AKO, M.
KÔDAI MATH. SEM. REP.
24 (1972), 482—505

FIBRED SPACES WITH ALMOST COMPLEX STRUCTURES

By Mitsue Ako

Dedicated to Professor Yosio Mutō on his sixtieth birthday

Introduction.

Many papers on the theory of submersion, together with immersions, have been published in recent years (e.g. [1], [4], [8], [9], [17]). A mapping σ from a manifold $\tilde{M}^n$ onto a manifold M^m is called a *submersion* if its differential σ_* is of rank m at any point of $\tilde{M}^n$, where n is larger than m. It seems, generally speaking, that there are two directions of investigating submersions. One is to discuss the existence of a submersion in a given manifold and the other is to study a manifold in which a submersion is assumed to be given a priori. The submersion has also been studied as a fibred space. The concept of a fibred space has been used, since 1922, in unified field theories and in the theory of projective connections.

The purpose of the present paper is to study fibred spaces with a projectable Riemannian metric and a projectable almost complex structure. In §§1 and 2 definitions and lemmas are stated in the most general case for the later use. We discuss in § 3, by use of tensor analysis, the properties of a fibred Riemannian manifold in detail. The structure equations for a fibred space are prepared in § 4. In § 5, we assume that $\tilde{M}$ and fibres are both of even dimensional and we introduce in $\tilde{M}$ an almost complex structure. First we assume that each fibre is an invariant subspace of $\tilde{M}$ and next we treat with more general case. For the case in which the dimension of a fibre is odd, especially 1-dimensional, see [7], where an almost contact structure is introduced in $\tilde{M}$.

§ 1. Preliminaries.

Let $\tilde{M}$ and M be differentiable[1] manifolds of dimension n and m respectively, where n is larger than m. We assume that there is given a differentiable submersion σ from $\tilde{M}$ to M, that is, σ is a differentiable mapping from $\tilde{M}$ onto M whose differential σ_* is of rank m at each point $\tilde{P}$ of $\tilde{M}$. Therefore, the complete inverse image $\mathcal{F}_P$ of $P \in M$ is an $n-m$ dimensional closed submanifold of $\tilde{M}$. We call $\mathcal{F}_P$ a *fibre* over P. Throughout this paper we assume that every fibre is

Received October 13, 1971.

1) Differentiability is always assumed to be of C^∞.

connected.[2] Let $'M$ be the disjoint union of all $\mathcal{F}_P$, then $'M$ is regulary imbedded submanifold of $\tilde{M}$. We call $(\tilde{M}, M, 'M, \sigma)$ a *fibred space* over M. A vector in $\tilde{M}$ is said to be *vertical*, if it is tangent to $'M$. In other words, a vertical vector is a vector which is tangent to $\tilde{M}$ at a point $\tilde{P}$ and belongs to the kernel of σ_* at the point $\tilde{P}$. If a (local) vector field is a (local) field of vertical vectors, then it is called a (local) *vertical vector field*. Since the rank of σ_* is m, there are $n-m$ linearly independent vertical vector fields in a neighborhood of every point of $\tilde{M}$, which will be denoted by C_α.[3] C_α define an $n-m$ dimensional distribution: $\tilde{P} \rightarrow T_{\tilde{P}}{}^V(\tilde{M})$ which is completely integrable and therefore the set of all vertical vector fields of $\tilde{M}$ is a subalgebra of Lie algebra of all vector fields of $\tilde{M}$. A complementary subspace $T_{\tilde{P}}{}^H(\tilde{M})$ of $T_{\tilde{P}}{}^V(\tilde{M})$ in $T_{\tilde{P}}(\tilde{M})$ defines an m dimensional distribution which is called a *horizontal distribution* or a *field of horizontal planes*. We can choose, in a neighborhood of each point $\tilde{P}$ of $\tilde{M}$, m linearly independent vector fields E_a[4] which span the horizontal planes at $\tilde{P}$. We fix, from now on, a field of horizontal planes which can be arbitrarily chosen. Thus n vectors E_a and C_α form a basis of $T_{\tilde{P}}(\tilde{M})$ at each point $\tilde{P}$ of $\tilde{M}$. The inverse of (E_a, C_α) is denoted by $\begin{pmatrix} E^a \\ C^\alpha \end{pmatrix}$. Then any tensor $\tilde{T}$ of type (r, s) in $\tilde{M}$ is expressed as

$$\begin{aligned} T = {} & T_{a_1\cdots a_s}{}^{b_1\cdots b_r} E^{a_1}\otimes\cdots\otimes E^{a_s}\otimes E_{b_1}\otimes\cdots\otimes E_{b_r} + T_{a_1\cdots a_s}{}^{\beta_1\cdots\beta_r} E^{a_1}\otimes\cdots \\ & \otimes E^{a_s}\otimes C_{\beta_1}\otimes\cdots\otimes C_{\beta_r} + T_{\alpha_1\cdots\alpha_s}{}^{b_1\cdots b_r} C^{\alpha_1}\otimes\cdots\otimes C^{\alpha_s}\otimes E_{b_1}\otimes\cdots\otimes E_{b_r} \\ & + T_{\alpha_1\cdots\alpha_s}{}^{\beta_1\cdots\beta_r} C^{\alpha_1}\otimes\cdots\otimes C^{\alpha_s}\otimes C_{\beta_1}\otimes\cdots\otimes C_{\beta_r}. \end{aligned}$$

The first and the last terms in the right hand side are called the *horizontal part* and the *vertical part* of $\tilde{T}$ and denoted by $\tilde{T}^H$ and $\tilde{T}^V$ respectively. The horizontal part and the vertical part of a tensor field in $\tilde{M}$ can be defined in the same way. We denote by $\mathcal{T}_s^r(\tilde{M})$ the space of all tensor fields of type (r, s) in $\tilde{M}$ and put $\mathcal{T}(\tilde{M}) = \sum_{r,s} \mathcal{T}_s^r(\tilde{M})$. Then we have

$$\tilde{X} = \tilde{X}^H + \tilde{X}^V \qquad \text{for} \quad \tilde{X} \in \mathcal{T}_0^1(\tilde{M}) \tag{1.1}$$

and

$$\tilde{\omega} = \tilde{\omega}^H + \tilde{\omega}^V \qquad \text{for} \quad \tilde{\omega} \in \mathcal{T}_1^0(\tilde{M}). \tag{1.2}$$

The facts expressed by equations (1.1) and (1.2) are called the *canonical decomposition* of a vector field $\tilde{X}$ and a 1-form $\tilde{\omega}$ respectively. If we define

$$\tilde{f}^H = \tilde{f}^V = \tilde{f} \qquad \text{for} \quad \tilde{f} \in \mathcal{T}_0^0(\tilde{M}),$$

---

2) This assumption is indispensable for a geometric object which is projectable (See below).

3) Greek indices $\alpha, \beta, \cdots$ run over the range $1, 2, \cdots, n-m$. Strictly speaking, $C_\alpha \in T_0^1('M)$ and $\iota^* C_\alpha \in T_0^1(\tilde{M})$, where ι_* is the differential of the imbedding ι: $'M \rightarrow M$. But we shall omit ι_* as far as there is no fear of confusion.

4) Latin indices $a, b, \cdots, g$ run over the range $1, 2, \cdots, m$, while $h, i, j, \cdots$ over the range $1, 2, \cdots, n$.

we have

$$(\tilde{T}\otimes\tilde{S})^H=\tilde{T}^H\otimes\tilde{S}^H, \qquad (\tilde{T}\otimes\tilde{S})^V=\tilde{T}^V\otimes\tilde{S}^V$$

for any $\tilde{T},\tilde{S}\in\mathcal{T}(\tilde{M})$.

A tensor field $\tilde{T}$ in $\tilde{M}$ is said to be *projectable* if it satisfies

$$(\text{P.1}) \qquad (\mathcal{L}_{\tilde{X}}\tilde{T}^H)^H=0$$

for any vertical vector field $\tilde{X}$, where $\mathcal{L}_{\tilde{X}}$ denotes the Lie derivative with respect to $\tilde{X}$. If $\tilde{T}$ is a local tensor field defined in some neighborhood $\mathcal{U}$ and satisfies (P.1) in $\mathcal{U}$, then $\tilde{T}$ is also said to be *projectable*. It can be shown that the condition (P.1) reduces to $\mathcal{L}_{\tilde{X}}\tilde{T}^H=0$ for $\tilde{T}\in\mathcal{T}^0_s(\tilde{M})$, and to $(\mathcal{L}_{\tilde{X}}\tilde{Y})^H=0$ for $\tilde{Y}\in\mathcal{T}^1_0(\tilde{M})$, because the distribution $\tilde{P}\to T^V_{\tilde{P}}(\tilde{M})$ is completely integrable. The set of all projectable tensor fields is denoted by $\mathcal{P}(\tilde{M})$ and we put $\mathcal{P}^r_s(\tilde{M})=\mathcal{P}(\tilde{M})\cap\mathcal{T}^r_s(\tilde{M})$.

This fact is expressed as follows

LEMMA 1.1. [14] *If $\tilde{X}\in\mathcal{T}^V{}^1_0(\tilde{M})$ and $\tilde{Y}\in\mathcal{P}^1_0(\tilde{M})$, then $[\tilde{X},\tilde{Y}]$ is vertical. Conversely if $[\tilde{X},\tilde{Y}]$ is vertical for any $\tilde{X}\in\mathcal{T}^V{}^1_0(\tilde{M})$ and if every fibre is connected, then $\tilde{Y}\in\mathcal{P}^1_0(\tilde{M})$.*

We need following lemmas which give other expressions of (P.1).

LEMMA 1.2. [6] *$\mathcal{P}^1_0(\tilde{M})$, the set of all projectable vector fields, is subalgebra of $\mathcal{T}^1_0(\tilde{M})$.*

The proof is easily given by means of Jacobi identity and Lemma 1.1.

We shall show in § 4 that $[C_\alpha, E_a]$ is vertical, and thus E_a is projectable.

LEMMA 1.3. [6] *$\tilde{Y}\in\mathcal{P}^1_0(\tilde{M})$ if and only if $\tilde{Y}\tilde{f}\in\mathcal{P}^0_0(\tilde{M})$ for any $\tilde{f}\in\mathcal{P}^0_0(\tilde{M})$.*

LEMMA 1.4. [6] *If $Y\in\mathcal{P}^1_0(\tilde{M})$, then $\sigma_*\tilde{Y}$ is constant along each fibre.*

Lemma 1.4 enables us to define a homomorphism π: $\mathcal{P}^1_0(\tilde{M})\to\mathcal{T}^1_0(M)$ in such a way that π is the restriction of σ_* on $\mathcal{P}^1_0(\tilde{M})$, that is, $Y=\pi\tilde{Y}=\sigma_*\tilde{Y}$ for $\tilde{Y}\in\mathcal{P}^1_0(\tilde{M})$. π is called a *projection*. Clearly, the kernel of π is $\mathcal{T}^V{}^1_0(M)$. Thus $\mathcal{P}^H{}^1_0(\tilde{M})$ is isomorphic to $\mathcal{T}^1_0(M)$. We define an isomorphism

$$L:\ \mathcal{T}^1_0(M)\to\mathcal{P}^H{}^1_0(\tilde{M}),$$

which is the inverse of π restricted to $\mathcal{P}^H{}^1_0(\tilde{M})$, that is, we define $X^L\in\mathcal{P}^H{}^1_0(\tilde{M})$ for $X\in\mathcal{T}^1_0(M)$ in such a way that

$$(1.3) \qquad \pi X^L=X.$$

LEMMA 1.5. *L is naturally extended to an isomorphism from $\mathcal{T}^r_s(M)$ to $\mathcal{P}^H{}^r_s(\tilde{M})$ as follows:*

$$f^L=f\circ\sigma \qquad f\in\mathcal{T}^0_0(M)$$

$$\omega^L={}^*\sigma\omega \qquad \omega\in\mathcal{T}^0_1(M)$$

and

$$(S\otimes T)^L = S^L \otimes T^L \qquad S, T\in \mathcal{T}(M),$$

where $*\sigma$ *is the dual mapping of* σ_*.

