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KŌDAI MATH. SEM. REP.
25 (1973), 48—62

MANIFOLDS WITH ANTINORMAL (f, g, u, v, λ)-STRUCTURE

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To Professor Shigeru Ishihara on his fiftieth birthday

0. Introduction.

It is now well known that submanifolds of codimension 2 of an almost Hermitian manifold and hypersurfaces of an almost contact metric manifold admit an (f, g, u, v, λ)-structure, that is, a set of a tensor field f of type (1, 1), a Riemannian metric g, two 1-forms u and v and a function λ satisfying

$$(0.1)\qquad \begin{aligned} &f^2X=-X+u(X)U+v(X)V,\\ &g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y),\\ &u(fX)=\lambda v(X),\quad v(fX)=-\lambda u(X),\\ &u(U)=1-\lambda^2,\quad u(V)=0,\quad v(U)=0,\quad v(V)=1-\lambda^2 \end{aligned}$$

for arbitrary vector fields X and Y, U and V being vector fields defined by $u(X)=g(U, X)$ and $v(X)=g(V, X)$ respectively. If the tensor defined by

$$(0.2)\qquad S(X, Y)=N(X, Y)+(du)(X, Y)U+(dv)(X, Y)V,$$

$N(X, Y)$ being the Nijenhuis tensor formed with f, vanishes, the (f, g, u, v, λ)-structure is said to be *normal*.

In the sequel we assume that the dimension of the manifold denoted by M is greater than 2.

Okumura and one of the present authors [8] proved

THEOREM 0.1. *Let M be a complete differentiable manifold with normal (f, g, u, v, λ)-structure satisfying*

$$du=2\omega,\qquad dv=2\phi\omega,$$

ω being a 2-form defined by $\omega(X, Y)=g(fX, Y)$ and ϕ a function on M. If $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then M is isometric to an even-dimensional sphere.

Received December 24, 1971.

The present authors [6] proved

THEOREM 0.2. *Let M be a complete differentiable manifold with normal (f, g, u, v, λ)-structure satisfying*

$$\mathcal{L}_U g = -2c\lambda g \quad or \quad dv = 2c\omega,$$

$\mathcal{L}_U$ denoting the Lie derivation with respect to the vector field U and c a non-zero constant. If $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then M is isometric to an even-dimensional sphere.

Okumura and one of the present authors [9] proved

THEOREM 0.3. *Let a complete differentiable submanifold M of codimension 2 of an even-dimensional Euclidean space E be such that the connection induced in the normal bundle of M is trivial. If the (f, g, u, v, λ)-structure induced on M is normal, $\lambda(1-\lambda^2)$ being almost everywhere non-zero, then M is a sphere, a plane, or a product of a sphere and a plane.*

A typical example of an even-dimensional differentiable manifold with a normal (f, g, u, v, λ)-structure is an even-dimensional sphere S^{2n}.

$S^n \times S^n$ is also a typical example of an even-dimensional differentiable manifold which admits an (f, g, u, v, λ)-structure, but the structure is not normal. Blair, Ludden and one of the present authors [1, 2] proved

THEOREM 0.4. *If M is a complete orientable hypersurface of S^{2n+1} of constant scalar curvature satisfying $fK + Kf = 0$, K being the Weingarten tensor and $\lambda \neq constant$, $\lambda(1-\lambda^2)$ being almost everywhere non-zero, then M is a natural sphere S^{2n} or $S^n \times S^n$.*

The (f, g, u, v, λ)-structure induced on an orientable hypersurface of $S^{2n+1}(1)$ with induced metric tensor g_{ji} and the second fundamental tensor k_{ji} satisfies

$$\nabla_j f_i{}^h = -g_{ji}u^h + \delta_j^h u_i - k_{ji}v^h + k_j{}^h v_i, \tag{0.3}$$

$$\nabla_j u_i = f_{ji} - \lambda k_{ji}, \tag{0.4}$$

$$\nabla_j v_i = -k_{jt}f_i{}^t + \lambda g_{ji}, \tag{0.5}$$

$$\nabla_j \lambda = k_{ji}u^i - v_j, \tag{0.6}$$

where $f_i{}^h$, u_i, v_i and λ are components of f, u, v and λ respectively, ∇_j being the operator of covariant differentiation with respect to g_{ji}. Here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$.

