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MINIMAL SURFACES IN 4-DIMENSIONAL RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

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For surfaces immersed in a Euclidean 4-space E^4, Little [5] proved the following

THEOREM. *Let x: $M \to E^4$ be an immersion of a compact orientable surface in E^4. Suppose that K_N, the curvature of the connection in the normal bundle, is everywhere positive (negative), then x is inflection free immersion and furthermore $\chi(N) = -2\chi(M)$ $(\chi(N) = 2\chi(M))$, where $\chi(M)$ and $\chi(N)$ denote the Euler characteristics of M and the normal bundle over M respectively.*

Furthermore he brought forward a problem to find examples of immersions with everywhere positive K_N.

In the present paper, for compact surfaces minimally immersed in a 4-dimensional Riemannian manifold of constant curvature, we shall find surfaces with non-zero constant K_N. Our main result is the following

THEOREM A. *Let M be a 2-dimensional compact Riemannian manifold which is minimally immersed in a 4-dimensional unit sphere S^4. If K_N is non-zero constant everywhere on M, then we may regard M as a Veronese surface in S^4.*

Solving directly the differential equations (2.6) or (2.7) of Theorem B in the same way as [4] or [7], we can verify that M may be regarded as a Veronese surface in S^4.

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§ 1. Preliminaries. Let $\bar{M} = \bar{M}^4(\bar{c})$ be a 4-dimensional Riemannian manifold of constant curvature $\bar{c}$ and $M = M^2$ be a 2-dimensional compact Riemannian manifold immersed in $\bar{M}$ with the induced Riemannian structure through an immersion x: $M \to \bar{M}$. Let $F(M)$ and $F(\bar{M})$ be the orthonormal frame bundles over M and $\bar{M}$ respectively. We denote by F_ν the bundle of normal frames. Let B be the set of all element $b = (p, e_1, e_2, e_3, e_4)$ such that $(p, e_1, e_2) \in F(M)$ and $(p, e_1, e_2, e_3, e_4) \in F(\bar{M})$, identifying $p \in M$ with $x(p)$ and e_i with $dx(e_i)$, $i = 1, 2$. Then B is naturally considered as a smooth submanifold of $F(\bar{M})$. Let $\bar{\omega}_A$,

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$\bar{\omega}_{AB}=-\bar{\omega}_{BA}$, $A, B=1, 2, 3, 4$, be the basic and connection forms of $\bar{M}$ on $F(\bar{M})$ which satisfy the structure equations

$$d\bar{\omega}_A=\sum_B \bar{\omega}_{AB}\wedge\bar{\omega}_B, \qquad d\bar{\omega}_{AB}=\sum_C \bar{\omega}_{AC}\wedge\bar{\omega}_{CB}-\bar{c}\bar{\omega}_A\wedge\bar{\omega}_B. \tag{1.1}$$

Then, deleting the bars of $\bar{\omega}_A$, $\bar{\omega}_{AB}$ on B, we have

$$\omega_\alpha=0, \qquad \omega_{i\alpha}=\sum_j A_{\alpha ij}\omega_j, \qquad A_{\alpha ij}=A_{\alpha ji}, \tag{1.2}$$

and

$$\begin{aligned} &d\omega_i=\omega_{ij}\wedge\omega_j,\\ &d\omega_{ij}=\sum_\alpha \omega_{i\alpha}\wedge\omega_{\alpha j}-\bar{c}\omega_i\wedge\omega_j,\\ &d\omega_{i\alpha}=\omega_{ij}\wedge\omega_{j\alpha}+\omega_{i\beta}\wedge\omega_{\beta\alpha},\\ &d\omega_{34}=\sum_i \omega_{3i}\wedge\omega_{i4}, \qquad i\neq j, \qquad \alpha\neq\beta, \end{aligned} \tag{1.3}$$

where throughout this paper we use the following ranges of indices:

$$1\leqq i,\ j\leqq 2; \qquad 3\leqq\alpha,\ \beta\leqq 4.$$

Now, ω_{12} and ω_{34} are the connection forms in $F(M)$ and F_ν respectively and $d\omega_{12}$ and $d\omega_{34}$ are curvature forms of these bundles respectively. Making use of (1.2) and (1.3), we may write $d\omega_{12}$ and $d\omega_{34}$ as follows:

$$d\omega_{12}=-K\omega_1\wedge\omega_2, \qquad d\omega_{34}=-K_N\omega_1\wedge\omega_2, \tag{1.4}$$

where K is the Gaussian curvature of M. We regard the second equality as the definition of K_N and call it *the curvature of the connection in the normal bundle*, or simply the *normal scalar curvature* of M in this paper.