Proof. Obviously f^L is constant on each fibre and thus $f^L\in \mathcal{P}^0_0(\tilde{M}) = \mathcal{P}^{H0}_0(\tilde{M})$. From the definitions of X^L and ω^L, we see $\omega^L(X^L)=(\omega(X))^L$, thus we have

$$\mathcal{L}_{\tilde{V}}\omega^L(X^L) = (\mathcal{L}_{\tilde{V}}\omega^L)(X^L) - \omega^L(\mathcal{L}_{\tilde{V}}X^L) = (\mathcal{L}_{\tilde{V}}\omega^L)^H(X^L) = 0$$

for any $\tilde{V}\in \mathcal{T}^{V1}_0(\tilde{M})$ and any $X\in \mathcal{T}^1_0(M)$. This shows $\omega^L\in \mathcal{P}^{H0}_1(\tilde{M})$. Since any tensor given by tensor product of projectable tensors are also projectable, the lemma is proved.

From now on, simplifying the notation, we use σ in place of σ_* and $*\sigma$.

The projection π defined by (1.3) is also extendable to a homomorphism: $\mathcal{P}^r_s(\tilde{M})\to \mathcal{T}^r_s(M)$ which we call again *projection* and denote by the same letter π. The definition of π is as follows: If $\tilde{f}\in \mathcal{P}^0_0(\tilde{M})$, $\tilde{f}$ is constant on each fibre and thus there exists a function f in M such that $\tilde{f} = f\sigma$. We define

$$\pi\tilde{f} = f.$$

For $\tilde{\omega}\in \mathcal{P}^0_1(\tilde{M})$ we define $\pi\tilde{\omega}$ by the following equation:

$$(\pi\tilde{\omega})(X) = \tilde{\omega}(X^L),$$

where X is an arbitrary element of $\mathcal{T}^1_0(M)$. We define π inductively by

$$\pi(\tilde{S}\otimes\tilde{T}) = (\pi\tilde{S})\otimes(\pi\tilde{T})$$

for $\tilde{S}, \tilde{T}\in \mathcal{P}(\tilde{M})$.

Next we consider the case in which $\tilde{M}$ admits an affine connection $\tilde{\nabla}$. If the vector field $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ is projectable for any $\tilde{X}, \tilde{Y}\in \mathcal{P}^{H1}_0(\tilde{M})$, then $\tilde{\nabla}$ is said to be *projectable*. We define ∇ by

$$\nabla_X Y = \pi(\tilde{\nabla}_{X^L} Y^L) \tag{1.4}$$

for arbitrary X and Y of $\mathcal{T}^1_0(M)$.

LEMMA 1.6. *The* ∇ *defined by* (1.4) *is an affine connection in M. If* $\tilde{\nabla}$ *is torsionless, so is* ∇.

We call ∇ the *induced connection* and denote it by $\pi(\tilde{\nabla})$.

Proof. It is obvious that $\nabla_X Y\in \mathcal{T}^1_0(M)$. First we show

1) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$

and

2) $\nabla_{fX+gY}X = f\nabla_X Z + g\nabla_Y Z$

for any f and g of $\mathcal{T}_0^0(M)$ and Z of $\mathcal{T}_0^1(M)$. Since $\tilde{\nabla}$ is an affine connection, we see

$$\tilde{\nabla}_{X^L}(f^L Y^L)=f^L\tilde{\nabla}_{X^L}Y^L+(X^L f^L)Y^L$$

and

$$\tilde{\nabla}_{f^L X^L+g^L Y^L}Z^L=f^L\tilde{\nabla}_{X^L}Z^L+g^L\tilde{\nabla}_{Y^L}Z^L.$$

Thus we have

$$\nabla_X(fY)=\pi(\tilde{\nabla}_{X^L}(fY)^L)=\pi(\tilde{\nabla}_{X^L}(f^L Y^L))=f\nabla_X Y+(Xf)Y$$

and

$$\begin{aligned}\nabla_{fX+gY}Z&=\pi(\tilde{\nabla}_{(fX+gY)^L}Z^L)=\pi(f^L\tilde{\nabla}_{X^L}Z^L+g^L\tilde{\nabla}_{Y^L}Z^L)\\&=f\nabla_X Z+g\nabla_Y Z,\end{aligned}$$

which prove that ∇ is an affine connection in M. A similar computation shows

$$T(X, Y)=\pi\tilde{T}(X^L, Y^L),$$

where T and $\tilde{T}$ are torsion tensors of ∇ and $\tilde{\nabla}$ respectively.

§ 2. Projectable Riemannian metric.

We assume, in this section and in the following, that there is given a projectable Riemannian metric $\tilde{g}$.[5] By the definition, $\tilde{g}$ satisfies

$$\mathcal{L}_{\tilde{V}}\tilde{g}^H=0$$

for any vertical vector field $\tilde{V}$. This means

$$\|\tilde{X}\|_{\tilde{g}}=\|\tilde{Y}\|_{\tilde{g}}$$

for any two horizontal and at the same time projectable vector fields $\tilde{X}$ and $\tilde{Y}$ along a fibre with the same projection, where $\|\tilde{x}\|_{\tilde{g}}$ is the length of $\tilde{X}$ with respect to $\tilde{g}$. Without loss of generality, we can assume that $\tilde{g}$ satisfies conditions

(2.1) $$\tilde{g}(C_\alpha, E_a)=0.$$

Lie derivatives of $\tilde{g}$ with respect to a vertical vector field $\tilde{V}$ are given as follows:

(2.2) $$(\mathcal{L}_{\tilde{V}}\tilde{g})^H=0, \qquad (\mathcal{L}_{V}\tilde{g})^V\iota=\mathcal{L}_{\tilde{V}}{}'g,$$

where $'g$ is the metric tensor induced on $'M$ from $\tilde{g}$ and ι is the injection of $'M$ into $\tilde{M}$, and

5) Reinhart called such a metric a *bundle-like metric* [10].

$$(\mathcal{L}_{\tilde{V}}\tilde{g})(\tilde{Y}^H, \tilde{X}^V) = -\tilde{g}([\tilde{V}, \tilde{Y}^H], \tilde{X}^V). \tag{2.3}$$

On the other hand, $\tilde{g}$ induces a Riemannian metric g in M by means of the projection π, since $\tilde{g}$ is assumed to be projectable. Thus we have

$$g(X, Y)^L = \tilde{g}(X^L, Y^L), \qquad X, Y \in \mathcal{T}^1_0(M), \tag{2.4}$$

or

$$g(\pi\tilde{X}, \pi\tilde{Y}) = \pi(\tilde{g}(\tilde{X}, \tilde{Y})), \qquad \tilde{X}, \tilde{Y} \in \mathcal{P}^{H1}_0(\tilde{M}). \tag{2.4$'$}$$

PROPOSITION 2.1. *Let $\tilde{\nabla}$ be the Riemannian connection with respect to $\tilde{g}$, that is,*

$$\tilde{\nabla}_{\tilde{X}}\tilde{g} = 0 \qquad \text{and} \qquad 2\tilde{T}(\tilde{X}, \tilde{Y}) = \tilde{\nabla}_{\tilde{X}}\tilde{Y} - \tilde{\nabla}_{Y}\tilde{X} - [\tilde{X}, \tilde{Y}] = 0.$$

Then $\tilde{\nabla}$ is projectable (see § 3) *and the induced connection $\nabla = \pi(\tilde{\nabla})$ is also the Riemannian connection with respect to the induced metric $g = \pi\tilde{g}$.*

Proof. We take X, Y and $Z \in \mathcal{T}^1_0(M)$ arbitrarily. We have, from the definition,

$$X(g(Y, Z)) = \pi(X^L(g(Y, Z))^L).$$

On the other hand

$$X^L(g(Y, Z))^L = X^L(\tilde{g}(Y^L, Z^L)) = (\tilde{\nabla}_{X^L}\tilde{g})(Y^L, Z^L) + \tilde{g}(\tilde{\nabla}_{X^L}Y^L, Z^L) + \tilde{g}(Y^L, \tilde{\nabla}_{X^L}Z^L)$$
$$= \tilde{g}(\tilde{\nabla}_{X^L}Y^L, Z^L) + \tilde{g}(Y^L, \tilde{\nabla}_{X^L}Z^L),$$

since $\tilde{\nabla}$ is the Riemannian connection. From this equation we have

$$\pi(X^L(g(Y, Z))^L) = g(\pi(\tilde{\nabla}_{X^L}Y^L), Z) + g(Y, \pi(\tilde{\nabla}_{X^L}Z^L))$$
$$= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y, Z) - (\nabla_X g)(Y, Z).$$

Thus we find

$$\nabla_X g = 0.$$

The proof of the latter part of the proposition is given in Lemma 1.6. q.e.d.

Let $\tilde{\nabla}$ be the Riemannian connection in $\tilde{M}$. If we put, for $X, Y \in \mathcal{T}^1_0(M)$,

$$\tilde{\nabla}_{X^L}Y^L - (\nabla_X Y)^L = H_1(X, Y) \tag{2.5}$$

or

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} - (\nabla_{\pi\tilde{X}}\pi\tilde{Y})^L = H_1(\pi\tilde{X}, \pi\tilde{Y}) \qquad \text{for} \quad \tilde{X}, \tilde{Y} \in \mathcal{P}^{H1}_0(\tilde{M}), \tag{2.5$'$}$$

then $H_1(X, Y)$ is a vector field in $\tilde{M}$. By a straightforward computation we have

$$\tilde{g}(\tilde{\nabla}_{X^L}Y^L, Z^L) = g(\nabla_X Y, Z)\circ\sigma$$

for vector fields X, Y and Z in M. This shows

$$\tilde{g}(\tilde{\nabla}_{X^L}Y^L-(\nabla_X Y)^L, Z^L)=0,$$

from which we have

$$\tilde{g}(H_1(X, Y), Z^L)=0$$

and the same, Z^L being replaced by $\tilde{Z}^H$, holds too. Thus $H_1(X, Y)$ is a vertical vector field in M. On the other hand $(\nabla_X Y)^L$ is horizontal, and consequently

$$\tilde{\nabla}_{X^L}Y^L=(\nabla_X Y)^L+H_1(X, Y)$$

is the canonical decomposition of the vector field $\tilde{\nabla}_{X^L}Y^L$. $H_1(X, Y)$ defines a tensor field $\tilde{H}$ of type (1, 2) in $\tilde{M}$ in the following way:

$$\begin{aligned}
&\tilde{H}(\tilde{X}, \tilde{Y})=H_1(\pi\tilde{X}, \pi\tilde{Y}), && \tilde{X}, \tilde{Y}\in \mathcal{P}^H{}^1_0(\tilde{M}),\\
(2.6)\qquad &\tilde{H}(\tilde{X}^H, \tilde{Y}^H)=(\tilde{\nabla}_{\tilde{X}^H}\tilde{Y}^H)^V, && \tilde{X}, \tilde{Y}\in \mathcal{T}^1_0(\tilde{M}),\\
&\tilde{H}(\tilde{X}, \tilde{Y}^V)=\tilde{H}(\tilde{X}^V, \tilde{Y})=0, && \tilde{X}, \tilde{Y}\in \mathcal{T}^1_0(\tilde{M}).
\end{aligned}$$

We must show that $(\tilde{\nabla}_{\tilde{X}^H}\tilde{Y}^H)^V$ defines a tensor field in $\tilde{M}$. For that it is sufficient to show that $(\nabla_{\tilde{X}^H}\tilde{Y}^H)^V$ is bilinear with respect to $\tilde{X}$ and $\tilde{Y}$. For $\tilde{\rho}, \tilde{\tau}\in \mathcal{T}^0_0(\tilde{M})$ we see

$$\tilde{\nabla}_{\rho\tilde{X}^H}(\tilde{\tau}\tilde{Y}^H)=\tilde{\rho}(\tilde{X}^H\tilde{\tau})\tilde{Y}^H+\tilde{\rho}\tilde{\tau}\tilde{\nabla}_{\tilde{X}^H}\tilde{Y}^H.$$

Thus we have

$$(\tilde{\nabla}_{\rho\tilde{X}^H}(\tilde{\tau}\tilde{Y}^H))^V=\tilde{\rho}\tilde{\tau}(\tilde{\nabla}_{\tilde{X}^H}\tilde{Y}^H)^V,$$

which shows that $(\nabla_{\tilde{X}^H}\tilde{Y}^H)^V$ is bilinear with respect to $\tilde{X}$ and $\tilde{Y}$.