One of the present authors [4] proved

THEOREM 0.5. *Suppose that a complete orientable $2n$-dimensional differentiable manifold M^{2n} is immersed in $S^{2n+1}(1)$ as a hypersurface. If (f, g, u, v, λ)-structure induced on this hypersurface is such that $\lambda \neq const.$ and $\lambda(1-\lambda^2)$ is almost every-*

where non-zero and if it satisfies $\nabla_i\lambda = cv_i$, *c being a non-zero constant, then c must be* -1 *or* -2 *and when* $c=-1$, M^{2n} *is isometric to* $S^{2n}(1)$ *and when* $c=-2$, M^{2n} *is isometric to* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

THEOREM 0.6. *If* M^{2n} *is a complete orientable hypersurface of* $S^{2n+1}(1)$ *satisfying* $f_t{}^h k_i{}^t + k_t{}^h f_i{}^t = 0$ *and* $K(\gamma)=const.$, *where* $f_i{}^h$ *is the tensor field of type* $(1,1)$ *defining the* (f, g, u, v, λ)*-structure induced on* M^{2n}, $\lambda(1-\lambda^2)$ *being almost everywhere non-zero,* k_{ji} *the second fundamental tensor of the hypersurface and* $K(\gamma)$ *the sectional curvature of* M^{2n} *with respect to the section* γ *spanned by* u^h *and* v^h, *then* M^{2n} *is isometric to a natural sphere* $S^{2n}(1)$ *or to* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

THEOREM 0.7. *Assume that a complete 2n-dimensional differentiable manifold* M^{2n} *admits an* (f, g, u, v, λ)*-structure such that* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero, and*

$$\nabla_j u_i - \nabla_i u_j = 2f_{ji}, \qquad \nabla_i\lambda = -v_i$$

or

$$\nabla_j u_i - \nabla_i u_j = 2f_{ji}, \qquad \nabla_i\lambda = -2v_i.$$

At a point at which $\lambda \neq 0$, *we define a tensor field* k_{ji} *of type* $(0,2)$ *by*

$$\nabla_j u_i + \nabla_i u_j = -2\lambda k_{ji}$$

and assume that u_i *satisfies*

$$\nabla_k\nabla_j u_i = -g_{kj}u_i + g_{ki}u_j - k_{kj}v_i + k_{ki}v_j + 2v_k k_{ji}.$$

Then M^{2n} *is isometric to* $S^{2n}(1)$ *if* $\nabla_i\lambda = -v_i$ *and isometric to* $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$ *if* $\nabla_i\lambda = -2v_i$.

We note here that Theorems 0.3～0.6 state properties of (f, g, u, v, λ)-structures induced on submanifolds of codimension 2 of a Euclidean space E^{2n+2} or on hypersurfaces of a sphere $S^{2n+1}(1)$, while Theorems 0.1, 0.2 and 0.7 state intrinsic properties of (f, g, u, v, λ)-structures of manifolds themselves.

In the present paper we first of all show that for an (f, g, u, v, λ)-structure induced on a hypersurface of $S^{2n+1}(1)$ the conditions

$$f_t{}^h k_i{}^t + k_t{}^h f_i{}^t = 0 \tag{0.7}$$

and

$$S_{ji}{}^h = 2v_j(\nabla_i v^h - \lambda\delta_i^h) - 2v_i(\nabla_j v^h - \lambda\delta_j^h) \tag{0.8}$$

are equivalent.

Since the commutativity of f and K and the condition $S=0$ are equivalent for a hypersurface of $S^{2n+1}(1)$ and an (f, g, u, v, λ)-structure satisfying $S=0$ is said to be normal, we say that an (f, g, u, v, λ)-structure satisfying (0.7) or (0.8) is *antinormal*. (See [2], [3], [4], [5]).

We study in the present paper properties of (f, g, u, v, λ)-structures which are antinormal in this sense.

1. A necessary and sufficient condition to be $fK+Kf=0$.

We prove in this section

THEOREM 1.1. *In an orientable hypersurface M with an (f, g, u, v, λ)-structure of $S^{2n+1}(1)$ (or of a Sasakian manifold) such that $\lambda(1-\lambda^2)$ is almost everywhere non-zero, the conditions* (0.7) *and* (0.8) *are equivalent.*

Proof. We know that the (f, g, u, v, λ)-structure induced on an orientable hypersurface of $S^{2n+1}(1)$ or of a Sasakian manifold satisfies (0.3)~(0.6).

We substitute these into

$$S_{ji}{}^h=f_j{}^t\nabla_t f_i{}^h-f_i{}^t\nabla_t f_j{}^h-(\nabla_j f_i{}^t-\nabla_i f_j{}^t)f_t{}^h +(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h \tag{1.1}$$

and find

$$S_{jih}=-v_j(k_{it}f_h{}^t+k_{ht}f_i{}^t)+v_i(k_{jt}f_h{}^t+k_{ht}f_j{}^t)$$

or, using (0.5),

$$S_{jih}=v_j(\nabla_i v_h+\nabla_h v_i-2\lambda g_{ih})-v_i(\nabla_j v_h+\nabla_h v_j-2\lambda g_{jh}), \tag{1.2}$$

where $S_{jih}=S_{ji}{}^t g_{th}$.

Suppose now that (0.7) is satisfied. Then we have

$$k_{jt}f_i{}^t-k_{it}f_j{}^t=0 \tag{1.3}$$

and consequently, we have, from (0.5),

$$\nabla_j v_i-\nabla_i v_j=0.$$

Thus (1.2) gives (0.8).

