YANO, K., AND S. ISHIHARA
KŌDAI MATH. SEM. REP.
24 (1972), 422—429

NOTES ON HYPERSURFACES OF AN ODD-DIMENSIONAL SPHERE

BY KENTARO YANO AND SHIGERU ISHIHARA

Dedicated to Professor Y. Muto on his sixtieth birthday

Blair [1, 2, 3, 4, 5], Ki [6], Ludden [1, 2, 3, 4, 5], Okumura [7, 8] and one of the present authors [2, 3, 4, 5, 6, 7, 8] started the study of a structure induced on a hypersurface of an almost contact manifold or a submanifold of codimension 2 of an almost complex manifold. When the ambient manifold admits a Riemannian metric, the structure induced is called an (f, g, u, v, λ)-structure [7, 8], where f is a tensor field of type (1, 1), g the induced Riemannian metric, u and v two 1-forms and λ a function.

Since the odd-dimensional sphere S^{2n+1} has an almost contact structure naturally induced from the Kähler structure of Euclidean space E^{2n+2}, a hypersurface immersed in S^{2n+1} admits a so-called (f, g, u, v, λ)-structure.

In [3], Blair, Ludden and one of the present authors proved

THEOREM. *If M^{2n} is a complete orientable hypersurface of S^{2n+1} of constant scalar curvature satisfying $Kf+fK=0$ and $\lambda \neq$ constant, where K is the Weingarten map of the embedding, then M^{2n} is a natural sphere S^{2n} or $M^{2n}=S^n\times S^n$.*

The purpose of the present notes is to show that if M^{2n} is a real analytic complete orientable hypersurface of a unit sphere $S^{2n+1}(1)$ satisfying $Kf+fK=0$ and $\lambda\neq$ constant and if

$$K_{ji}=\frac{1}{2n}kg_{ji}$$

holds at a point of M^{2n} at which $1-\lambda^2\neq 0$, K_{ji} and k being the Ricci tensor and the scalar curvature of M^{2n} respectively, then M^{2n} is, provided $n>1$, either a great sphere $S^{2n}(1)$ of $S^{2n+1}(1)$ or the product of two n-dimensional spheres $S^n(1/\sqrt{2})$ of radius $1/\sqrt{2}$.

§ 1. Preliminaries.

We consider a $2n$-dimensional submanifold M^{2n} immersed differentiably in a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$ embedded in a $(2n+2)$-dimensional Eucli-

Received August 13, 1971.

dean space E^{2n+2} and denote by X: $M^{2n}\to E^{2n+2}$ the immersion of M^{2n} into E^{2n+2}, where X is regarded as the position vector with its initial point at the origin of E^{2n+2} and its terminal point at a point of $X(M^{2n})$. Submanifolds we consider are assumed to be orientable, connected and differentiable and of class C^∞. Suppose that M^{2n} is covered by a system of coordinate neighborhoods $\{U; x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$, and that M^{2n} is orientable. Then, denoting by C the unit normal $-X$ to S^{2n+1} defined globally along $X(M^{2n})$, we can choose another unit normal D to $X(M^{2n})$ globally along $X(M^{2n})$ in such a way that C and D are mutually orthogonal along $X(M^{2n})$. If we put

$$X_i=\partial_i X, \qquad \partial_i=\partial/\partial x^i, \tag{1.1}$$

then components g_{ji} of the induced metric tensor of M^{2n} are given by

$$g_{ji}=X_j\cdot X_i,$$

where the dot denotes the inner product in E^{2n+2}.

We denote by $\{{}_j{}^h{}_i\}$ the Christoffel symbols formed with g_{ji} and by ∇_j the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$. We then have the equations of Gauss

$$\nabla_j X_i=\partial_j X_i-\left\{ {h \atop j\ i} \right\} X_h=g_{ji}C+k_{ji}D, \tag{1.2}$$

where k_{ji} are components of the second fundamental tensor with respect to the unit normal D, and the equations of Weingarten

$$\nabla_j C=-X_j, \qquad \nabla_j D=-k_j{}^i X_i, \tag{1.3}$$

where $k_j{}^i=k_{jt}g^{ti}$ and $(g^{ji})=(g_{ji})^{-1}$, because the connection $\tilde{\nabla}$ induced in the normal bundle of the submanifold M^{2n} relative to E^{2n+2} is locally flat and C and D are parallel with respect to $\tilde{\nabla}$. We also have the structure equations of the submanifold M^{2n}, i.e., the equations of Gauss

$$K_{kji}{}^h=\delta_k^h g_{ji}-\delta_j^h g_{ki}+k_k{}^h g_{ji}-k_j{}^h g_{ki}, \tag{1.4}$$