Thus we can define

$$\tilde{H}(\tilde{X}, \tilde{Y})=(\tilde{\nabla}_{\tilde{X}^H}\tilde{Y}^H)^V \qquad \tilde{X}, \tilde{Y}\in \mathcal{T}^1_0(\tilde{M}).$$

The following propositions are well known:

PROPOSITION 2.2. *$\tilde{H}(\tilde{X}, \tilde{Y})$ is skew-symmetric.*

Proof. It is sufficient to show that $\tilde{H}(\tilde{X}, \tilde{Y})$ is skew-symmetric for $\tilde{X}, \tilde{Y}\in \mathcal{P}^H{}^1_0(\tilde{M})$, because, as we noted above, there are m linearly independent vector fields which belong to $\mathcal{P}^H{}^1_0(\tilde{M})$ and any horizontal vector field is a linear combination of these m vector fields. We have, for any $\tilde{X}\in \mathcal{P}^H{}^1_0(\tilde{M})$ and any $\tilde{V}\in \mathcal{T}^V{}^1_0(\tilde{M})$,

$$\begin{aligned}
0=\tilde{V}\tilde{g}(\tilde{X}, \tilde{X})&=2\tilde{g}(\tilde{\nabla}_{\tilde{V}}\tilde{X}, \tilde{X})=2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{V}+[\tilde{V}, \tilde{X}], \tilde{X})\\
&=2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{V}, \tilde{X})=-2\tilde{g}(\tilde{V}, \tilde{\nabla}_{\tilde{X}}\tilde{X})=-2\tilde{g}(\tilde{V}, \tilde{H}(\tilde{X}, \tilde{X})).
\end{aligned}$$

Thus we have

$$\tilde{H}(\tilde{X}, \tilde{X})=0. \qquad \text{q.e.d.}$$

From the definition of $\tilde{H}(\tilde{X}, \tilde{Y})$ and Proposition 2.2, we get

$$[\tilde{X}, \tilde{Y}]^V=2\tilde{H}(\tilde{X}, \tilde{Y}) \qquad \text{for} \quad \tilde{X}, \tilde{Y}\in \mathcal{T}^{H}{}^1_0(\tilde{M}).$$

Thus we have

PROPOSITION 2.3. *$\tilde{H}(\tilde{X}, \tilde{Y})=0$ if and only if $\mathcal{T}^{H}{}^1_0(\tilde{M})$ is a Lie subalgebra of $\mathcal{T}^1_0(\tilde{M})$. Thus the integral submanifold of the horizontal distribution is totally geodesic.*

In the case of Proposition 2.3, that is, when the horizontal distribution is integrable, $\tilde{M}$ is said to be *locally trivial.*

We fix, for a while, $\tilde{X}\in \mathcal{P}^{H}{}^1_0(\tilde{M})$ and $\tilde{V}\in \mathcal{T}^{V}{}^1_0(\tilde{M})$ and let

$$\tilde{\nabla}_V\tilde{X}=-H_2(\tilde{V}, \tilde{X})-L_1(\tilde{V}, \tilde{X}) \tag{2.7}$$

be the canonical decomposition. Then we see that

$$\tilde{\nabla}_{\tilde{X}}V=-H_2(\tilde{V}, \tilde{X})-(L_1(\tilde{V}, \tilde{X})+[\tilde{V}, \tilde{X}]) \tag{2.8}$$

is the canonical decomposition, since $[\tilde{V}, \tilde{X}]$ is vertical. Further, if we take $\tilde{U}\in \mathcal{T}^{V}{}^1_0(\tilde{M})$, we have the canonical decomposition

$$\tilde{\nabla}_U\tilde{V}=L_2(\tilde{U}, \tilde{V})+\iota_*('\nabla_{\tilde{U}}\tilde{V}), \tag{2.9}$$

where $'\nabla$ is the induced connection on $'M$ from $\tilde{\nabla}$ and ι_* is the differential of ι: $'M\to\tilde{M}$. $L_2(\tilde{U}, \tilde{V})$ is symmetric, because $\mathcal{T}^{V}{}^1_0(\tilde{M})$ is a subalgebra of $\mathcal{T}^1_0(\tilde{M})$. We have, by a direct computation,

$$\tilde{g}(\tilde{H}(\tilde{X}, \tilde{Y}), \tilde{V})=\tilde{g}(H_2(\tilde{V}, \tilde{X}), \tilde{Y}), \qquad \tilde{Y}\in \mathcal{T}^1_0(\tilde{M})$$

and

$$\tilde{g}(L_1(\tilde{V}, \tilde{X}), \tilde{U})=\tilde{g}(\tilde{X}, L_2(\tilde{V}, \tilde{U})).$$

The four formulas (2.6)$\sim$(2.9) correspond to the equations of Gauss and Weingarten for a submanifold and are called the equations of *Co-Gauss* and *Co-Weingarten.*

To conclude this section, we consider the Lie derivative of $\tilde{g}$ with respect to a horizontal vector field. The following formulas, especially (2.12), will be useful to discuss isometric fibres (cf. Mutō [5]).

$$(\mathcal{L}_{X^L}\tilde{g})(Y^L, Z^L)=(\mathcal{L}_X g)(Y, Z)\circ\sigma, \tag{2.10}$$

$$(\mathcal{L}_{X^L}\tilde{g})(Y^L, \tilde{Z}^V)=-2\tilde{g}(H_1(X, Y), \tilde{Z}^V), \qquad \tilde{Z}\in \mathcal{T}^1_0(\tilde{M}). \tag{2.11}$$

and

$$(\mathcal{L}_{X^L}\tilde{g})(\tilde{Y}^V, \tilde{Z}^V)=-2\tilde{g}(\tilde{X}, L_2(\tilde{Y}^V, \tilde{Z}^V)). \tag{2.12}$$

Equations (2.10)~(2.12) will be used, in Corollary 3.1, to obtain a condition that E_a's are Killing vector fields in $\tilde{M}$.

§ 3. Expressions in terms of local coordinate system.

As a continuation of the preceding section, we discuss in this chapter the fibred space with projectable Riemannian metric in detail by means of a local coordinate system. Let $(\tilde{x}^i)$ and (x^a) be local coordinate systems of $\tilde{M}$ and M respectively. Then the submersion σ: $\tilde{M}\rightarrow M$ is represented by equations $x^a=x^a(\tilde{x}^i)$ whose Jacobian matrix $\partial x^a/\partial\tilde{x}^i$ is of rank m at any point of $\tilde{M}$. The vertical vector fields C_α with components $C^i{}_\alpha$ satisfy $\partial x^a/\partial\tilde{x}^i\ C^i{}_\alpha=0$.[6] On the other hand, if we represent by $('x^\alpha)$ a local coordinate system of $\mathcal{F}_p$, then we have $\partial x^a/\partial' x^\alpha=0$, since x^a=const. on each fibre. Thus we can choose C_α as vectors with components $C^i{}_\alpha=\partial\tilde{x}^i/\partial' x^\alpha$. We may put $E_i{}^a=\partial x^a/\partial\tilde{x}^i$, because, for a fixed a, the transformation law of $\partial x^a/\partial\tilde{x}^i$ under the change of a local coordinate system is just the same as that of a covariant vector in $\tilde{M}$. We denote by $E^i{}_a$ the components of E_a and by $C_i{}^\beta$ those of C^β, then we have

$$E^i{}_aE_i{}^b=\delta^b_a,\qquad E^i{}_aC_i{}^\beta=0,\qquad E_i{}^bC^i{}_\alpha=0\quad\text{and}\quad C_i{}^\beta C^i{}_\alpha=\delta^\beta_\alpha.$$

(E_a, C_α) is a so-called non-holonomic frame. Since we can identify $\tilde{x}^a$ with x^a, we may choose a local coordinate system $(\tilde{x}^i)$ in $\tilde{M}$ in such a way that each fibre is expressed by equations x^a=const.

The horizontal distribution is defined by Pfaffian equations

$$\omega^\alpha=C_i{}^\alpha dx^i=0$$

which can also be written as

$$\Pi_a{}^\alpha dx^a+d'x^\alpha=0$$

in the natural frame. Thus, the non-holonomic frame has the following components with respect to the natural frame.

$$C_\beta=\begin{pmatrix}0\\ \delta^\alpha_\beta\end{pmatrix},\qquad E_b=\begin{pmatrix}\delta^a_b\\ -\Pi^\alpha_b\end{pmatrix},$$

$$C^\alpha=(\Pi^\alpha_a, \delta^\alpha_\beta),\qquad E^a=(\delta^a_b, 0).$$

We remark that we can choose $E^i{}_a$ and $C_i{}^\beta$ in such a way that $E^i{}_a=A_{ba}\tilde{g}^{ji}E_j{}^b$ and $C_i{}^\beta=B^{\beta\alpha}\tilde{g}_{ji}C^j{}_\alpha$. Thus we have $A_{ba}=\tilde{g}_{ji}E^j{}_bE^i{}_a$ and $B^{\beta\alpha}=\tilde{g}^{ji}C_j{}^\beta C_i{}^\alpha$. On the other hand $\tilde{g}^H{}_{ji}=A_{ba}E_j{}^bE_i{}^a$ and, by the assumption $(\mathcal{L}_{C_\alpha}\tilde{g}^H)^H=0$, $\mathcal{L}_{C_\alpha}A_{ba}$ must be zero. This means that A_{ba} are projectable functions. Thus there exists a Riemannian metric g in M such that $g_{ba}=A_{ba}\circ\sigma$[7], where g_{ba} are components of g. We denote

6) We shall use, in the sequel, the summation convention.

7) In the sequel, we identify g_{ba} (resp. $'g^{\beta\alpha}$) with A_{ba} (resp. $B^{\beta\alpha}$).

by g^{ba} the inverse of g_{ba}, i.e. $g^{ba}g_{ac}=\delta^b_c$. Since $\mathcal{F}_P$ is a submanifold of $\tilde{M}$ for each $P\in M$, $\tilde{g}_{ji}C^j{}_\beta C^i{}_\alpha$ is regarded as the induced metric in $'M$ which is denoted by $'g$. Let $'g^{\beta\gamma}$ be the inverse of $'g_{\alpha\beta}$, i.e. $'g_{\alpha\beta}{}'g^{\beta\gamma}=\delta^\gamma_\alpha$, then $B^{\beta\alpha}={}'g^{\beta\alpha}\circ\iota$.