Conversely suppose that (0.8) is satisfied. Then substituting (0.3)~(0.6) into (0.8), we find

$$v_j(k_{it}f_h{}^t-k_{ht}f_i{}^t)-v_i(k_{jt}f_h{}^t-k_{ht}f_j{}^t)=0. \tag{1.4}$$

Transvecting v^j to (1.4), we find

$$(1-\lambda^2)(k_{it}f_h{}^t-k_{ht}f_i{}^t)=v_i\alpha_h,$$

where

$$\alpha_h=(k_{jt}f_h{}^t-k_{ht}f_j{}^t)v^j,$$

from which

$$v_i\alpha_h+v_h\alpha_i=0$$

and consequently $\alpha_i=0$. Thus we have

$$(1-\lambda^2)(k_{it}f_h{}^t-k_{ht}f_i{}^t)=0,$$

from which

$$k_{it}f_h{}^t-k_{ht}f_i{}^t=0,$$

and we have (0.7). Thus the theorem is proved.

Combining Theorems 0.6 and 1.1, we have

THEOREM 1.2. *If M^{2n} is a complete orientable hypersurface of $S^{2n+1}(1)$ with antinormal (f, g, u, v, λ)-structure and with $K(\gamma)=const.$ $\lambda(1-\lambda^2)$ being almost everywhere non-zero, where $K(\gamma)$ is the sectional curvature with respect to the section γ spanned by u^h and v^h, then M^{2n} is isometric to the unit sphere $S^{2n}(1)$ or to $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.*

2. Lemmas.

The present authors [6] proved following general formulas which an (f, g, u, v, λ)-structure satisfies, that is,

$$\begin{aligned}(2.1)\qquad &S_{jih}-(f_j{}^tf_{tih}-f_i{}^tf_{tjh})\\ &=-(f_j{}^t\nabla_hf_{ti}-f_i{}^t\nabla_hf_{tj})+u_j(\nabla_iu_h)-u_i(\nabla_ju_h)+v_j(\nabla_iv_h)-v_i(\nabla_jv_h)\end{aligned}$$

and

$$\begin{aligned}&\{S_{jih}-(f_j{}^tf_{tih}-f_i{}^tf_{tjh})\}u^j\\ (2.2)\qquad &=(\nabla_iu_h+\nabla_hu_i)-u_i(\nabla_tu_h+\nabla_hu_t)u^t+\lambda f_i{}^t(\nabla_tv_h+\nabla_hv_t)\\ &\quad-\lambda^2(\nabla_iu_h-\nabla_hu_i)-(\lambda f_i{}^t+v_iu^t)(\nabla_tv_h-\nabla_hv_t),\end{aligned}$$

where

$$(2.3)\qquad f_{jih}=\nabla_jf_{ih}+\nabla_if_{hj}+\nabla_hf_{ji}.$$

We now prove a series of lemmas.

LEMMA 2.1. *Assume that a differentiable manifold admits an (f, g, u, v, λ)-structure such that $\lambda(1-\lambda^2)$ is almost everywhere non-zero,*

$$(2.4)\qquad \nabla_ju_i-\nabla_iu_j=2f_{ji}$$

and

(2.5) $$S_{ji}{}^{h}=2v_{j}(\nabla_{i}v^{h}-\lambda\delta_{i}^{h})-2v_{i}(\nabla_{j}v^{h}-\lambda\delta_{j}^{h}).$$

At a point at which $\lambda\neq 0$, *we define a tensor field* k_{ji} *of type* $(0,2)$ *by*

(2.6) $$\nabla_{j}u_{i}+\nabla_{i}u_{j}=-2\lambda k_{ji}.$$

Then we have

(2.7) $$\nabla_{j}u_{i}=f_{ji}-\lambda k_{ji},$$

(2.8) $$\nabla_{j}v_{i}=-k_{jt}f_{i}{}^{t}+\lambda g_{ji}$$

and

(2.9) $$\nabla_{j}\lambda=k_{ji}u^{i}-v_{j}.$$

Proof. Equation (2.7) follows from (2.4) and (2.6). Transvecting u^{i} to (2.7) and using $u_{i}u^{i}=1-\lambda^{2}$, we find

$$-\lambda\nabla_{j}\lambda=\lambda v_{j}-\lambda k_{ji}u^{i},$$

from which (2.9) follows.