§ 2. Minimal surfaces with constant normal scalar curvature $K_N\neq 0$.

In this section, we assume that M is minimal in $\bar{M}$, i.e., trace $A_\alpha=0$, and K_N is non-zero constant on M. Then, making use of (1.2), (1.3), (1.4) and trace $A_\alpha=0$, we obtain

$$K_N=2(A_{311}A_{412}-A_{312}A_{411}). \tag{2.1}$$

Hence it must be

$$m\text{-index}_p M=2 \text{ at each point } p\in M. \tag{2.2}$$

For any matrices $A, B\in S_2$, we define the inner product of A and B by $\langle A, B\rangle=(1/2)\,\mathrm{trace}\,AB$. With respect to this inner product, we denote the square of the length of the system of second fundamental forms by S, i.e.

$$S=\sum_{\alpha=3}^{4}\|A_\alpha\|^2=\frac{1}{2}\sum_{\alpha,i,j}A_{\alpha ij}A_{\alpha ij}.$$

LEMMA 1. *If M is minimal in $\bar{M}$, the set $E=\{p\in M|\langle A_3, A_4\rangle=0,\ ||A_3||=||A_4||$ at $p\}$ coincides with the set $\{p\in M|S^2-K_N^2=0$ at $p\}$.*

Proof. Since trace $A_\alpha=0$, we have

$$\begin{aligned}S^2-K_N^2&=(||A_3||^2+||A_4||^2)^2-\{2(A_{311}A_{412}-A_{312}A_{411})\}^2\\&=\{(A_{311}+A_{412})^2+(A_{312}-A_{411})^2\}\{(A_{311}-A_{412})^2+(A_{312}+A_{411})^2\}.\end{aligned}\tag{2.3}$$

Hence, if we have $S^2-K_N^2=0$ at p, then we have

$$A_{311}+A_{412}=A_{312}-A_{411}=0 \qquad \text{or} \qquad A_{311}-A_{412}=A_{312}+A_{411}=0,$$

which implies that $\langle A_3, A_4\rangle=0$ and $||A_3||=||A_4||$. It follows easily from the above equations that if $\langle A_3, A_4\rangle=0$ and $||A_3||=||A_4||$, then we have $S^2-K_N^2=0$. Q.E.D.

Since S and K_N are differentiable functions on M, E is closed in M. Hence $M-E$ is open in M.

LEMMA 2. *If M is minimal in $\bar{M}$ and $K_N\neq 0$, then in $M-E$ we can choose locally frames $b\in B$ such that*

$$\begin{aligned}&\omega_{13}=h_1\omega_1, \qquad \omega_{14}=h_2\omega_2,\\&\omega_{23}=-h_1\omega_2, \qquad \omega_{24}=h_2\omega_1, \qquad h_1^2>h_2^2>0.\end{aligned}\tag{2.4}$$

Proof. Putting $\bar{e}_3=e_3\cos\theta+e_4\sin\theta$ and $\bar{e}_4=-e_3\sin\theta+e_4\cos\theta$, we have

$$\langle\bar{A}_3, \bar{A}_4\rangle=\langle A_3, A_4\rangle\cos 2\theta+\frac{||A_4||^2-||A_3||^2}{2}\sin 2\theta.$$

Since we have

$$\langle A_3, A_4\rangle^2+(||A_4||^2-||A_3||^2)^2\neq 0 \qquad \text{on} \quad M-E,$$

we can choose locally differentiable frame fields e_3 and e_4 such that $\langle A_3, A_4\rangle=0$ and may suppose that $||A_3||^2-||A_4||^2>0$. Therefore we can choose locally frame fields e_1 and e_2 such that $\omega_{13}=h_1\omega_1$ and $\omega_{23}=-h_1\omega_2$, $h_1\neq 0$. Then it follows from $\langle A_3, A_4\rangle=0$ and trace $A_4=0$ that we have $\omega_{14}=h_2\omega_2$ and $\omega_{24}=h_2\omega_1$, $h_2\neq 0$. Since $||A_3||^2-||A_4||^2>0$, we get $h_1^2>h_2^2>0$. Q.E.D.