By a straightforward computation, we have

$$\mathcal{L}_{C_\alpha}E^a=0,\quad \mathcal{L}_{C_\alpha}C_\beta=0,\quad \mathcal{L}_{C_\alpha}E_a=-\Pi_a{}^\beta{}_\alpha C_\beta \quad\text{and}\quad \mathcal{L}_{C_\alpha}C^\beta=\Pi_a{}^\beta{}_\alpha E^a, \tag{3.1}$$

where we have put

$$\Pi_a{}^\beta{}_\alpha=\partial_\alpha \Pi_a{}^\beta, \tag{3.2}$$

since $(\partial\tilde{x}^i/\partial' x^\alpha)$ are chosen as components of C_α. We also have

$$\mathcal{L}_{C_\alpha}\tilde{g}_{ji}=\Pi_{b\beta\alpha}(E_j{}^bC_i{}^\beta+C_j{}^\beta E_i{}^b)+(\mathcal{L}_{C_\alpha}{}'g_{\gamma\beta})C_j{}^\gamma C_i{}^\beta, \tag{3.3}$$

where $\Pi_{b\beta\alpha}=\Pi_b{}^\gamma{}_\alpha{}'g_{\gamma\beta}$.

Equations (3.1) and (3.2) give

PROPOSITION 3.1. *E_a commute with C_α if and only if the functions $\Pi_a{}^\beta$ are constant with respect to C_α for all β. Furthermore, E_a commute with each C_α if and only if $\Pi_a{}^\beta$ are projectable functions.*

(3.3) shows that the question whether C_α is a Killing vector field in $\tilde{M}$ is equivalent to the question whether it is a Killing vector field on $\mathcal{F}_P$ when C_α commutes with any E_b. Thus we have

PROPOSITION 3.2. *In order that C_α is a Killing vector field in $\tilde{M}$, it is necessary and sufficient that* 1) *C_α commute with any E_b and* 2) *C_α is a Killing vector field on $\mathcal{F}_P$.*

We call $\tilde{g}$ an *invariant metric*, if $\mathcal{L}_{C_\alpha}\tilde{g}_{ji}$ is vertical. Thus we have

PROPOSITION 3.3. *$\tilde{M}$ has an invariant Riemannian metric if and only if C_α commute with any E_b $(\alpha=1,\cdots,n-m)$.*

Now we give formulas for the covariant differentiation with respect to Riemannian connection in M. From (2.5)~(2.9) we may put

$$\begin{aligned}
\tilde{\nabla}_iE^j{}_b&=\{a{}^c{}b\}E_i{}^aE^j{}_c-h_b{}^c{}_\alpha E^j{}_cC_i{}^\alpha+h_{ab}{}^\beta E_i{}^aC^j{}_\beta-l_\alpha{}^\beta{}_bC^j{}_\beta C_i{}^\alpha,\\
\tilde{\nabla}_iE_j{}^c&=-\{a{}^c{}b\}E_i{}^aE_j{}^b+h_b{}^c{}_\alpha(E_j{}^bC_i{}^\alpha+E_i{}^bC_j{}^\alpha)-l_{\beta\alpha}{}^cC_j{}^\beta C_i{}^\alpha,\\
\tilde{\nabla}_iC^j{}_\beta&=-h_a{}^b{}_\beta E^j{}_bE_i{}^a-(l_\beta{}^\alpha{}_a-\Pi_a{}^\alpha{}_\beta)E_i{}^aC^j{}_\alpha+l_{\alpha\beta}{}^bE^j{}_bC_i{}^\alpha+{}'\{\gamma{}^\alpha{}\beta\}C_i{}^\gamma C^j{}_\alpha,\\
\tilde{\nabla}_iC_j{}^\beta&=-h_{ab}{}^\beta E_i{}^aE_j{}^b+(l_\alpha{}^\beta{}_a-\Pi_a{}^\beta{}_\alpha)E_i{}^aC_j{}^\alpha+l_\alpha{}^\beta{}_bE_j{}^bC_i{}^\alpha-{}'\{\gamma{}^\beta{}\alpha\}C_i{}^\gamma C_j{}^\alpha,
\end{aligned} \tag{3.4}$$

where

$$h_{ab}{}^\alpha=H_{ji}{}^mE^j{}_aE^i{}_bC_m{}^\alpha=H_{ji}{}^mC^j{}_\gamma E^i{}_bE_m{}^c{}'g^{\gamma\alpha}g_{ca}, \tag{3.5}$$

$$l_\alpha{}^b{}_\beta=L_{2ji}{}^mC^j{}_\alpha C^i{}_\beta E_m{}^b=L_{2ji}{}^mC^j{}_\alpha E^i{}_aC_m{}^\gamma g^{ab}{}'g_{\gamma\beta} \tag{3.6}$$

and $'\{_\gamma{}^\beta{}_\alpha\}$ are coefficients of the induced Riemannian connection on $\mathcal{F}_P$. The coefficients $\{_b{}^c{}_a\}$ are given by

$$\{_a{}^c{}_b\} = -E^i{}_a E^j{}_b \tilde{\nabla}_i E_j{}^c = E_j{}^c E^i{}_a \tilde{\nabla}_i E^j{}_b \tag{3.7}$$

and symmetric with respect to b and a, since $E_j{}^c = \partial x^c / \partial \tilde{x}^j$ and the Riemannian connection is symmetric.

Now we prove that the Riemannian connection with respect to $\tilde{g}_{ji}$ is projectable. The definition of $\tilde{\nabla}$ being projectable is given in § 1 by the equation

$$(\mathcal{L}_{\tilde{V}} \tilde{\nabla}_{\tilde{X}} \tilde{Y})^H = 0$$

for any $\tilde{V} \in \mathcal{T}^V{}^1_0(\tilde{M})$ and any $\tilde{X}, \tilde{Y} \in \mathcal{P}^H{}^1_0(\tilde{M})$. The equation above can be written as

$$\mathcal{L}_{C_\alpha} \{_b{}^c{}_a\} = 0 \tag{3.8}$$

in the local coordinate system if we take account of (3.7). Thus we have

LEMMA 3.1. *The Riemannian connection with respect to $\tilde{g}$ is projectable if and only if functions $\{_a{}^c{}_b\}$ are all projectable.*

On the other hand, we have

$$\mathcal{L}_{C_\alpha} \{_c{}^a{}_b\} = (\mathcal{L}_{C_\alpha} \{_j{}^h{}_i\}) E^j{}_c E^i{}_b E_h{}^a + \Pi_c{}^\beta{}_\alpha h_b{}^a{}_\beta + \Pi_b{}^\beta{}_\alpha h_c{}^a{}_\beta, \tag{3.9}$$

if we take account of (3.4) and the well-known equations

$$\mathcal{L}_V \{_i{}^j{}_k\} \tilde{Y}^k = \mathcal{L}_V (\tilde{\nabla}_i \tilde{Y}^j) - \tilde{\nabla}_i (\mathcal{L}_V \tilde{Y}^j)$$

(cf. [11]). To prove that the right hand side of (3.9) vanishes, we substitute (3.3) into the equations

$$\mathcal{L}_{C_\alpha} \{_j{}^h{}_i\} = \frac{1}{2} \tilde{g}^{hk} \{\tilde{\nabla}_j (\mathcal{L}_{C_\alpha} \tilde{g}_{ik}) + \tilde{\nabla}_i (\mathcal{L}_{C_\alpha} \tilde{g}_{jk}) - \tilde{\nabla}_k (\mathcal{L}_{C_\alpha} \tilde{g}_{ji})\}$$

and then take account of (3.4). Thus we have

PROPOSITION 3.4. *The Riemannian connection with respect to $\tilde{g}_{ji}$ is projectable.*

Thus M has the Riemannian connection which is induced from $\tilde{g}_{ji}$ and therefore we can consider structure equations in the fibred space. They are called equations of *Co-Gauss*, of *Co-Codazzi* and of *Co-Ricci* corresponding to the equations of Gauss, of Codazzi and of Ricci for a submanifold. We shall give them in the next section.

We have seen in § 2, that the horizontal distribution is integrable if and only if $h_{ba}{}^\alpha = 0$ and in § 3, that C_α commute with E_b if and only if $\Pi_b{}^\beta$ are constant with respect to C_α for all β. It might be interesting to show the relation between $h_{ba}{}^\alpha$ and $\Pi_a{}^\alpha$. Taking account of (3.1) and (3.4), we have

$$h_{ba}{}^\alpha = \Pi_{[b}{}^\beta \Pi_{a]}{}^\alpha{}_\beta + \Pi_{[b}{}^\alpha{}_{,a]}, \tag{3.10}$$

where [ ] denotes skew-symmetrization and comma denotes partial differentiation. Thus we have

PROPOSITION 3.5. *If Π_a^α are constant, then the horizontal distribution is integrable and the integral submanifold is totally geodesic. Conversely, if the horizontal distribution is integrable, then we can choose a local coordinate system in which $\Pi_b^\alpha=0$.*

Using Jacobi identity with respect to the triple (C_α, E_b, E_a), we have

$$\partial_\alpha h_{ba}{}^\beta = -K_{ba\alpha}{}^\beta, \tag{3.11}$$

where $K_{ba\alpha}{}^\beta$ is the so-called curvature of Π_b^α defined by

$$K_{ba\alpha}{}^\beta = \partial_{[b}\Pi_{a]}{}^\beta{}_\alpha - \Pi_{[b}{}^\gamma \partial_{|\gamma|}\Pi_{a]}{}^\beta{}_\alpha + \Pi_{[b}{}^\beta{}_{|\gamma|}\Pi_{a]}{}^\gamma{}_\alpha. \tag{3.12}$$

Thus we have

PROPOSITION 3.6. *$h_{ba}{}^\alpha$ are projectable functions if and only if the curvature of Π_b^γ vanishes.*

When the curvature of Π_a^α vanishes, $h_{ba}{}^\alpha$ induce on M $(n-m)$ vector-valued 2-forms which we denote by the same letter h. From this fact and equations (3.10) and (3.12) we have

PROPOSITION 3.7. *If $\Pi_a{}^\alpha{}_\beta=0$, then $h_{ba}{}^\alpha$ induce on M vector-valued 2-forms which are closed.*

Jacobi identity for the triple (E_c, E_b, E_a) shows

$$\partial_{[c}h_{ba]}{}^\alpha + h_{[cb}{}^\beta \Pi_{a]}{}^\alpha{}_\beta + \Pi_{[c}{}^\beta K_{ba]\beta}{}^\alpha = 0, \tag{3.13}$$

and these equations give another proof of Proposition 3.7.

Here we consider the case in which the horizontal distribution gives an isometric correspondence between two neighboring fibres. In this case, fibres are called *isometric fibres* by Mutō [5]. The condition for fibres to be isometric is given by

$$(\mathcal{L}_{E_a}\tilde{g})^V = 0, \tag{3.14}$$

or equivalently by

$$'g_{\beta\alpha,a} - \Pi_a{}^\gamma{}' g_{\beta\alpha,\gamma} - {}'g_{\gamma\alpha}\Pi_a{}^\gamma{}_\beta - {}'g_{\beta\gamma}\Pi_a{}^\gamma{}_\alpha = 0. \tag{3.15}$$

On the other hand, since we have seen, in (2.12) and (3.4), that

$$(\mathcal{L}_{E_a}\tilde{g})^V = -2l_{\beta\alpha a}C_j{}^\beta C_i{}^\alpha \tag{3.16}$$

and $l_{\beta\alpha}{}^a$ are components of the second fundamental tensor on $\mathcal{F}_P$ with respect to the normal vector E_a, we have

PROPOSITION 3.8. [5] *If the horizontal distribution gives an isometric correspondence between two neighboring fibres, then any fibre* $\mathcal{F}_P$ *is a totally geodesic submanifold of* $\tilde{M}$.

We also have, from this and Proposition 2.3,

THEOREM 3.1. *If* $\tilde{M}$ *has isometric fibres and the horizontal distribution is integrable, then* $\tilde{M}$ *is locally the Riemannian product of* $\mathcal{F}_P$ *and* $\hat{M}$, *where* $\hat{M}$ *is diffeomorphic to* M.