Differentiating (2.4) covariantly, we find

$$\nabla_{k}\nabla_{j}u_{i}-\nabla_{k}\nabla_{i}u_{j}=2\nabla_{k}f_{ji},$$

from which

(2.10) $$f_{kji}=\nabla_{k}f_{ji}+\nabla_{j}f_{ik}+\nabla_{i}f_{kj}=0.$$

Thus substituting (2.5), (2.7) and (2.10) into (2.2), we obtain

$$\begin{aligned}&-2v_{i}(\nabla_{j}v_{h}-\lambda g_{jh})u^{j}\\&=-2\lambda k_{ih}+2\lambda u_{i}k_{th}u^{t}+\lambda f_{i}{}^{t}(\nabla_{t}v_{h}+\nabla_{h}v_{t})\\&\quad-2\lambda^{2}f_{ih}-\lambda f_{i}{}^{t}(\nabla_{t}v_{h}-\nabla_{h}v_{t})-v_{i}(\nabla_{t}v_{h}-\nabla_{h}v_{t})u^{t},\end{aligned}$$

from which

$$\begin{aligned}&-2\lambda k_{ih}+2\lambda u_{i}k_{th}u^{t}+2\lambda f_{i}{}^{t}\nabla_{h}v_{t}-2\lambda^{2}f_{ih}\\&+v_{i}(u^{t}\nabla_{t}v_{h})-v_{i}(\nabla_{h}u_{t})v^{t}-2\lambda v_{i}u_{h}=0,\end{aligned}$$

or

(2.11) $$\begin{aligned}&-2\lambda k_{ih}+2\lambda u_{i}k_{th}u^{t}+2\lambda f_{i}{}^{t}\nabla_{h}v_{t}-2\lambda^{2}f_{ih}\\&+v_{i}(u^{t}\nabla_{t}v_{h})+\lambda v_{i}k_{th}v^{t}-\lambda v_{i}u_{h}=0,\end{aligned}$$

by virtue of (2.7).

Transvecting (2.11) with v^{i}, we find

$$-2\lambda k_{th}v^t+2\lambda^2u^t\nabla_h v_t-2\lambda^3u_h$$

$$+(1-\lambda^2)(u^t\nabla_t v_h)+\lambda(1-\lambda^2)k_{th}v^t-\lambda(1-\lambda^2)u_h=0,$$

from which, using $u^t\nabla_h v_t=-v^t(\nabla_h u_t)=-v^t(f_{ht}-\lambda k_{ht})=\lambda u_h+\lambda k_{th}v^t$

$$u^t\nabla_t v_h=\lambda(u_h+k_{th}v^t). \tag{2.12}$$

Substituting (2.12) into (2.11), we find

$$-2\lambda k_{ih}+2\lambda u_i k_{th}u^t+2\lambda f_i{}^t\nabla_h v_t-2\lambda^2 f_{ih}+2\lambda v_i k_{th}v^t=0.$$

or

$$f_i{}^t\nabla_h v_t=k_{ih}+\lambda f_{ih}-u_i k_{th}u^t-v_i k_{th}v^t,$$

from which, transvecting with $f_k{}^i$,

$$(-\delta_k^t+u_k u^t+v_k v^t)\nabla_h v_t$$

$$=k_{ht}f_k{}^t+\lambda(-g_{kh}+u_k u_h+v_k v_h)-\lambda v_k k_{th}u^t+\lambda u_k k_{th}v^t,$$

or, using (2.7) and (2.9),

$$-\nabla_h v_k-u_k(f_{ht}-\lambda k_{ht})v^t-\lambda v_k(k_{ht}u^t-v_h)$$

$$=k_{ht}f_k{}^t-\lambda g_{kh}+\lambda u_k u_h+\lambda v_k v_h-\lambda v_k k_{th}u^t+\lambda u_k k_{th}v^t,$$

from which,

$$\nabla_h v_k=-k_{ht}f_k{}^t+\lambda g_{hk}$$

which proves (2.8).

Substituting (2.8) into (2.12), we find

$$u^t(-k_{ts}f_h{}^s+\lambda g_{th})=\lambda(u_h+k_{th}v^t),$$

or

$$k_{ts}u^t f_h{}^s+\lambda k_{th}v^t=0, \tag{2.13}$$

from which, transvecting v^h,

$$k_{ji}u^j u^i+k_{ji}v^j v^i=0. \tag{2.14}$$

LEMMA 2.2. *Under the same assumptions as those in Lemma* 2.1, *we have*

$$k_{jt}f_i{}^t-k_{it}f_j{}^t=0 \tag{2.15}$$

and

$$\nabla_k f_{ji}=-g_{kj}u_i+g_{ki}u_j-k_{kj}v_i+k_{ki}v_j. \tag{2.16}$$

Proof. Substituting (2.5) and (2.10) into (2.1), we find

$$\begin{aligned} &f_j{}^t\nabla_h f_{ti} - f_i{}^t\nabla_h f_{tj} \\ &= u_j(\nabla_i u_h) - u_i(\nabla_j u_h) - v_j(\nabla_i v_h - 2\lambda g_{ih}) + v_i(\nabla_j v_h - 2\lambda g_{jh}). \end{aligned} \tag{2.17}$$

We compute the first member of (2.17) as follows.

$$\begin{aligned} &f_j{}^t\nabla_h f_{ti} - f_i{}^t\nabla_h f_{tj} \\ &= \nabla_h(f_j{}^t f_{ti}) + 2f_i{}^t\nabla_h f_{jt} \\ &= \nabla_h(-g_{ji} + u_j u_i + v_j v_i) + 2f_i{}^t\nabla_h f_{jt} \\ &= (\nabla_h u_j)u_i + u_j(\nabla_h u_i) + (\nabla_h v_j)v_i + v_j(\nabla_h v_i) + 2f_i{}^t\nabla_h f_{jt}. \end{aligned}$$

