q^2=0$.

We first prove

THEOREM 7.1. *If there exists in* $T(M)$ *a tensor field* P *of type* (1,1) *such that*

$$P^2=0, \qquad PQ+QP=1,$$

then P *has components of the form*

$$P=\begin{pmatrix} \Gamma & E \\ -\Gamma^2 & -\Gamma \end{pmatrix} \tag{7.2}$$

with respect to the induced coordinate system, Γ *being an* $n\times n$ *matrix.*

Conversely, tensor fields P and Q in $T(M)$ having components of the form

$$P=\begin{pmatrix} \Gamma & E \\ -\Gamma^2 & -\Gamma \end{pmatrix}, \quad Q=\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} \tag{7.3}$$

with respect to the induced coordinate system respectively satisfy

$$P^2=0, \qquad Q^2=0, \qquad PQ+QP=1.$$

Proof. We put

$$P=\begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{pmatrix},$$

where Γ_1, Γ_2, Γ_3 and Γ_4 are $n\times n$ matrices. Then from $PQ+QP=1$, that is,

$$\begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}\begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{pmatrix}=\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},$$

we have

$$\begin{pmatrix} \Gamma_2 & 0 \\ \Gamma_4+\Gamma_1 & \Gamma_2 \end{pmatrix}=\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix},$$

from which

$$\Gamma_2=E, \qquad \Gamma_4=-\Gamma_1$$

Thus P must be of the form

$$P=\begin{pmatrix} \Gamma & E \\ \Gamma' & -\Gamma \end{pmatrix},$$

where Γ and Γ' are $n\times n$ matrices.

From $P^2=0$, we have

$$\begin{pmatrix} \Gamma^2+\Gamma' & 0 \\ \Gamma'\Gamma-\Gamma\Gamma' & \Gamma^2+\Gamma' \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

from which we have

$$\Gamma^2+\Gamma'=0, \qquad \Gamma'\Gamma-\Gamma\Gamma'=0,$$

that is, $\Gamma'=-\Gamma^2$, and consequently P is of the form

$$P=\begin{pmatrix} \Gamma & E \\ -\Gamma^2 & -\Gamma \end{pmatrix}.$$

The converse is easy to check.

If the manifold M admits a linear connection or a non-linear connection [4], we can construct P given in Theorem 7.1 and P and Q satisfy $P^2=0$, $Q^2=0$, $PQ+QP=1$. Moreover the tensor field Q satisfies $[Q,Q]=0$. This is the reason why we have assumed $[Q,Q]=0$ in §5 and §6. Theorems in §5 and §6 we proved under the assumption $[Q,Q]=0$ are consequently valid in this section too.

It might be interesting to give the integrability condition of the horizontal distribution in terms of the induced coordinate system.

We first note that the complementary projection operators PQ and QP discussed in §5 have components of the form

$$PQ=\begin{pmatrix} E & 0 \\ -\Gamma & 0 \end{pmatrix} \quad \text{and} \quad QP=\begin{pmatrix} 0 & 0 \\ \Gamma & E \end{pmatrix} \tag{7.4}$$

respectively. Let a vector X in $T(M)$ have components

$$X=\begin{pmatrix} X^h \\ X^{\bar{h}} \end{pmatrix}$$

with respect to an induced coordinate system $(x^h, x^{\bar{h}})$, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$ and the indices $\bar{h}, \bar{i}, \bar{j}, \bar{k}, \cdots$ the range $\{n+1, n+2, \cdots, 2n\}$. If X is a horizontal vector, that is, a vector in the horizontal distribution, then we have

$$QPX=\begin{pmatrix} 0 & 0 \\ \Gamma & E \end{pmatrix}\begin{pmatrix} X^h \\ X^{\bar{h}} \end{pmatrix}=0,$$

that is

$$X^{\bar{h}}+\Gamma_i^h X^i=0,$$

Γ_i^h being components of Γ. Thus a horizontal vector X has components of the form

$$X=\begin{pmatrix} X^h \\ -\Gamma_i^h X^i \end{pmatrix}. \tag{7.4}$$

Let a vector Y in $T(M)$ have components

$$Y=\begin{pmatrix} Y^h \\ Y^{\bar{h}} \end{pmatrix}$$

with respect to the induced coordinate system. If Y is a vertical vector, that is, a vector in the vertical distribution, then we have

$$PQY=\begin{pmatrix} E & 0 \\ \Gamma & 0 \end{pmatrix}\begin{pmatrix} Y^h \\ Y^{\bar h} \end{pmatrix}=0,$$