Proof. Propositions 2.3 and 3.8 show that $\tilde{M}$ is a local product of two submanifolds $\mathcal{F}_P$ and $\hat{M}$ which is an integral submanifold of the horizontal distribution. Since we can choose a local coordinate system in which $\Pi_a{}^\alpha=0$ (see Proposition 3.5), we have

$$\partial_a{}' g_{\beta\alpha}=0. \tag{3.17}$$

On the other hand, Riemannian metric $\tilde{g}$ is assumed to be projectable, and hence

$$\partial_\alpha g_{ba}=0 \tag{3.18}$$

holds. (3.17) and (3.18) show that $\tilde{M}$ is locally the Riemannian product of $\mathcal{F}_P$ and the integral submanifold $\hat{M}$ of the horizontal distribution. Thus $\hat{M}$ is diffeomorphic to M.

On the other hand, equations (2.10)$\sim$(2.12) show that

$$\mathcal{L}_{E_a}\tilde{g}_{ji}=(\mathcal{L}_{E_a}g_{cb})E_j{}^cE_i{}^b+2h_{ba\beta}(C_j{}^\beta E_i{}^a+C_i{}^\beta E_j{}^a)-2l_{\beta\alpha a}C_j{}^\beta C_i{}^\alpha,$$

from which and Theorem 3.1 we have

COROLLARY 3.1. *In order that* E_a'*s are Killing vector fields in* M, *it is necessary and sufficient that* $\tilde{M}$ *is locally the Riemannian product of* $\mathcal{F}_P$ *and* $\hat{M}$, *where* $\hat{M}$ *is diffeomorphic to* M *and has a flat metric and* $\mathcal{F}_P$ *is a totally geodesic submanifold of* $\tilde{M}$.

As we have seen in §1, $h_{ba}{}^\alpha$ are components of vector fields in $'M$ with respect to index α. By a straightforward computation we have

$$\mathcal{L}_{h_{ba}}{}' g_{\beta\alpha}=-\mathcal{L}_{E_b}l_{\beta\alpha a}+\mathcal{L}_{E_a}l_{\beta\alpha b}-l_{\gamma\beta b}\Pi_a{}^\gamma{}_\alpha-l_{\gamma\alpha b}\Pi_a{}^\gamma{}_\beta+l_{\gamma\alpha a}\Pi_b{}^\gamma{}_\beta+l_{\gamma\beta a}\Pi_b{}^\gamma{}_\alpha, \tag{3.19}$$

from which and Proposition 3.8 we have

PROPOSITION 3.9. *If* $\tilde{M}$ *has isometric fibres, then the vector fields* $h_{ba}{}^\alpha$ *in* $'M$ *are Killing vector fields on each* $\mathcal{F}_P$ *for all* a *and* b.

Next we consider the case in which the horizontal distribution defines a conformal correspondence between two neighboring fibres (Mutō [5] called such fibres *similar fibres*.) Such a correspondence is defined by the condition

$$(\mathcal{L}_{E_a}\tilde{g}_{ji})^V=2\rho_a\tilde{g}_{ji}. \tag{3.20}$$

From (3.16) and (3.20), we have

(3.21) $$l_{\beta\alpha a}=-\rho_a{}'g_{\beta\alpha},$$

which proves that $'M$ is a totally umbilical submanifold of $\tilde{M}$. On the other hand, the mean curvature vector field has a special meaning for a totally umbilical submanifold. The mean curvature vector field is given by $-\rho_a g^{ab}E_b{}^i$ when $\tilde{M}$ has similar fibres. Thus we have, taking account of (3.4),

PROPOSITION 3.10. *If $\tilde{M}$ has similar fibres, each $\mathcal{F}_P$ is a totally umbilical submanifold of $\tilde{M}$. The normal components of the covariant derivatives along $\mathcal{F}_P$ of the mean curvature vector field vanish if ρ_a are projectable functions for all a and $h_b{}^{a\alpha}\rho_a=0$.*

Substituting (3.21) into (3.19) we have

(3.22) $$\mathcal{L}_{h_{ba}}{}'g_{\beta\alpha}=(\mathcal{L}_{E_b}\rho_a-\mathcal{L}_{E_a}\rho_b)'g_{\beta\alpha}$$

and thus

PROPOSITION 3.11. *If $\tilde{M}$ has similar fibres, the vector fields $h_{ba}{}^\alpha$ in $'M$ are conformal Killing vector fields on $\mathcal{F}_P$.*

COROLLARY 3.2. *If the correspondence between two neighboring fibres defined by the horizontal distribution is homothetic, the vector field $h_{ba}{}^\alpha$ in $'M$ are Killing vector fields on $\mathcal{F}_P$.*

§ 4. Structure equations.

First of all, we recall the definition of van der Waerden-Bortolotti covariant differentiation. It is a kind of differentiation of a object which has various kind of indices. (For details, see, e.g. [11, Ch. V]). Let us denote the formal tensor product by $\mathcal{T}=\mathcal{T}(\tilde{M})\,\#\,\mathcal{T}^H(\tilde{M})\,\#\,\mathcal{T}^V(\tilde{M})$. Van der Waerden-Bortolotti covariant derivative $\overset{*}{\nabla}_{\tilde{X}}$ with respect to $\tilde{X}\in\mathcal{T}_0^1(\tilde{M})$ is a derivation in $\mathcal{T}$ which has following properties:

1) $\overset{*}{\nabla}_{\tilde{X}}\tilde{T}=\tilde{\nabla}_{\tilde{X}}\tilde{T}$ for $\tilde{T}\in\mathcal{T}(\tilde{M})$,

2) $\overset{*}{\nabla}_{\tilde{X}}\tilde{T}=(\tilde{\nabla}_{\tilde{X}}\tilde{T})^H$ for $\tilde{T}\in\mathcal{T}^H(\tilde{M})$,

3) $\overset{*}{\nabla}_{\tilde{X}}\tilde{T}=(\tilde{\nabla}_{\tilde{X}}\tilde{T})^V$ for $\tilde{T}\in\mathcal{T}^V(\tilde{M})$.

$\overset{*}{\nabla}_{\tilde{X}}$ is decomposed into $\overset{*}{\nabla}_{\tilde{X}^V}$ and $\overset{*}{\nabla}_{\tilde{X}^H}$. $\overset{*}{\nabla}_{\tilde{X}^V}$ is nothing but van der Waerden-Bortolotti covariant derivative along a fibre as a submanifold of $\tilde{M}$ which is familiar to us and is called that of the first kind. $\overset{*}{\nabla}_{\tilde{X}^H}$ is called that of the second kind. Each expression in a local coordinate system is given as follows:

If we take a tensor field

$$\tilde{T}=\tilde{T}_i{}^j{}_h{}^k{}_s{}^t=\overset{*}{T}_i{}^j{}_a{}^b{}_\alpha{}^\beta E_h{}^a E^k{}_b C^t{}_\beta C_s{}^\alpha,$$

for example, van der Waerden-Bortolotti covariant derivative $\overset{*}{\nabla}_l$ is defined by

$$
\begin{aligned}
\overset{*}{\nabla}_l \overset{*}{T}_i{}^j{}_a{}^b{}_\alpha{}^\beta &= \partial_l \overset{*}{T}_i{}^j{}_a{}^b{}_\alpha{}^\beta + \begin{Bmatrix} j \\ l \quad m \end{Bmatrix} \overset{*}{T}_i{}^m{}_a{}^b{}_\alpha{}^\beta - \begin{Bmatrix} m \\ l \quad i \end{Bmatrix} \overset{*}{T}_m{}^j{}_a{}^b{}_\alpha{}^\beta \\
&\quad + E_l{}^c \left(\begin{Bmatrix} b \\ c \quad e \end{Bmatrix} \overset{*}{T}_i{}^j{}_a{}^e{}_\alpha{}^\beta - \begin{Bmatrix} e \\ c \quad a \end{Bmatrix} \overset{*}{T}_i{}^j{}_e{}^b{}_\alpha{}^\beta \right) \qquad (4.1) \\
&\quad + C_l{}^\gamma \left(' \begin{Bmatrix} \beta \\ \gamma \quad \varepsilon \end{Bmatrix} \overset{*}{T}_i{}^j{}_a{}^b{}_\alpha{}^\varepsilon - ' \begin{Bmatrix} \varepsilon \\ \gamma \quad \alpha \end{Bmatrix} \overset{*}{T}_i{}^j{}_a{}^b{}_\varepsilon{}^\beta \right).
\end{aligned}
$$

If we put conventionally

$$
\begin{aligned}
\overset{*}{\nabla}_j E^h{}_b &= \partial_j E^h{}_b + \begin{Bmatrix} h \\ j \quad i \end{Bmatrix} E^i{}_b - \begin{Bmatrix} a \\ c \quad b \end{Bmatrix} E_j{}^c E^h{}_a, \\
\overset{*}{\nabla}_j E_i{}^a &= \partial_j E_i{}^a - \begin{Bmatrix} h \\ j \quad i \end{Bmatrix} E_h{}^a + \begin{Bmatrix} a \\ c \quad b \end{Bmatrix} E_j{}^c E_i{}^b, \qquad (4.2) \\
\overset{*}{\nabla}_j C^i{}_\alpha &= \partial_j C^i{}_\alpha + \begin{Bmatrix} i \\ j \quad l \end{Bmatrix} C^l{}_\alpha - ' \begin{Bmatrix} \varepsilon \\ \gamma \quad \alpha \end{Bmatrix} C_\varepsilon{}^i C_j{}^\gamma, \\
\overset{*}{\nabla}_j C_i{}^\alpha &= \partial_j C_i - \begin{Bmatrix} l \\ j \quad i \end{Bmatrix} C_l{}^\alpha + ' \begin{Bmatrix} \alpha \\ \gamma \quad \varepsilon \end{Bmatrix} C_i{}^\varepsilon C_j{}^\gamma,
\end{aligned}
$$

then we have

$$
\begin{aligned}
\overset{*}{\nabla}_l T_i{}^j{}_h{}^k{}_s{}^t &= (\overset{*}{\nabla}_l \overset{*}{T}_i{}^j{}_a{}^b{}_\alpha{}^\beta) E_h{}^a E^k{}_b C_s{}^\alpha C^t{}_\beta \\
&\quad + \overset{*}{T}_i{}^j{}_a{}^b{}_\alpha{}^\beta \{ (\overset{*}{\nabla}_l E_h{}^a) E^k{}_b C_s{}^\alpha C^t{}_\beta + E_h{}^a (\overset{*}{\nabla}_l E^k{}_b) C_s{}^\alpha C^t{}_\beta + E_h{}^a E^k{}_b (\overset{*}{\nabla}_l C_s{}^\alpha) C^t{}_\beta + E_h{}^a E^k{}_b C_s{}^\alpha \overset{*}{\nabla} C^t{}_\beta \}.
\end{aligned}
$$