Thus (2.17) becomes

$$\begin{aligned} 2f_i{}^t\nabla_h f_{jt} = {} & u_j(\nabla_i u_h - \nabla_h u_i) - u_i(\nabla_j u_h + \nabla_h u_j) \\ & - v_j(\nabla_i v_h + \nabla_h v_i - 2\lambda g_{ih}) + v_i(\nabla_j v_h - \nabla_h v_j - 2\lambda g_{jh}). \end{aligned}$$

Substituting (2.7) and (2.8) into this, we find

$$\begin{aligned} 2f_i{}^t\nabla_h f_{jt} = {} & 2u_j f_{ih} + 2\lambda u_i k_{jh} - v_j(-k_{it}f_h{}^t - k_{ht}f_i{}^t) \\ & + v_i(-k_{jt}f_h{}^t + k_{ht}f_j{}^t - 2\lambda g_{jh}), \end{aligned}$$

from which, transvecting $f_k{}^i$, we obtain

$$\begin{aligned} &2(-\delta_k^t + u_k u^t + v_k v^t)\nabla_h f_{jt} \\ &= 2u_j(-g_{kh} + u_k u_h + v_k v_h) + 2\lambda^2 v_k k_{jh} \\ &\quad + v_j\{k_{ts}f_k{}^t f_h{}^s + k_{ht}(-\delta_k^t + u_k u^t + v_k v^t)\} - \lambda u_k(k_{ht}f_j{}^t - k_{jt}f_h{}^t - 2\lambda g_{jh}). \end{aligned} \tag{2.18}$$

We compute the first member of (2.18) as follows:

$$\begin{aligned} &2(-\delta_k^t + u_k u^t + v_k v^t)\nabla_h f_{jt} \\ &= -2\nabla_h f_{jk} + 2u_k\{\nabla_h(f_{jt}u^t) - f_j{}^t(\nabla_h u_t)\} + 2v_k\{\nabla_h(f_{jt}v^t) - f_j{}^t(\nabla_h v_t)\} \\ &= -2\nabla_h f_{jk} + 2u_k\{(\nabla_h\lambda)v_j + \lambda(\nabla_h v_j) - f_j{}^t(\nabla_h u_t)\} - 2v_k\{(\nabla_h\lambda)u_j + \lambda(\nabla_h u_j) + f_j{}^t(\nabla_h v_t)\}, \end{aligned}$$

or, using (2.7), (2.8) and (2.9),

$$\begin{aligned} &2(-\delta_k^t + u_k u^t + v_k v^t)\nabla_h f_{jt} \\ &= -2\nabla_h f_{jk} + 2u_k\{(k_{ht}u^t - v_h)v_j + \lambda(-k_{ht}f_j{}^t + \lambda g_{hj}) \\ &\quad - (g_{jh} - u_j u_h - v_j v_h - \lambda k_{ht}f_j{}^t)\} \\ &\quad - 2v_k\{(k_{ht}u^t - v_h)u_j + \lambda(f_{hj} - \lambda k_{hj}) \\ &\quad + (k_{hj} - k_{ht}u^t u_j - k_{ht}v^t v_j + \lambda f_{jh})\} \end{aligned}$$

$$=-2\nabla_h f_{jk}+2u_k v_j k_{ht}u^t-2(1-\lambda^2)u_k g_{jh}+2u_k u_j u_h$$
$$+2v_k u_j v_h-2(1-\lambda^2)v_k k_{hj}+2v_k v_j k_{ht}v^t.$$

Thus (2.18) becomes

$$-2\nabla_h f_{jk}+2u_k v_j k_{ht}u^t-2(1-\lambda^2)u_k g_{jh}+2u_k u_j u_h$$
$$+2v_k u_j v_h-2(1-\lambda^2)v_k k_{hj}+2v_k v_j k_{ht}v^t$$
$$=2u_j(-g_{kh}+u_k u_h+v_k v_h)+2\lambda^2 v_k k_{jh}$$
$$+v_j\{k_{ts}f_k{}^t f_h{}^s+k_{ht}(-\delta_k^t+u_k u^t+v_k v^t)\}$$
$$-\lambda u_k(k_{ht}f_j{}^t-k_{jt}f_h{}^t-2\lambda g_{jh}),$$

or

$$2\nabla_h f_{jk}=2u_j g_{kh}-2u_k g_{jh}-2v_k k_{hj}+v_j k_{hk}$$
$$+u_k v_j k_{ht}u^t+v_k v_j k_{ht}v^t \tag{2.19}$$
$$-v_j k_{ts}f_k{}^t f_h{}^s+\lambda u_k(k_{ht}f_j{}^t-k_{jt}f_h{}^t).$$