Now, making use of the above two Lemmas, we shall prove the following

THEOREM B. *Let M be a 2-dimensional compact Riemannian manifold which is minimally immersed in a 4-dimensional Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$. If K_N is non-zero constant on M, then we have*

$$S=\frac{2\bar{c}}{3}>0 \qquad \textit{and} \qquad K=\text{constant}>0.\tag{2.5}$$

Furthermore, the Frenet formulas of M are given by the following:

(I) $K_N>0$:

$$\begin{aligned} &dx=R((e_1+ie_2)(\omega_1-i\omega_2)),\\ (2.6)\qquad &\bar{D}(e_1+ie_2)=-i(e_1+ie_2)\omega_{12}+h(e_3+ie_4)(\omega_1-i\omega_2),\\ &\bar{D}(e_3+ie_4)=-2i(e_3+ie_4)\omega_{12}-h(e_1+ie_2)(\omega_1+i\omega_2),\end{aligned}$$

where we may put $K_N=2h^2$ *and* $\bar{D}$ *denotes the covariant differentiation of* $\bar{M}$.

(II) $K_N<0$:

$$\begin{aligned} &dx=R((e_1+ie_2)(\omega_1-i\omega_2)),\\ (2.7)\qquad &\bar{D}(e_1+ie_2)=-i(e_1+ie_2)\omega_{12}+h(e_3-ie_4)(\omega_1-i\omega_2),\\ &\bar{D}(e_3-ie_4)=-2i(e_3-ie_4)\omega_{12}-h(e_1+ie_2)(\omega_1+i\omega_2),\end{aligned}$$

where we may put $K_N=-2h^2$.

Proof. In the first place we shall prove that $M=E$, namely, $S^2=K_N^2$ on M. Supposing that the open subset $M-E$ of M is not empty, the continuous function $S^2-K_N^2$ is positive there from (2.3). Hence the function $S^2-K_N^2$ takes its positive maximum at some point p_0 in $M-E$. We choose a neighbourhood U of p_0 in $M-E$ such that we have frames $b\in B$ over U which satisfy (2.4). Then, making use of (2.4) and (1.3), we obtain

$$(2.8)\qquad \begin{aligned} &dh_1\wedge\omega_1+(2h_1\omega_{12}-h_2\omega_{34})\wedge\omega_2=0, \cdots\cdots ①\\ &dh_1\wedge\omega_2-(2h_1\omega_{12}-h_2\omega_{34})\wedge\omega_1=0, \cdots\cdots ②\end{aligned}$$

$$(2.9)\qquad \begin{aligned} &dh_2\wedge\omega_1+(2h_2\omega_{12}-h_1\omega_{34})\wedge\omega_2=0, \cdots\cdots ③\\ &dh_2\wedge\omega_2-(2h_2\omega_{12}-h_1\omega_{34})\wedge\omega_1=0, \cdots\cdots ④\end{aligned}$$

and

$$K_N=2h_1h_2.$$

Since K_N is non-zero constant, making ①$\times h_2+$③$\times h_1$ and ②$\times h_2+$④$\times h_1$, we obtain

$$(2.10)\qquad 2K_N\omega_{12}=S\omega_{34}.$$

From (2.10), we get

$$(2.11)\qquad 2K_Nd\omega_{12}=dS\wedge\omega_{34}-K_NS\omega_1\wedge\omega_2.$$

Also, making ①$\times h_1+$③$\times h_2$ and ②$\times h_1+$④$\times h_2$, we have

$$(2.12)\qquad \begin{aligned} dS\wedge\omega_1 + \frac{4(S^2-K_N^2)}{S}\omega_{12}\wedge\omega_2 = 0, \\ dS\wedge\omega_2 - \frac{4(S^2-K_N^2)}{S}\omega_{12}\wedge\omega_1 = 0. \end{aligned}$$