Van der Waerden-Bortolotti covariant derivative of the first kind $\overset{*}{\nabla}_\alpha$ for $\tilde{T}$ is defined by the covariant derivative along a fibre, i.e., we put $\overset{*}{\nabla}_\alpha = C^j{}_\alpha \overset{*}{\nabla}_j$. Then we have

$$
\begin{aligned}
\overset{*}{\nabla}_\alpha E^h{}_b &= C^j{}_\alpha \partial_j E^h{}_b + \begin{Bmatrix} h \\ j \quad i \end{Bmatrix} E^i{}_b C^j{}_\alpha, \\
\overset{*}{\nabla}_\alpha E_i{}^a &= C^j{}_\alpha \partial_j E_i{}^a - \begin{Bmatrix} h \\ j \quad i \end{Bmatrix} E_h{}^a C^j{}_\alpha, \qquad (4.3) \\
\overset{*}{\nabla}_\alpha C^i{}_\beta &= C^j{}_\alpha \partial_j C^i{}_\beta + \begin{Bmatrix} i \\ j \quad l \end{Bmatrix} C^l{}_\beta C^j{}_\alpha - ' \begin{Bmatrix} \varepsilon \\ \alpha \quad \beta \end{Bmatrix} C^i{}_\varepsilon, \\
\overset{*}{\nabla}_\alpha C_i{}^\beta &= C^j{}_\alpha \partial_j C_i{}^\beta - \begin{Bmatrix} l \\ j \quad i \end{Bmatrix} C_l{}^\beta C^j{}_\alpha + ' \begin{Bmatrix} \beta \\ \alpha \quad \varepsilon \end{Bmatrix} C_i{}^\varepsilon,
\end{aligned}
$$

and

$$\overset{*}{\nabla}_\gamma \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta} = C^l{}_\gamma\Big(\partial_l \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta} + \begin{Bmatrix} j \\ l\ m \end{Bmatrix} \overset{*}{T}_{i}{}^{m}{}_{a}{}^{b}{}_{\alpha}{}^{\beta} - \begin{Bmatrix} m \\ l\ i \end{Bmatrix} \overset{*}{T}_{m}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta}\Big) \tag{4.4}$$
$$+' \begin{Bmatrix} \beta \\ \gamma\ \varepsilon \end{Bmatrix} \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\varepsilon} -' \begin{Bmatrix} \varepsilon \\ \gamma\ \alpha \end{Bmatrix} \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\varepsilon}{}^{\beta}$$

and thus

$$C^l{}_\gamma \tilde{\nabla}_l T_i{}^j{}_h{}^k{}_s{}^t = (\overset{*}{\nabla}_\gamma \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta}) E_h{}^a E^k{}_b C_s{}^\alpha C^t{}_\beta$$
$$+ \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta}\{(\overset{*}{\nabla}_\gamma E_h{}^a) E^k{}_b C_s{}^\alpha C^t{}_\beta + E_h{}^a (\overset{*}{\nabla}_\gamma E^k{}_b) C_s{}^\alpha C^t{}_\beta \tag{4.5}$$
$$+ E_h{}^a E^k{}_b (\overset{*}{\nabla}_\gamma C_s{}^\alpha) C^t{}_\beta + E_h{}^a E^k{}_b C_s{}^\alpha \overset{*}{\nabla}_\gamma C^t{}_\beta\}.$$

The last one is defined along horizontal plane fields which is called van der Waerden-Bortolotti covariant derivative of the second kind and denoted by $\overset{*}{\nabla}_c$. $\overset{*}{\nabla}_c$ is defined by $\overset{*}{\nabla}_c = E^l{}_c \overset{*}{\nabla}_l$ and thus we have

$$\overset{*}{\nabla}_c E^i{}_a = E^j{}_c \partial_j E^i{}_a + \begin{Bmatrix} i \\ j\ h \end{Bmatrix} E^j{}_c E^h{}_a - \begin{Bmatrix} b \\ c\ a \end{Bmatrix} E^i{}_b,$$
$$\overset{*}{\nabla}_c E_i{}^a = E^j{}_c \partial_j E_i{}^a - \begin{Bmatrix} h \\ j\ i \end{Bmatrix} E^j{}_c E_h{}^a + \begin{Bmatrix} a \\ c\ b \end{Bmatrix} E_i{}^b, \tag{4.6}$$
$$\overset{*}{\nabla}_c C^i{}_\alpha = E^j{}_c \partial_j C^i{}_\alpha + \begin{Bmatrix} i \\ j\ l \end{Bmatrix} E^j{}_c C^l{}_\alpha,$$
$$\overset{*}{\nabla}_c C_i{}^\alpha = E^j{}_c \partial_j C_i{}^\alpha - \begin{Bmatrix} l \\ j\ i \end{Bmatrix} E^j{}_c C_l{}^\alpha.$$

If we define, for $\tilde{T}$

$$\overset{*}{\nabla}_c \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta} = E^l{}_c\Big(\partial_l \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta} - \begin{Bmatrix} m \\ l\ i \end{Bmatrix} \overset{*}{T}_{m}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta} + \begin{Bmatrix} j \\ l\ m \end{Bmatrix} \overset{*}{T}_{i}{}^{m}{}_{a}{}^{b}{}_{\alpha}{}^{\beta}\Big) \tag{4.7}$$
$$+ \begin{Bmatrix} b \\ c\ d \end{Bmatrix} \overset{*}{T}_{i}{}^{j}{}_{a}{}^{d}{}_{\alpha}{}^{\beta} - \begin{Bmatrix} d \\ c\ a \end{Bmatrix} \overset{*}{T}_{i}{}^{j}{}_{d}{}^{b}{}_{\alpha}{}^{\beta},$$

then we have

$$E_c{}^l \overset{*}{\nabla}_l \tilde{T}_i{}^j{}_h{}^k{}_s{}^t = (\overset{*}{\nabla}_c \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta}) E_h{}^a E^k{}_b C_s{}^\alpha C^t{}_\beta$$
$$+ \overset{*}{T}_{i}{}^{j}{}_{a}{}^{b}{}_{\alpha}{}^{\beta}\{(\overset{*}{\nabla}_c E_h{}^a) E^k{}_b C_s{}^\alpha C^t{}_\beta + E_h{}^a (\overset{*}{\nabla}_c E^k{}_b) C_s{}^\alpha C^t{}_\beta$$
$$+ E_h{}^a E^k{}_b (\overset{*}{\nabla}_c C_s{}^\alpha) C^t{}_\beta + E_h{}^a E^k{}_b C_s{}^\alpha \overset{*}{\nabla}_c C^t{}_\beta\}.$$

From (3.4), we have

$$
(4.9)\qquad
\begin{aligned}
\tilde{\nabla}_j E_i^{\ a} &= \overset{*}{\nabla}_j E_i^{\ a} - \left\{ {a \atop c\ b} \right\} E_j^{\ c} E_i^{\ b}, \\
\tilde{\nabla}_j E^h{}_b &= \overset{*}{\nabla}_j E^h{}_b + \left\{ {a \atop c\ b} \right\} E_j^{\ c} E^h{}_a, \\
\tilde{\nabla}_j C_i^{\ \alpha} &= \overset{*}{\nabla}_j C_i^{\ \alpha} - {}'\left\{ {\alpha \atop \gamma\ \varepsilon} \right\} C_j^{\ \gamma} C_i^{\ \varepsilon}, \\
\tilde{\nabla}_j C^h{}_\beta &= \overset{*}{\nabla}_j C^h{}_\beta + {}'\left\{ {\varepsilon \atop \gamma\ \beta} \right\} C_j^{\ \gamma} C^h{}_\varepsilon
\end{aligned}
$$

and therefore

$$
(4.10)\qquad
\begin{aligned}
\overset{*}{\nabla}_j E_i^{\ a} &= h_b{}^a (E_i^{\ b} C_j^{\ \beta} + E_j^{\ b} C_i^{\ \alpha}) - l_{\beta\alpha}{}^a C_j^{\ \beta} C_i^{\ \alpha}, \\
\overset{*}{\nabla}_j E^h{}_b &= h_b{}^a E^h{}_a C_j^{\ \alpha} - l_\alpha{}^\beta{}_b C^h{}_\beta C_j^{\ \alpha} + h_{ab}{}^\beta C^h{}_\beta E_j^{\ a}, \\
\overset{*}{\nabla}_j C_i^{\ \alpha} &= -h_{ab}{}^\alpha E_j^{\ a} E_i^{\ b} + (l_\beta{}^\alpha{}_a - \Pi_a{}^\alpha{}_\beta) E_j^{\ a} C_i^{\ \beta} + l_\beta{}^\alpha{}_b C_j^{\ \beta} E_i^{\ b}, \\
\overset{*}{\nabla}_j C^h{}_\beta &= -h_a{}^b{}_\beta E^h{}_b E_j^{\ a} - (l_\beta{}^\alpha{}_a - \Pi_a{}^\alpha{}_\beta) C^h{}_\alpha E_j^{\ a} + l_{\alpha\beta}{}^b E^h{}_b C_j^{\ \alpha}.
\end{aligned}
$$

We also have from (4.3) and (4.6),

$$
(4.11)\qquad
\begin{aligned}
\overset{*}{\nabla}_\gamma E_i^{\ a} &= h_b{}^a{}_\gamma E_i^{\ b} - l_{\beta\gamma}{}^a C_i^{\ \beta}, \\
\overset{*}{\nabla}_\gamma E^h{}_b &= -h_b{}^c{}_\gamma E_c^{\ h} - l_\gamma{}^\beta{}_b C^h{}_\beta, \\
\overset{*}{\nabla}_\gamma C_j^{\ \alpha} &= l_\gamma{}^\alpha{}_b E_i^{\ b}, \\
\overset{*}{\nabla}_\gamma C^h{}_\beta &= l_{\gamma\beta}{}^b E^h{}_b
\end{aligned}
$$

and

$$
(4.12)\qquad
\begin{aligned}
\overset{*}{\nabla}_c E_i^{\ a} &= h_c{}^a{}_\alpha C_i^{\ \alpha}, \\
\overset{*}{\nabla}_c E^h{}_b &= h_{cb}{}^\beta C^h{}_\beta, \\
\overset{*}{\nabla}_c C_i^{\ \alpha} &= h_{cb}{}^\alpha E_i^{\ b} + (l_\beta{}^\alpha{}_c - \Pi_c{}^\alpha{}_\beta) C_i^{\ \beta}, \\
\overset{*}{\nabla}_c C^h{}_\beta &= -h_c{}^b{}_\beta E^h{}_b - (l_\beta{}^\alpha{}_c - \Pi_c{}^\alpha{}_\beta) C^h{}_\alpha.
\end{aligned}
$$

(4.11) are nothing but the equations of Gauss and Weingarten for a fibre as a submanifold of $\tilde{M}$, where we see that $l_\gamma{}^\alpha{}_b$ and $-h_b{}^c{}_\gamma$ are respectively the second and the third fundamental tensors with respect to a normal vector field E_b.

On the other hand, if the horizontal distribution is integrable, then equations (4.12) are those of Gauss and Weingarten for the integral submanifold $\hat{M}$. Thus $-l_{\beta\alpha c}$ are components of the third fundamental tensors on $\hat{M}$.