Taking the skew-symmetric part of (2.19) with respect to h and k and using $\nabla_h f_{jk}-\nabla_k f_{jh}=-\nabla_j f_{kh}$, we find

$$-2\nabla_j f_{kh}=-2(u_k g_{jh}-u_h g_{jk})-2(v_k k_{hj}-v_h k_{kj})+v_j(u_k k_{ht}u^t-u_h k_{kt}u^t$$
$$+v_k k_{ht}v^t-v_h k_{kt}v^t)+\lambda u_k(k_{ht}f_j{}^t-k_{jt}f_h{}^t)-\lambda u_h(k_{kt}f_j{}^t-k_{jt}f_k{}^t). \tag{2.20}$$

Now, transvecting u^j to (2.19) and taking account of (2.13), we find

$$u^t\nabla_j f_{th}=(1-\lambda^2)g_{jh}-u_j u_h-v_h k_{jt}u^t. \tag{2.21}$$

On the other hand, transvecting u^k to (2.20) and taking account of (2.13), we find

$$-2u^t\nabla_j f_{th}=-2(1-\lambda^2)g_{jh}+2u_j u_h+2v_h k_{jt}u^t$$
$$+v_j\{(1-\lambda^2)k_{ht}u^t-u_h k_{ts}u^t u^s-v_h k_{ts}u^t v^s\} \tag{2.22}$$
$$+\lambda(1-\lambda^2)(k_{ht}f_j{}^t-k_{jt}f_h{}^t).$$

Adding twice of (2.21) and (2.22), we find

$$v_j\{(1-\lambda^2)k_{ht}u^t-u_h k_{ts}u^t u^s-v_h k_{ts}u^t v^s\}+\lambda(1-\lambda^2)(k_{ht}f_j{}^t-k_{jt}f_h{}^t)=0, \tag{2.23}$$

from which, taking the symmetric part,

$$v_j\{(1-\lambda^2)k_{ht}u^t-u_h k_{ts}u^t u^s-v_h k_{ts}u^t v^s\}$$
$$+v_h\{(1-\lambda^2)k_{jt}u^t-u_j k_{ts}u^t u^s-v_j k_{ts}u^t v^s\}=0,$$

Transvecting this with v^j, we find

$$(1-\lambda^2)k_{ht}u^t=k_{ts}u^t u^s u_h+k_{ts}u^t v^s v_h. \tag{2.24}$$

Substituting (2.24) into (2.23), we find

$$k_{jt}f_h{}^t-k_{ht}f_j{}^t=0, \tag{2.25}$$

which proves (2.15).

From (2.13), we have

$$(1-\lambda^2)k_{ts}u^t f_h{}^s+\lambda(1-\lambda^2)k_{th}v^t=0.$$

Substituting (2.24) into this equation, we find

$$\lambda k_{ts}u^t u^s v_h-\lambda k_{ts}u^t v^s u_h+\lambda(1-\lambda^2)k_{th}v^t=0,$$

from which,

$$(1-\lambda^2)k_{jt}v^t=k_{ts}u^t v^s u_j-k_{ts}u^t u^s v_j. \tag{2.26}$$

Substituting (2.24), (2.25) and (2.26) into (2.20)$\times(1-\lambda^2)$, we find

$$2(1-\lambda^2)\nabla_j f_{kh}=2(1-\lambda^2)(u_k g_{jh}-u_h g_{jk})+2(1-\lambda^2)(v_k k_{hj}-v_h k_{kj}),$$

which proves (2.16).

LEMMA 2.3. *Under the same assumptions as those in Lemma* 2.1, *we have, at a point at which* $1-\lambda^2\neq 0$,

$$k_t{}^t=0, \tag{2.27}$$

$$k_{jt}u^t=\beta v_j, \tag{2.28}$$

$$k_{jt}v^t=\beta u_j, \tag{2.29}$$

$$\nabla_j\lambda=(\beta-1)v_j, \tag{2.30}$$

where

$$\beta=\frac{1}{1-\lambda^2}k_{ts}u^t v^s.$$

Proof. Differentiating (2.9) covariantly and using (2.7) and (2.8), we find

$$\nabla_k\nabla_j\lambda=(\nabla_k k_{ji})u^i+k_j{}^t(f_{kt}-\lambda k_{kt})+k_{kt}f_j{}^t-\lambda g_{kj},$$

from which,

$$(\nabla_k k_{ji}-\nabla_j k_{ki})u^i=0. \tag{2.31}$$

From (2.24) and (2.26), we have

(2.32) $$k_{jt}u^t=\alpha u_j+\beta v_j$$

and

(2.33) $$k_{jt}v^t=\beta u_j-\alpha v_j$$

respectively, where

$$\alpha=\frac{1}{1-\lambda^2}k_{ts}u^t u^s.$$

Differentiating (2.32) covariantly and using (2.7) and (2.8), we find

$$(\nabla_k k_{jt})u^t+k_j{}^t(f_{kt}-\lambda k_{kt})$$
$$=(\nabla_k\alpha)u_j+\alpha(f_{kj}-\lambda k_{kj})+(\nabla_k\beta)v_j+\beta(-k_{kt}f_j{}^t+\lambda g_{kj}),$$

from which, taking the skew-symmetric part and using (2.31),

(2.34) $$(\nabla_k\alpha)u_j-(\nabla_j\alpha)u_k+(\nabla_k\beta)v_j-(\nabla_j\beta)v_k+2\alpha f_{kj}=0.$$