Applying Cartan's Lemma to (2.8), we may put

$$dh_1 = A\omega_1 + B\omega_2,$$

$$2h_1\omega_{12} - h_2\omega_{34} = B\omega_1 - A\omega_2,$$

which imply, together with (2.10), that

$$\frac{2h_1(h_1^2-h_2^2)}{S}\omega_{12} = B\omega_1 - A\omega_2.$$

Since $h_1 \neq 0$ and $h_1^2 - h_2^2 > 0$, we may write $\omega_{12} = A_1\omega_1 + B_1\omega_2$ in U, where A_1 and A_2 are differentiable functions in U. Then, from (2.10) we obtain

$$(2.13)\qquad \omega_{34} = \frac{2K_N}{S}\omega_{12} = \frac{2K_N}{S}(A_1\omega_1 + B_1\omega_2).$$

Hence, by means of (2.11), (2.12) and (2.13), we obtain

$$2K_N d\omega_{12} = -\left\{\frac{8K_N(S^2-K_N^2)(A_1^2+B_1^2)}{S^2} + SK_N\right\}\omega_1\wedge\omega_2.$$

Since $d\omega_{12} = -K\omega_1\wedge\omega_2$, $K_N \neq 0$ and $S>0$, we get

$$2K = \frac{8(S^2-K_N^2)(A_1^2+B_1^2)}{S^2} + S > 0,$$

that is, we get

$$(2.14)\qquad K>0 \quad \text{in } U.$$

On the other hand, making ①$\times h_1 -$ ③$\times h_2$ and ②$\times h_1 -$ ④$\times h_2$, we have

$$(2.15)\qquad \begin{aligned} d(h_1^2-h_2^2)\wedge\omega_1 + 4(h_1^2-h_2^2)\omega_{12}\wedge\omega_2 = 0, \\ d(h_1^2-h_2^2)\wedge\omega_2 - 4(h_1^2-h_2^2)\omega_{12}\wedge\omega_1 = 0. \end{aligned}$$

which imply that there exists a neighbourhood V of p_0 with isothermal coordinates (u, v) such that

$$(2.16)\qquad ds^2 = \lambda\{du^2+dv^2\}, \quad \omega_1 = \sqrt{\lambda}\,du, \quad \omega_2 = \sqrt{\lambda}\,dv, \quad \sqrt{h_1^2-h_2^2}\,\lambda = 1,$$

where $\lambda = \lambda\ (u, v)$ is a positive function in V. With respect to the isothermal coordinates, as is well known, the Gaussian curvature K is given by the following equation

$$K=-\frac{1}{2\lambda}\Delta\log\lambda.$$

Since $\sqrt{h_1^2-h_2^2}\,\lambda=1$ and $(h_1^2-h_2^2)^2=S^2-K_N^2$, we get

$$K=\frac{\sqrt{h_1^2-h_2^2}}{8}\Delta\log(S^2-K_N^2). \tag{2.17}$$

From (2.14) and (2.17), we obtain the inequality

$$\Delta\log(S^2-K_N^2)>0 \quad \text{in} \quad V,$$

which implies that the function $\log(S^2-K_N^2)$ is a subharmonic function in V. On the other hand, by the assumption the function $S^2-K_N^2$ takes its positive maximum at p_0, and hence $\log(S^2-K_N^2)$ takes also its maximum at p_0. By a well-known theorem on subharmonic functions it must be constant in V, which implies, together with (2.17),

$$K=0 \quad \text{in} \quad V.$$

This contradicts (2.14). Thus it must be $M-E=\emptyset$, that is $M=E$. Hence we get

$$S^2=K_N^2 \quad \text{on} \quad M. \tag{2.18}$$

Since K_N is constant, S is also so. As stated in Lemma 1, on $E=M$ we have $\langle A_3, A_4\rangle=0$ and $||A_3||=||A_4||$ for any frames $b\in B$. Hence we can choose frames $b\in B$ in a neighbourhood of a point of M such that

$$\begin{aligned} &\omega_{13}=h_1\omega_1, \qquad \omega_{14}=h_2\omega_2, \qquad h_1^2=h_2^2\neq 0,\\ &\omega_{23}=-h_1\omega_2, \qquad \omega_{24}=h_2\omega_1, \end{aligned} \tag{2.19}$$

where h_1 and h_2 are local differentiable functions in M. In this case, we have $S=h_1^2+h_2^2$ and $K_N=2h_1h_2$, so that h_1 and h_2 are constant, because S and K_N are constant. Making use of (2.19) and (1.3), we obtain