that is, $Y^h=0$. Thus a vertical vector Y has components of the form

$$Y=\begin{pmatrix} 0 \\ Y^{\bar h} \end{pmatrix}. \tag{7.5}$$

We have, for a general vector X,

$$PX=\begin{pmatrix} \Gamma & E \\ -\Gamma^2 & -\Gamma \end{pmatrix}\begin{pmatrix} X^h \\ X^{\bar h} \end{pmatrix}=\begin{pmatrix} X^{\bar h}+\Gamma^h_i X^i \\ -\Gamma^h_k(X^{\bar k}+\Gamma^k_i X^i) \end{pmatrix},$$

which shows that the operator P transforms a vector into a horizontal vector and a horizontal vector into a zero vector, the fact which we stated in §5.

Also, we have, for a general vector Y,

$$QY=\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}\begin{pmatrix} Y^h \\ Y^{\bar h} \end{pmatrix}=\begin{pmatrix} 0 \\ Y^h \end{pmatrix},$$

which shows that the operator Q transforms a vector into a vertical vector and a vertical vector into a zero vector, the fact which we also stated in §5.

Since the horizontal distribution is given by

$$dx^{\bar h}+\Gamma^h_j dx^j=0 \tag{7.5}$$

in terms of the induced coordinate system $(x^h, x^{\bar h})$, the integrability condition of the horizontal distribution is given by [4]

$$R_{kj}{}^h=0, \tag{7.6}$$

where

$$R_{kj}{}^h=\partial_k\Gamma_j{}^h-\partial_j\Gamma^h_k+\Gamma^h_{kt}\Gamma_j{}^t-\Gamma^h_{jt}\Gamma_k{}^t \tag{7.7}$$

and

$$\Gamma^h_{kt}=\partial_{\bar t}\Gamma^h_k, \qquad \partial_k=\partial/\partial x^k, \qquad \partial_{\bar i}=\partial/\partial x^{\bar i}.$$

We have

THEOREM 7.2. *The vector field*

$$[PQ, PQ](X, Y)$$

is vertical for arbitrary vector fields X and Y and

$$[PQ, PQ]=0$$

is equivalent to

$$R_{kj}{}^{h}=0.$$

Proof. The theorem follows from Theorem 5.1, Corollaries 1 and 2, but we shall give here a proof in terms of induced coordinate system.

Putting

$$X=\begin{pmatrix} X^{h} \\ X^{\bar{h}} \end{pmatrix}, \qquad Y=\begin{pmatrix} Y^{h} \\ Y^{\bar{h}} \end{pmatrix}$$

and using $(PQ)^2=PQ$, we have

$$\begin{aligned}
&[PQ, PQ](X, Y) \\
&=[PQX, PQY]-PQ[PQX, Y]-PQ[X, PQY]+PQ[X, Y] \\
&=\left[\begin{pmatrix} X^{h} \\ -\Gamma_{k}{}^{h}X^{k} \end{pmatrix}, \begin{pmatrix} Y^{h} \\ -\Gamma_{j}{}^{h}Y^{j} \end{pmatrix}\right]-\begin{pmatrix} E & 0 \\ -\Gamma & 0 \end{pmatrix}\left[\begin{pmatrix} X^{h} \\ -\Gamma_{k}{}^{h}X^{k} \end{pmatrix}, \begin{pmatrix} Y^{h} \\ Y^{\bar{h}} \end{pmatrix}\right] \\
&\quad -\begin{pmatrix} E & 0 \\ -\Gamma & 0 \end{pmatrix}\left[\begin{pmatrix} X^{h} \\ X^{\bar{h}} \end{pmatrix}, \begin{pmatrix} Y^{h} \\ -\Gamma_{j}{}^{h}Y^{j} \end{pmatrix}\right]+\begin{pmatrix} E & 0 \\ -\Gamma & 0 \end{pmatrix}\left[\begin{pmatrix} X^{h} \\ X^{\bar{h}} \end{pmatrix}, \begin{pmatrix} Y^{h} \\ Y^{\bar{h}} \end{pmatrix}\right],
\end{aligned}$$

from which, by a straightforward computation,

$$[PQ, PQ](X, Y)=\begin{pmatrix} 0 \\ -R_{kj}{}^{h}X^{k}Y^{j} \end{pmatrix},$$

which proves the theorem.

We have proved in §5 (see, Corollary 1 to Theorem 5.2 and Corollary to Theorem 5.3) that $[P, P]=0$, $[Q, Q]=0$, or $[P, Q]=0$, $[Q, Q]=0$ are sufficient conditions for the horizontal distribution to be integrable, but they are not necessary in general. The next theorem gives an explanation of this situation.