Transvecting u^j to (2.34), we see that $\nabla_k\alpha$ is written in the form

$$\nabla_k\alpha=au_k+bv_k,$$

and transvecting v^j to (2.34), we see that $\nabla_k\beta$ is written in the form

$$\nabla_k\beta=cu_k+dv_k.$$

Substituting these into (2.34), we have

$$(b-c)(v_k u_j-u_k v_j)+2\alpha f_{kj}=0,$$

from which, we have $\alpha=0$. This proves (2.28) and (2.29).
Transvecting $f_k{}^h$ to (2.25), we find

$$k_{jt}(-\delta_k^t+u_k u^t+v_k v^t)-k_{ts}f_j{}^t f_k{}^s=0,$$

or using (2.28) and (2.29),

$$-k_{jk}+\beta(u_j v_k+u_k v_j)-k_{ts}f_j{}^t f_k{}^s=0,$$

from which, transvecting g^{jk},

$$-k_t{}^t-k_{ts}(g^{ts}-u^t u^s-v^t v^s)=0,$$

that is, $k_t{}^t=0$ and (2.27) is proved.
Finally, from (2.9) and (2.28), we have

$$\nabla_j\lambda=k_{jt}u^t-v_j=(\beta-1)v_j$$

which proves (2.30).

3. Theorems on (f, g, u, v, λ)-structures.

In this section we first prove

THEOREM 3.1. *Suppose that a complete differentiable manifold M admits an (f, g, u, v, λ)-structure such that $\lambda(1-\lambda^2)$ is almost everywhere non-zero,*

$$\nabla_j u_i - \nabla_i u_j = 2f_{ji} \tag{3.1}$$

and

$$S_{ji}{}^h = 2v_j(\nabla_i v^h - \lambda\delta_i^h) - 2v_i(\nabla_j v^h - \lambda\delta_j^h). \tag{3.2}$$

At a point at which $\lambda \neq 0$, we define a tensor field k_{ji} of type $(0, 2)$ by

$$\nabla_j u_i + \nabla_i u_j = -2\lambda k_{ji}. \tag{3.3}$$

If u^h and k_{ji} satisfy

$$u^j \nabla_j u_i = 0 \tag{3.4}$$

and

$$\nabla_k k_{ji} - \nabla_j k_{ki} = 0, \tag{3.5}$$

then the manifold is isometric to $S^n(1/\sqrt{2}) \times S^n(1/\sqrt{2})$.

Proof. Since the assumptions of Lemma 2.1 are satisfied, the conclusions of Lemmas 2.1, 2.2 and 2.3 are all valid.

From (2.7), (2.28) and (3.4), we have

$$0 = u^j \nabla_j u_i = -\lambda v_i - \lambda\beta v_i = -\lambda(1+\beta)v_i,$$

from which $\beta = -1$. Thus, (2.28), (2.29) and (2.30) become respectively

$$k_{jt}u^t = -v_j, \tag{3.6}$$

$$k_{jt}v^t = -u_j, \tag{3.7}$$

$$\nabla_j \lambda = -2v_j. \tag{3.8}$$

Differentiating (3.7) covariantly and substituting (2.7) and (2.8), we find

$$(\nabla_k k_j{}^t)v_t + k_j{}^t(-k_{ks}f_t{}^s + \lambda g_{kt}) = \lambda k_{kj} - f_{kj},$$

from which, taking the skew-symmetric part and using (3.5),

$$k_j{}^t k_k{}^s f_{ts} = f_{kj},$$

or, using (2.15),

$$k_j{}^t k_t{}^s f_{ks} = f_{kj}. \tag{3.9}$$

Transvecting (3.9) with $f_i{}^k$, we find

$$k_j{}^t k_t{}^s(-g_{is}+u_iu_s+v_iv_s)=-g_{ji}+u_ju_i+v_jv_i,$$

or, using (3.6) and (3.7),

$$k_j{}^t k_{ti}=g_{ji}. \tag{3.10}$$

Differentiating (3.10) covariantly, we have

$$(\nabla_k k_j{}^t)k_{ti}+k_j{}^t(\nabla_k k_{ti})=0. \tag{3.11}$$

Since $\nabla_k k_{ji}$ is symmetric in all indices, (3.11) can be written as

$$k_j{}^t(\nabla_i k_{tk})+k_i{}^t(\nabla_j k_{tk})=0, \tag{3.12}$$

which shows that $k_j{}^t(\nabla_k k_{ti})$ is skew-symmetric in j and k.