where

$$K_{kji}{}^h=\partial_k\left\{ {h \atop j\ i} \right\}-\partial_j\left\{ {h \atop k\ i} \right\}+\left\{ {h \atop k\ t} \right\}\left\{ {t \atop j\ i} \right\}-\left\{ {h \atop j\ t} \right\}\left\{ {t \atop k\ i} \right\}$$

are components of the curvature tensor of M^{2n}, and the equations of Codazzi

$$\nabla_k k_{ji}-\nabla_j k_{ki}=0. \tag{1.5}$$

Now, the $(2n+2)$-dimensional Euclidean space E^{2n+2} has a natural Kähler structure F, i.e., a tensor field F of type (1, 1) with constant components such that

$$F^2=-1, \qquad (FX)\cdot X=0, \qquad (FX)\cdot(FY)=X\cdot Y$$

for any vector fields X and Y in E^{2n+2}, where 1 denotes the unit tensor of type (1, 1). Thus we can put

$$(1.6) \qquad \begin{aligned} FX_i &= f_i{}^h X_h + u_i C + v_i D, \\ FC &= -u^i X_i + \lambda D, \\ FD &= -v^i X_i - \lambda C, \end{aligned}$$

where $f_i{}^h$ are components of a tensor field of type (1, 1), u_i and v_i components of 1-forms and λ a function in M^{2n}, u^h and v^h being defined respectively by

$$u^h = g^{ht} u_t, \qquad v^h = g^{ht} v_t.$$

From equations (1. 6), we find

$$(1.7) \qquad \begin{aligned} & f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h, \\ & u_i f_j{}^i = \lambda v_j, \qquad v_i f_j{}^i = -\lambda u_j, \\ & f_i{}^h u^i = -\lambda v^h, \qquad f_i{}^h v^i = \lambda u^h, \\ & u_i u^i = v_i v^i = 1-\lambda^2, \qquad u_i v^i = 0, \\ & g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i. \end{aligned}$$

The set of a tensor field $f_i{}^h$, a Riemannian metric g_{ji}, two 1-forms u_i and v_i and a function λ is called a *(f, g, u, v, λ)-structure* in M^{2n} [6, 7, 8], if they satisfy the equations (1. 7).

Differentiating (1. 6) covariantly and taking account of equations (1. 2) of Gauss and equations (1. 3) of Weingarten, we find

$$(1.8) \qquad \begin{aligned} & \nabla_j f_i{}^h = -g_{ji} u^h + \delta_j^h u_i - k_{ji} v^h + k_j{}^h v_i, \\ & \nabla_j u_i = f_{ji} - \lambda k_{ji}, \\ & \nabla_j v_i = -k_{jt} f_i{}^t + \lambda g_{ji}, \\ & \nabla_j \lambda = -v_j + k_j{}^t u_t, \end{aligned}$$

where $f_{ji} = f_j{}^t g_{ti}$ are skew-symmetric [6, 7, 8].

Denoting by M_0 and M_1 the submanifold of M^{2n} defined respectively by

$$M_0 = \{p \in M^{2n} | \lambda(p) \neq 0\}, \qquad M_1 = \{p \in M^{2n} | 1-\lambda(p)^2 \neq 0\},$$

we assume that $M_0 \cap M_1$ is dense in M^{2n}, i.e., that $\lambda(1-\lambda^2) \neq 0$ holds almost everywhere in M^{2n}.

§ 2. Certain hypersufaces of an odd-dimensional unit sphere.

We assume in this section that the two tensor fields f_i^h and k_i^h of type (1, 1) are anti-commutative, i.e.,

$$f_j^t k_t^h + k_j^t f_t^h = 0, \tag{2. 1}$$

which is equivalent to

$$k_{jt} f_i^t - k_{it} f_j^t = 0, \tag{2. 2}$$

since f_{ji} is skew-symmetric. Transvecting (2. 2) with u^i, we obtain

$$-\lambda(k_{jt}v^t) - (k_{it}u^i) f_j^t = 0 \tag{2. 3}$$

and, transvecting (2. 2) with v^i,

$$\lambda(k_{jt}u^t) - (k_{it}v^i) f_j^t = 0. \tag{2. 4}$$

Transvecting (2. 3) with v^j, we have

$$-\lambda k_{ji}v^j v^i - \lambda k_{ji}u^j u^i = 0,$$

from which,

$$k_{ji}v^j v^i + k_{ji}u^j u^i = 0. \tag{2. 5}$$

Next, changing indices in (2. 3), we have

$$\lambda(k_{st}v^t) + (k_{it}u^i) f_s^t = 0$$

and, transvecting this with f_j^s,

$$(1-\lambda^2) k_{ji}u^i = (k_{ts}u^t u^s) u_j + (k_{ts}u^t v^s) v_j.$$

Similarly, using (2. 4), we obtain

$$(1-\lambda^2) k_{ji}v^i = (k_{ts}u^t v^s) u_j + (k_{ts}v^t v^s) v_j.$$

Thus, in M_1, we can put

$$k_i^t u_t = \alpha u_i + \beta v_i, \tag{2. 6}$$

$$k_i^t v_t = \beta u_i - \alpha v_i \tag{2. 7}$$

because of (2. 5), where α and β are functions defined in M_1.