$$\begin{aligned} &2h_1\omega_{12}-h_2\omega_{34}=0,\\ &2h_2\omega_{12}-h_1\omega_{34}=0, \end{aligned} \tag{2.20}$$

from which we get

$$2K_N\omega_{12}=S\omega_{34}. \tag{2.21}$$

Then we have

$$2K_N d\omega_{12}=Sd\omega_{34}=-SK_N\omega_1\wedge\omega_2,$$

which implies

$$K=\frac{S}{2}>0.$$

On the other hand, making use of (1.3), we get

$$d\omega_{12}=-(\bar{c}-S)\omega_1\wedge\omega_2,$$

which implies

$$K=\bar{c}-S.$$

Thus we get

$$S=\frac{2\bar{c}}{3}, \tag{2.22}$$

hence $\bar{c}$ is a positive constant. Since K_N is non-zero, we can consider the following two cases:

Case (I) $K_N>0$; In this case, we have $h_1=h_2$. From (2.21) we obtain

$$\omega_{34}=2\omega_{12}.$$

Also we may write $K_N=2h^2$, where $h_1=h_2=h$. Then we get the following Frenet formulas of M:

$$dx=e_1\omega_1+e_2\omega_2,$$

$$\bar{D}e_1=\omega_{12}e_2+h\omega_1e_3+h\omega_2e_4,$$

$$\bar{D}e_2=-\omega_{12}e_1-h\omega_2e_3+h\omega_1e_4,$$

$$\bar{D}e_3=2\omega_{12}e_4-h\omega_1e_1+h\omega_2e_2,$$

$$\bar{D}e_4=-2\omega_{12}e_3-h\omega_2e_1-h\omega_1e_2,$$

which are reduced to (2.6).

Case (II) $K_N<0$. Since we have $h_1=-h_2$, from (2.21) we get

$$\omega_{34}=-2\omega_{12}$$

and we may set $h=h_1=-h_2$, i.e., $K_N=-2h^2$. Then we get the following Frenet formulas of M:

$$dx=e_1\omega_1+e_2\omega_2,$$

$$\bar{D}e_1=\omega_{12}e_2+h\omega_1e_3-h\omega_2e_4,$$

$$\bar{D}e_2=-\omega_{12}e_2-h\omega_2e_3-h\omega_1e_4,$$

$$\bar{D}e_3=-2\omega_{12}e_4-h\omega_1e_1+h\omega_2e_2,$$

$$\bar{D}e_4=2\omega_{12}e_3+h\omega_2e_1+h\omega_1e_2,$$

which are reduced to (2.7). Q.E.D.

Since $\bar{c}>0$, we may put $\bar{c}=1$. Then we may regard as $\bar{M}^4=S^4$ (unit sphere) and have

$$S=\frac{2}{3}, \quad K=\frac{1}{3} \quad \text{and} \quad K_N=\pm\frac{2}{3}.$$

Hence we can solve the differential equations (2.6) and (2.7) in the same way as [4] or [7] and verify that M may be regarded as a Veronese surface in S^4.

We remark that Theorem A also follows from Theorem B and the fact that S of [1] is identically 4/3 on M.

REFERENCES

[1] CHERN, S. S., M. DO CARM, AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length. Functional Analysis and Related Fields, Springer-Verlag, (1970), 59–75.

[2] ITOH, T., Complete surfaces in E^4 with constant mean curvature. Kōdai Math. Sem. Rep. **22** (1970) 150–158.

[3] ———, A note on minimal submanifolds with M-index 2. Kōdai Math. Sem. Rep. **23** (1971) 204–207.

[4] ———, Minimal surfaces with M-index 2, T_1-index 2 and T_2-index 2. Kōdai Math. Sem. Rep. **24** (1972) 1–16.

[5] LITTLE, J. A., On singularities of submanifolds of higher dimensional Euclidean spaces. Annli Math., (1969), 261–336.

[6] ŌTSUKI, T., On minimal submanifolds with M-index 2. J. Differential Geometry, **6** (1971), 193–211.

[7] ———, Minimal submanifolds with M-index 2 in Riemannian Manifolds of constant curvature. Tōhoku Math. J. **23** (1971), 371–402.

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