Now, from (3.11), we have, taking the skew-symmetric part with respect to k and j,

$$k_j{}^t(\nabla_k k_{ti})-k_k{}^t(\nabla_j k_{ti})=0,$$

or

$$k_j{}^t(\nabla_k k_{ti})=0, \tag{3.13}$$

from which, using (3.10),

$$\nabla_k k_{ti}=0. \tag{3.14}$$

On the other hand, differentiating (2.7) covariantly and using (2.15), (3.8) and (3.14), we obtain

$$\nabla_k\nabla_j u_i=-g_{kj}u_i+g_{ki}u_j-k_{kj}v_i+k_{ki}v_j+2v_k k_{ji}. \tag{3.15}$$

Thus the theorem follows from Theorem 0.7.

THEOREM 3.2. *Assume that a complete differentiable manifold M admits an (f, g, u, v, λ)-structure such that $\lambda(1-\lambda^2)$ is almost everywhere non-zero, and* (3.1), (3.2) *hold. At a point at which $\lambda\neq 0$, we define k_{ji} by* (3.3).

If the sectional curvature $K(\gamma)$ with respect to the section γ spanned by u^h and v^h is constant and

$$\nabla_k k_{ji}-\nabla_j k_{ki}=0, \tag{3.16}$$

then the manifold is isometric to a sphere $S^{2n}(1)$ or to $S^n(1/\sqrt{2})\times S^n(1/\sqrt{2})$.

Proof. In this case also, the conclusions of Lemmas 2.1, 2.2 and 2.3 are all valid.

Differentiating (2.7) covariantly and using (2.9), (2.15) and (2.28), we find

(3.17) $$\nabla_k \nabla_j u_i = -g_{kj}u_i + g_{ki}u_j - k_{kj}v_i + k_{ki}v_j + (1-\beta)v_k k_{ji} - \lambda \nabla_k k_{ji},$$

from which, using the Ricci identity,

$$-K_{kji}{}^h u_h = g_{ki}u_j - g_{ji}u_k + k_{ki}v_j - k_{ji}v_k + (1-\beta)(v_k k_{ji} - v_j k_{ki}),$$

$K_{kji}{}^h$ being the curvature tensor and consequently

(3.18) $$K(\gamma) = -\frac{K_{kjih}v^k u^j v^i u^h}{(1-\lambda^2)^2} = 1-\beta^2.$$

Since we have assumed that $K(\gamma)$ is constant, β must be also constant. From (2.29), we have

$$k_j{}^t v_t = \beta u_j.$$

Differentiating this covariantly and using (2.7) and (2.8), we find

$$(\nabla_k k_j{}^t)v_t + k_j{}^t(-k_{ks}f_t{}^s + \lambda g_{kt}) = \beta(f_{kj} - \lambda k_{kj}),$$

from which, taking the skew-symmetric part and using (3.16),

$$k_j{}^t k_{ks} f_t{}^s = -\beta f_{kj},$$

or, using (2.16),

(3.19) $$k_j{}^t k_t{}^s f_{ks} = -\beta f_{kj}.$$

Transvecting u^j to (3.19) and using (2.28) and (2.29), we find

$$\lambda\beta^2 v_k = -\lambda\beta v_k,$$

from which, using β=const.

(3.20) $$\beta=0 \quad \text{or} \quad \beta=-1.$$

Transvecting $f_i{}^k$ to (3.19), we find

$$k_j{}^t k_t{}^s(-g_{is} + u_i u_s + v_i v_s) = -\beta(-g_{ji} + u_j u_i + v_j v_i),$$

or, using (2.28) and (2.29)

$$-k_j{}^t k_{ti} + \beta^2(u_j u_i + v_j v_i) = \beta(g_{ji} - u_j u_i - v_j v_i),$$

that is,

(3.21) $$k_j{}^t k_{ti} = -\beta g_{ji} + \beta(\beta+1)(u_j u_i + v_j v_i).$$

Thus, if $\beta=0$, then $k_{ji}=0$ and in this case we have, from (2.30),

(3.22) $$\nabla_j \lambda = -v_i$$

and (3.17) becomes

(3.23) $$\nabla_k\nabla_j u_i = -g_{kj}u_i + g_{ki}u_j.$$

If $\beta=-1$, then

(3.24) $$k_j{}^t k_{ti} = g_{ji},$$

and in this case we have, from (2.30)

(3.25) $$\nabla_j\lambda = -2v_j.$$

In the proof of Theorem 3.1, we found that (3.16) and (3.24) imply $\nabla_k k_{ji}=0$. Thus (3.17) gives

(3.26) $$\nabla_k\nabla_j u_i = -g_{kj}u_i + g_{ki}u_j - k_{kj}v_i + k_{ki}v_j + 2v_k k_{ji}.$$

Equations (3.22), (3.23), (3.25), (3.26) and Theorem 0.7 prove the theorem.

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