Let us denote curvature tensors defined by $\tilde{g}, {}'g$ and g by $\tilde{K}, {}'K$ and K respectively. Since each fibre is a submanifold of $\tilde{M}$, we have equations of Gauss, Codazzi and Ricci as follows:

$$C^k{}_\nu C^j{}_\mu C^i{}_\lambda C_h{}^\kappa \tilde{K}_{kji}{}^h - {}'K_{\nu\mu\lambda}{}^\kappa = -l_{\mu\lambda}{}^a l_\nu{}^\kappa{}_a + l_{\nu\lambda}{}^a l_\mu{}^\kappa{}_a \qquad \text{(Gauss)}, \tag{4.13}$$

$$C^k{}_\nu C^j{}_\mu C^i{}_\lambda E_h{}^a \tilde{K}_{kji}{}^h = {}'\nabla_\nu l_{\mu\lambda}{}^a - {}'\nabla_\mu l_{\nu\lambda}{}^a - l_{\mu\lambda}{}^c h_c{}^a{}_\nu + l_{\nu\lambda}{}^c h_c{}^a{}_\mu \qquad \text{(Codazzi)} \tag{4.14}$$

and

$$C^k{}_\nu C^j{}_\mu E^i{}_b E_h{}^a \tilde{K}_{kji}{}^h = -\partial_\nu h_b{}^a{}_\mu + \partial_\mu h_b{}^a{}_\nu + h_c{}^a{}_\nu h_b{}^c{}_\mu - h_c{}^a{}_\mu h_b{}^c{}_\nu - l_{\nu\alpha}{}^a l_\mu{}^\alpha{}_b + l_{\mu\alpha}{}^a l_\nu{}^\alpha{}_b \qquad \text{(Ricci)}. \tag{4.15}$$

We also obtain relations between $\tilde{K}$ and K which correspond to three equations above and are called equations of Co-Gauss, Co-Codazzi and Co-Ricci respectively.

$$E^k{}_d E^j{}_c E^i{}_b E_h{}^a \tilde{K}_{kji}{}^h - K_{dcb}{}^a = -h_{cb}{}^\alpha h_d{}^a{}_\alpha + h_{db}{}^\alpha h_c{}^a{}_\alpha + 2h_{dc}{}^\alpha h_b{}^a{}_\alpha \qquad \text{(Co-Gauss)}, \tag{4.16}$$

$$\begin{aligned} E^k{}_d E^j{}_c E^i{}_b C_h{}^\alpha \tilde{K}_{kji}{}^h = {} & \overset{*}{\nabla}_d h_{cb}{}^\alpha - \overset{*}{\nabla}_c h_{db}{}^\alpha + 2h_{dc}{}^\gamma l_\gamma{}^\alpha{}_b - h_{cb}{}^\gamma (l_\gamma{}^\alpha{}_d - \Pi_d{}^\alpha{}_\gamma) \\ & + h_{db}{}^\gamma (l_\gamma{}^\alpha{}_c - \Pi_c{}^\alpha{}_\gamma) \qquad \text{(Co-Codazzi)} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} E^k{}_d E^j{}_c C^i{}_\beta C_h{}^\alpha \tilde{K}_{kji}{}^h = {} & -E^k{}_d \tilde{\nabla}_k (l_\beta{}^\alpha{}_c - \Pi_c{}^\alpha{}_\beta) + E^j{}_c \tilde{\nabla}_j (l_\beta{}^\alpha{}_d - \Pi_d{}^\alpha{}_\beta) \\ & - (l_\beta{}^\gamma{}_d - \Pi_d{}^\gamma{}_\beta)(l_\gamma{}^\alpha{}_c - \Pi_c{}^\alpha{}_\gamma) + (l_\beta{}^\gamma{}_c - \Pi_c{}^\gamma{}_\beta)(l_\gamma{}^\alpha{}_d - \Pi_d{}^\alpha{}_\gamma) \\ & + h_d{}^b{}_\beta h_{cb}{}^\alpha - h_{db}{}^\alpha h_c{}^b{}_\beta - 2h_{dc}{}^\gamma{}' \left\{ {\alpha \atop \gamma \;\; \beta} \right\} \qquad \text{(Co-Ricci)}. \end{aligned}$$

The following formula, together with formulas (4.13)~(4.18), is useful to compute the sectional curvature

$$\begin{aligned} C^k{}_\beta E^j{}_b C^i{}_\alpha E^h{}_a \tilde{K}_{kjih} = {} & -\overset{*}{\nabla}_\beta h_{ba\alpha} - \overset{*}{\nabla}_b l_{\beta\alpha a} \\ & + h_b{}^c{}_\alpha h_{ca\beta} - l_\alpha{}^\gamma{}_b l_{\gamma\beta a} + l_{\gamma\alpha a} \Pi_b{}^\gamma{}_\beta + l_{\gamma\beta a} \Pi_b{}^\gamma{}_\alpha . \end{aligned} \tag{4.19}$$

Thus, the sectional curvature $\tilde{K}(C_\alpha, E_a)$ with respect to the 2-plane spanned by C_α and E_a is given by

$$\tilde{K}(C_\alpha, E_a) = \frac{\overset{*}{\nabla}_a l_{\alpha\alpha a} + h_a{}^c{}_\alpha h_{ac\alpha} + l_\alpha{}^\gamma{}_a l_{\gamma\alpha a} - 2 l_{\gamma\alpha a} \Pi_a{}^\gamma{}_\alpha}{\|C_\alpha\|^2 \|E_a\|^2}, \tag{4.20}$$

where $\| \;\; \|$ denotes the length of a vector.

Taking account of the equation (3.16), we have

THEOREM 4.1. *If $\tilde{M}$ has non-positive sectional curvature with respect to the 2-plane spanned by C_α and E_a and has isometric fibres, then the horizontal distribution is integrable and $\tilde{M}$ is locally the Riemannian product of $\mathcal{F}_p$ and $\hat{M}$ which is diffeomorphic to M.*

For the proof of this theorem, see the proof of Theorem 3.1.

§ 5. Almost complex structures in fibred spaces.

We consider, in this section, an almost complex structure $\tilde{F}$ in $\tilde{M}$ which is assumed to be projectable. This means, by definition,

$$(\mathcal{L}_{\tilde{V}}\tilde{F}^{H})^{H}=0 \tag{5.1}$$

for any vertical vector field $\tilde{V}$. First we consider the case in which each fibre is an invariant submanifold of $\tilde{M}$. If we denote by $\tilde{F}_i{}^h$ the components of $\tilde{F}$ with respect to a local coordinate system, they are expressed as, using the non-holonomic frame (E_a, C_α),

$$\tilde{F}_i{}^h=f_b{}^a E^b{}_i E^h{}_a+f_\beta{}^\alpha C_i{}^\beta C^h{}_\alpha, \tag{5.2}$$

where $f_b{}^a$ are projectable functions by the assumption (5.1). We sometimes identify $f_b{}^a$ with their projections on M. By a straightforward computation we have

$$f_b{}^a f_a{}^c=-\delta_b^c, \qquad f_\beta{}^\alpha f_\alpha{}^\gamma=-\delta_\beta^\gamma, \tag{5.3}$$

These equations show that M and $'M$ have almost complex structures $f=(f_b{}^a)$ and $'f=(f_\beta{}^\alpha)$ respectively. Since each fibre is assumed to be an invariant submanifold of $\tilde{M}$ and there are many results ever obtained about an invariant submanifold of an almost complex space, we discuss here mainly the horizontal distributions.

If we denote respectively by $N(\tilde{F},\tilde{F})$, $N(f,f)$ and $N('f,'f)$ Nijenhuis tensors formed with $\tilde{F}$, f and $'f$, then the relation among those three tensors is as follows:

PROPOSITION 5.1. *$N(\tilde{F},\tilde{F})$ is zero if and only if*

1) $N(f,f)=0$, 2) $N('f,'f)=0$, 3) $f_c{}^a\Lambda_{a\beta}{}^\alpha-f_\gamma{}^\alpha\Lambda_{c\beta}{}^\gamma=0$,

where $\Lambda_{a\beta}{}^\alpha=(\mathcal{L}_{E_a}\tilde{F}_j{}^i)C^j{}_\beta C_i{}^\alpha$, *and*

4) $O_{cb}^{ed}h_{ed}{}^\alpha+f_\gamma{}^\alpha f_b{}^d O_{cd}^{ea}h_{ea}{}^\gamma=0$,

where O_{cb}^{ed} is the so-called pure operator (cf. [13]*) defined by*

$$O_{cb}^{ed}=\frac{1}{2}(\delta_c^e\delta_b^d-f_c{}^e f_b{}^d).$$

On the other hand, a straightforward computation shows that if $\Lambda_{a\beta}{}^\alpha=0$, then

$$\mathcal{L}_{h_{ba}}f_\beta{}^\alpha=0.$$

Thus we have

PROPOSITION 5.2. *If $\Lambda_{a\beta}{}^\alpha=0$ and $h_{ba}{}^\alpha\neq0$, then there exist at most $m(m-1)/2$ vertical almost analytic vector fields in $\mathcal{F}_P$.*

REMARK. We see, in Proposition 5.1, that $h_{ba}{}^{\alpha}$ is not zero in general even if $N(\tilde{F},\tilde{F})=0$. In the case in which the almost complex structure is integrable, that is, $N(\tilde{F},\tilde{F})=0$, $h_{ba}{}^{\alpha}$ are analytic vector fields in $\mathcal{F}_P$, if $\Lambda_{\alpha\beta}{}^{\alpha}=0$.

We refer here to the condition that C_α or E_a is to be an almost analytic vector field. The next proposition is a result of a direct computation:

PROPOSITION 5.3. *A necessary and sufficient condition that C_α is an almost analytic vector field is that*

1) $\partial_\alpha f_\beta{}^\gamma=0$

and

2) $f_b{}^a \Pi_a{}^\beta - f_\gamma{}^\beta \Pi_b{}^\gamma$ *are projectable functions.*

And a necessary and sufficient condition that E_a is an almost analytic vector field is that

1) $\partial_\alpha f_b{}^c=0$, 2) $\Lambda_{\alpha\beta}{}^{\alpha}=0$

and

3) $O_{ab}^{cd} h_{cd}{}^{\alpha} + f_\gamma{}^\alpha f_b{}^d O_{ad}^{ce} h_{ce}{}^{\gamma}=0.$

Thus, in the case in which E_a's are almost analytic vector fields, $N(\tilde{F},\tilde{F})$ vanishes if and only if $N('f,'f)$ vanishes.

Now we suppose that $\tilde{M}$ is a Kählerian manifold which is the most typical example of complex manifolds. In a Kählerian manifold we have $\tilde{\nabla}_j\tilde{F}_{ih}=0$, from which and the assumption (5.1) we have

$$\nabla_c f_b{}^a=0, \tag{5.5}$$

where ∇_c is the operator of covariant differentiation with respect to the connection induced on M from $\tilde{\nabla}$,

$$f_b{}^d h_{cd}{}^{\alpha} - f_\beta{}^\alpha h_{cb}{}^{\beta}=0, \tag{5.6}$$

$$f_b{}^d h_{cd}{}^{\alpha} - f_c{}^d h_{bd}{}^{\alpha}=0, \tag{5.7}$$

$$\Lambda_{c\beta}{}^{\alpha} - f_\beta{}^\gamma l_\gamma{}^\alpha{}_c + f_\gamma{}^\alpha l_\beta{}^\gamma{}_c=0, \tag{5.8}$$

$$-f_b{}^a l_{\alpha\beta}{}^b + f_\beta{}^\gamma l_{\alpha\gamma}{}^a=0 \tag{5.9}$$

and

$$'\nabla_\gamma f_\beta{}^\alpha=0, \tag{5.10}$$

where $'\nabla_\gamma$ is the operator of covariant differentiation with respect to the connection induced on $\mathcal{F}_P$ from $\tilde{\nabla}$. Equations (5.6) show that $f_b{}^d h_{cd}{}^{\alpha}$ is skew-symmetric in b and c, but it is also symmetric in b and c by equations (5.7). Thus we have

(5.11) $$h_{bc}{}^{\alpha}=0.$$

We also have, from equations (5.8) and (5.9),

(5.12) $$\Lambda_{c\beta}{}^{\alpha}=-2f_{\gamma}{}^{\alpha}l_{\beta}{}^{\gamma}{}_{c},$$

from which we have

LEMMA 5.1. *$\tilde{M}$ has isometric fibres if and only if $\Lambda_{c\beta}{}^{\alpha}=0$.*

Taking account of Theorem 3.1 and Lemma 5.1 we have

THEOREM 5.1. *In a fibred Kählerian space $\tilde{M}$ with a projectable metric and a projectable almost complex structure, if each fibre is a holomorphic submanifold of $\tilde{M}$, then the horizontal distribution is integrable, that is, $\tilde{M}$ is locally trivial. In this case $\tilde{M}$ is locally the Riemannian product of $\mathcal{F}_{P}$ and $\hat{M}$ if and only if $\Lambda_{c\beta}{}^{\alpha}=0$, where $\hat{M}$ is diffeomorphic to M. In the latter case we have $\partial_{c}f_{\gamma}{}^{\beta}=0$.*

Next we consider the case in which $\mathcal{F}_{P}$ is not an invariant manifold of $\tilde{M}$. We assume as before that $\tilde{M}$ has a projectable Riemannian metric and a projectable almost complex structure $\tilde{F}$. We assume, for the present, $\dim\mathcal{F}_{P}>\dim M$, because we can discuss analogously the case $\dim\mathcal{F}_{P}<\dim M$.