THEOREM 7.3. *The vector field* $[P, Q](X, Y)$ *is vertical for arbitrary vector fields* X *and* Y *if and only if*

(7.8) $$\Gamma^{h}_{ji}-\Gamma^{h}_{ij}=0$$

and vanishes if and only if

(7.9) $$\Gamma^{h}_{ji}-\Gamma^{h}_{ij}=0 \quad \textit{and} \quad R_{kj}{}^{h}=0.$$

Proof. Putting

$$X=\begin{pmatrix} X^{h} \\ X^{\bar{h}} \end{pmatrix}, \qquad Y=\begin{pmatrix} Y^{h} \\ Y^{\bar{h}} \end{pmatrix}$$

and using $PQ+QP=1$, we have

$$
\begin{aligned}
[P,Q](X,Y)=&[PX,QY]-P[QX,Y]-Q[X,PY]\\
&+[QX,PY]-Q[PX,Y]-P[X,QY]+[X,Y]\\
=&\left[\begin{pmatrix} X^{\bar h}+\Gamma_k^h X^k \\ -\Gamma_t^h(X^{\bar t}+\Gamma_k^t X^k)\end{pmatrix},\begin{pmatrix}0\\ Y^h\end{pmatrix}\right]-\begin{pmatrix}\Gamma & E\\ -\Gamma^2 & -\Gamma\end{pmatrix}\left[\begin{pmatrix}0\\ X^h\end{pmatrix},\begin{pmatrix}Y^h\\ Y^{\bar h}\end{pmatrix}\right]\\
&-\begin{pmatrix}0 & 0\\ E & 0\end{pmatrix}\left[\begin{pmatrix}X^h\\ X^{\bar h}\end{pmatrix},\begin{pmatrix}Y^{\bar h}+\Gamma_j{}^h Y^j\\ -\Gamma_t^h(Y^{\bar t}+\Gamma_j^t Y^j)\end{pmatrix}\right]\\
&+\left[\begin{pmatrix}0\\ X^h\end{pmatrix},\begin{pmatrix}Y^{\bar h}+\Gamma_j{}^h Y^j\\ -\Gamma_t^h(Y^{\bar t}+\Gamma_j{}^t Y^j)\end{pmatrix}\right]\\
&-\begin{pmatrix}0 & 0\\ E & 0\end{pmatrix}\left[\begin{pmatrix}X^{\bar h}+\Gamma_k{}^h X^k\\ -\Gamma_t{}^h(X^{\bar t}+\Gamma_k{}^t X^k)\end{pmatrix},\begin{pmatrix}Y^h\\ Y^{\bar h}\end{pmatrix}\right]\\
&-\begin{pmatrix}\Gamma & E\\ -\Gamma^2 & -\Gamma\end{pmatrix}\left[\begin{pmatrix}X^h\\ X^{\bar h}\end{pmatrix},\begin{pmatrix}0\\ Y^h\end{pmatrix}\right]+\left[\begin{pmatrix}X^h\\ X^{\bar h}\end{pmatrix},\begin{pmatrix}Y^h\\ Y^{\bar h}\end{pmatrix}\right],
\end{aligned}
$$

from which, by a straightforward computation,

$$
\begin{aligned}
&[P,Q](X,Y)\\
&=\begin{pmatrix} -S_{kj}X^kY^j \\ -R_{kj}{}^hX^kY^j+S_{kj}{}^t\Gamma_t^hX^kY^j+\Gamma_k{}^tS_{tj}{}^hX^kY^j \\ -\Gamma_j^tS_{tk}{}^hX^kY^j+S_{kj}{}^hX^kY^{\bar j}-S_{kj}{}^hX^{\bar k}Y^j\end{pmatrix},
\end{aligned}
\tag{7.10}
$$

where

$$S_{kj}{}^h=\Gamma_{kj}^h-\Gamma_{jk}^h,$$

which proves the theorem.

From Corollary 2 to Theorem 6.4 and Theorem 7.3, we have

THEOREM 7.4. *In the tangent bundle,* $[P,P]=0$ *and* $[P,Q]=0$ *are equivalent and they are equivalent to* (7.9).

Combining Corollary 3 to Theorem 6.3 and Theorem 7.3, we have

THEOREM 7.5. *The almost complex structure F induced in the tangent bundle $T(M)$ of a differentiable manifold M by $F=P-Q$ is integrable if and only if* (7.9) *holds* [1, 4, 6, 9].

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