We further assume that $\tilde{F}\tilde{V}$ is horizontal for any $\tilde{V}\in\mathcal{T}^{V}{}^{1}_{0}(\tilde{M})$. If there is a vertical vector field $\tilde{V}$ such that $\tilde{F}\tilde{V}$ is vertical, a certain submanifold of $\mathcal{F}_{P}$ is invariant under $\tilde{F}$ and in such a case we do over again the discussion mentioned above.

Renumbering $(E_{1},\cdots,E_{m})$, we can put

$$\tilde{F}E_{\bar{c}}=f_{\bar{c}}{}^{\bar{e}}E_{\bar{e}},\qquad \tilde{F}E_{b'}=f_{b'}{}^{\beta}C_{\beta},\qquad \tilde{F}C_{\gamma}=f_{\gamma}{}^{b'}E_{b'},$$

where $\bar{c}=1,2,\cdots,m-r$; $b'=m-r-1,\cdots,$; $r=\dim\mathcal{F}_{P}$. Thus $\tilde{F}$ is represented as

(5.12) $$F_{i}{}^{h}=f_{\bar{b}}{}^{\bar{a}}E_{i}{}^{\bar{b}}E^{h}{}_{\bar{a}}+f_{b'}{}^{d}E_{i}{}^{b'}C^{h}{}_{\alpha}+f_{\beta}{}^{a'}C_{i}{}^{\beta}E^{h}{}_{a'},$$

where $f_{\bar{b}}{}^{\bar{a}}$, $f_{b'}{}^{d}$ and $f_{\beta}{}^{a'}$ satisfy equations,

(5.13) $$f_{\bar{a}}{}^{\bar{c}}f_{\bar{b}}{}^{\bar{a}}=-\delta_{\bar{b}}^{\bar{c}},\qquad f_{b'}{}^{\alpha}f_{\alpha}{}^{c'}=-\delta_{b'}^{c'},\qquad f_{\beta}{}^{a'}f_{a'}{}^{\gamma}=-\delta_{\beta}^{\gamma}$$

and

(5.14) $$\partial_{\alpha}f_{\bar{b}}{}^{\bar{a}}=0.$$

Thus M has a so-called f-structure, f being given by a matrix of rank $m-r$

(5.15) $$\begin{pmatrix} f_{\bar{b}}{}^{\bar{a}} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the non-holonomic frame $(E_{a}, E_{b'})$.

By the definition of normality of f-structure in [3], the normality of f-structure given by (5.15) is equivalent to the integrability of the almost complex structure defined by (5.12). Thus we have, by a straightforward computation,

THEOREM 5.2. *The f-structure in M given by* (5.15) *is normal if and only if following conditions are satisfied.*

1) $f_{\bar{c}}{}^{\bar{e}}\partial_{\bar{e}}f_{\bar{b}}{}^{\bar{a}}-f_{\bar{b}}{}^{\bar{e}}\partial_{\bar{e}}f_{\bar{c}}{}^{\bar{a}}-f_{\bar{e}}{}^{\bar{a}}(\partial_{\bar{c}}f_{\bar{b}}{}^{\bar{a}}-\partial_{\bar{b}}f_{\bar{c}}{}^{\bar{a}})=0,$

2) $f_{\bar{b}}{}^{\bar{d}}h_{\bar{c}\bar{d}}{}^{\alpha}+f_{\bar{c}}{}^{\bar{d}}h_{\bar{d}\bar{b}}{}^{\alpha}=0,$

3) $\partial_{b'}f_{\bar{c}}{}^{\bar{a}}=0,$

4) $\partial_{\bar{c}}f_{b'}{}^{\alpha}-\Pi_{\bar{c}}{}^{\gamma}\partial_{\gamma}f_{b'}{}^{d}+f_{b'}{}^{\gamma}\Pi_{\bar{c}}{}^{\alpha}{}_{\gamma}+2f_{\bar{c}}{}^{\bar{e}}h_{\bar{e}b'}{}^{\alpha}=0,$

5) $\partial_{b'}f_{c'}{}^{\gamma}-\partial_{c'}f_{b'}{}^{\gamma}-\Pi_{b'}{}^{\beta}\partial_{\beta}f_{c'}{}^{\gamma}+\Pi_{c'}{}^{\beta}\partial_{\beta}f_{b'}{}^{\gamma}-f_{b'}{}^{\beta}\Pi_{c'}{}^{\gamma}{}_{\beta}+f_{c'}{}^{\beta}\Pi_{b'}{}^{\gamma}{}_{\beta}=0$

and

6) $f_{c'}{}^{\gamma}\partial_{\gamma}f_{b'}{}^{\alpha}-f_{b'}{}^{\gamma}\partial_{\gamma}f_{c'}{}^{\alpha}-2h_{c'b'}{}^{\alpha}=0.$

REMARK. The condition 1) in Theorem 5.2 is nothing but the integrability condition of the almost complex structure defined by $f_{\bar{b}}{}^{\bar{a}}$.

In particular, if $\tilde{M}$ is a Kählerian fibred space with a projectable Riemannian metric $\tilde{g}$ and the projectable almost complex structure $\tilde{F}$ defined by (5.12), then we get following identities

$$\nabla_{\bar{c}}f_{\bar{b}}{}^{\bar{a}}=0, \qquad \partial_{c'}f_{\bar{b}}{}^{\bar{a}}=0, \qquad \partial_{c'}g_{\bar{b}\bar{a}}=0, \qquad h_{\bar{c}\bar{b}}{}^{\alpha}=0,$$

$$\partial_{\bar{c}}g_{b'c'}=0, \qquad h_{c'\bar{a}}{}^{\alpha}=0, \qquad l_{\gamma\beta}{}^{\bar{a}}=0,$$

$$f_{b'}{}^{\varepsilon}h_{c'}{}^{a'}{}_{\varepsilon}+f_{\varepsilon}{}^{a'}h_{c'b'}{}^{\varepsilon}=0,$$

$$f_{e'}{}^{\alpha}l_{\gamma\beta}{}^{e'}+f_{\beta}{}^{e'}l_{\gamma}{}^{\alpha}{}_{e'}=0,$$

$$\partial_{\bar{c}}f_{b'}{}^{\alpha}-\Pi_{\bar{c}}{}^{\gamma}\partial_{\gamma}f_{b'}{}^{\alpha}+f_{b'}{}^{\varepsilon}\Pi_{\bar{c}}{}^{\alpha}{}_{\varepsilon}=0,$$

$$\overset{*}{\nabla}_{c'}f_{b'}{}^{\alpha}-f_{b'}{}^{\varepsilon}(l_{\varepsilon}{}^{\alpha}{}_{c'}-\Pi_{c'}{}^{\alpha}{}_{\varepsilon})=0$$

and

$$\overset{*}{\nabla}_{\gamma}f_{b'}{}^{\alpha}+f_{e'}{}^{\alpha}h_{b'}{}^{e'}{}_{\gamma}=0.$$

These equations are useful to prove the following:

THEOREM 5.3. *Let $\tilde{M}$ be a fibred Kählerian space with a projectable Riemannian metric $\tilde{g}$ and the projectable almost complex structure $\tilde{F}$ difined by* (5.12). *We denote by M_1 the distribution spanned by $E_{\bar{a}}$'s and by M_2 the distribution spanned by C_{α}'s and $E_{a'}$'s. Then M_1 and M_2 are both involutive distributions and their integral manifolds $\hat{M}_1$ and $\hat{M}_2$ are Kählerian submanifolds of $\tilde{M}$ which are totally geodesic and $\tilde{M}$ is the Riemannian product of $\hat{M}_1$ and $\hat{M}_2$.*

Proof. Since $h_{\bar{c}\bar{b}}{}^{\alpha}=0$, the distribution M_1 is integrable and its integral manifold $\hat{M}_1$ is totally geodesic. $\nabla_{\bar{c}} f_{\bar{b}}{}^{\bar{a}}=0$ means that $\hat{M}_1$ is a Kählerian submanifold of $\tilde{M}$ and $\partial_{c'} g_{\bar{b}\bar{a}}=0$ and $\partial_{\gamma} g_{\bar{b}\bar{a}}=0$ show that the metric induced on $\hat{M}_1$ is independent of $x^{c'}$ and $'x^{\alpha}$, $\hat{M}_2$ is totally geodesic because $l_{\gamma\beta\bar{a}}=0$ and $h_{c'\bar{a}}{}^{\alpha}=0$.

On the other hand, we can suppose $\Pi_c{}^{\alpha}=0$ and thus, taking account of (3.15) and $l_{\gamma\beta\bar{a}}=0$, we have $\partial_{\bar{a}} g_{\gamma\beta}=0$. Thus the metric of $\hat{M}_2$ is independent of $x^{\bar{a}}$ and therefore $\tilde{M}$ is the Riemannian product of $\hat{M}_1$ and $\hat{M}_2$.

REMARK. The almost complex structure induced on $\hat{M}_2$ is given by

$$\begin{pmatrix} 0 & f_{\beta}{}^{a'} \\ f_{b'}{}^{\alpha} & 0 \end{pmatrix}$$

and the connection which makes invariant the almost complex structure is given by the last two equations of the equations above.

BIBLIOGRAPHY

[1] EARLE, C. J., AND J. EELLS, JR., Foliations and fibrations. J. Diff. Geom. **1** (1967), 33–41.

[2] HERMANN, R., A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. Proc. Amer. Math. Soc. **11** (1960), 236–242.

[3] ISHIHARA, S., Normal structure f satisfying $f^3+f=0$. Kōdai Math. Sem. Rep. **18** (1966), 36–47.

[4] JAMES, I., AND E. THOMAS, Submersions and immersions of manifolds. Inventiones Math. **2** (1967), 171–177.

[5] MUTŌ, Y., On some properties of a fibred Riemannian manifold. Sci. Rep. Yokohama Nat. Univ. Sem. **1** (1952), 1–14.

[6] NAGANO, T., On fibred Riemann manifolds. Sci. Rep. Coll. Gen. Ed. Univ. of Tokyo **10** (1960), 17–27.

[7] OKUBO, T., Fibred spaces with almost Hermitian metrics whose base spaces admit almost contact metric structure. To appear in Math. Annalen.

[8] O'NEIL, B., The fundamental equations of a submersion. Mich. Math. J. **13** (1966), 459–469.

[9] PHILLIPS, A., Submersions of open manifolds. Topology **6** (1966), 171–206.

[10] REINHART, B. L., Foliated manifolds with bundle-like metrics. Ann. of Math. **69** (1959), 119–132.

[11] SCHOUTEN, J. A., Ricci-Calculus. 2nd Ed., Springer (1954).

[12] YANO, K., Theory of Lie derivatives and its applications. Amsterdam (1957).

[13] YANO, K., Differential geometry on complex and almost complex spaces. Pergamon Press (1965).

[14] YANO, K. AND E. T. DAVIES, On some local properties of fibred spaces, Kōdai Math. Sem. Rep. **11** (1959), 158–177.

[15] YANO, K. AND S. ISHIHARA, Fibred spaces with projectable Riemannian metric.

J. Diff. Geom. **1** (1967), 71–88.

[16] YANO, K. AND S. ISHIHARA, Fibred spaces with invariant Riemannian metric. Kōdai Math. Sem. Rep. **19** (1967), 317–360.

[17] YOKOTE, I., On some properties of curvatures of foliated Riemannian manifolds, Kōdai Math. Sem. Rep. **22** (1970), 1–29.

UNIVERSITY OF ELECTRO-COMMUNICATIONS.