Differentiating (2. 6) covariantly and using (1. 8), we have

$$(\nabla_j k_i^t) u_t + k_i^t (f_{jt} - \lambda k_{jt})$$
$$= (\nabla_j \alpha) u_i + \alpha(f_{ji} - \lambda k_{ji}) + (\nabla_j \beta) v_i + \beta(-k_{jt} f_i^t + \lambda g_{ji}),$$

from which, taking the skew-symmetric part with respect to j and i and taking account of equations (1. 5) of Codazzi and (2. 2),

$$(\nabla_j\alpha)u_i-(\nabla_i\alpha)u_j+(\nabla_j\beta)v_i-(\nabla_i\beta)v_j+2\alpha f_{ji}=0. \tag{2.8}$$

Transvecting this with $u^j v^i$, we find

$$(1-\lambda^2)\{-v^i\nabla_i\alpha+u^i\nabla_i\beta-2\alpha\lambda\}=0,$$

from which,

$$v^i\nabla_i\alpha-u^i\nabla_i\beta+2\alpha\lambda=0. \tag{2.9}$$

Transvecting (2. 8) with u^i, we obtain

$$(1-\lambda^2)\nabla_j\alpha-(u^i\nabla_i\alpha)u_j-(u^i\nabla_i\beta)v_j+2\alpha\lambda v_j=0,$$

from which, using (2. 9),

$$(1-\lambda^2)\nabla_j\alpha=(u^i\nabla_i\alpha)u_j+(v^i\nabla_i\alpha)v_j. \tag{2.10}$$

Similarly, tranvecting (2. 8) with v^j, we have

$$(1-\lambda^2)\nabla_j\beta=(u^i\nabla_i\beta)u_j+(v^i\nabla_i\beta)v_j. \tag{2.11}$$

Thus, multiplying (2. 8) by $(1-\lambda^2)$ and using (2. 10) and (2. 11), we have

$$2\alpha(1-\lambda^2)f_{ji}=(v^t\nabla_t\alpha-u^t\nabla_t\beta)(u_jv_i-u_iv_j).$$

Since the rank of f_{ji} is $2n-2$ in M_1, we find, if $n>1$,

$$\alpha=0, \qquad u^t\nabla_t\beta=0. \tag{2.12}$$

Thus equations (2. 6) and (2. 7) become respectively

$$k_i{}^t u_t=\beta v_i, \qquad k_i{}^t v_t=\beta u_i \tag{2.13}$$

and equations (2. 11) become

$$(1-\lambda^2)\nabla_j\beta=(v^i\nabla_i\beta)v_j. \tag{2.14}$$

Now, transvecting (2. 2) with $f_h{}^j$ and taking account of (2. 13), we obtain

$$k_{ts}f_i{}^t f_h{}^s+k_{ih}-\beta(u_iv_h+u_hv_i)=0,$$

from which, transvecting with g^{ih},

$$g^{ji}k_{ji}=0$$

in M_1. Since M_1 is dense in M^{2n}, we have

PROPOSITION 2. 1. *A hypersurface of a $(2n+1)$-dimensional unit sphere, for* *which f_i^h and k_i^h anticommute, is minimal if $n>1$.*

If we now differentiate the second equation of (2. 13) covariantly and take account of (1. 8), we find

$$(\nabla_j k_i^t)v_t + k_i^t(-k_{js}f_t^s + \lambda g_{jt}) = (\nabla_j\beta)u_i + \beta(f_{ji} - \lambda k_{ji}),$$

from which, taking the skew-symmetric part with respect to j and i and taking account of (1. 5) and (2. 2),

$$(\nabla_j\beta)u_i - (\nabla_i\beta)u_j - 2f_{ts}k_j^t k_i^s + 2\beta f_{ji} = 0. \tag{2. 15}$$

Transvecting this with u^i, we obtain

$$(1-\lambda^2)\nabla_j\beta - (u^i\nabla_i\beta)u_j + 2\beta^2\lambda v_j + 2\beta\lambda v_j = 0,$$

or, using (2. 12),

$$(1-\lambda^2)\nabla_j\beta = -2\beta(\beta+1)\lambda v_j. \tag{2. 16}$$

Thus we see that, if β is constant in M_1, then $\beta=0$ or $\beta=-1$ in $M_0 \cap M_1$.

We now suppose that $\beta=0$ or $\beta=-1$ at a point p belonging to $M_0 \cap M_1$. Then the equation (2. 16) shows that all of successive covariant derivatives of β vanish at the point p, i.e., that

$$\nabla_i\beta = 0, \qquad \nabla_j\nabla_i\beta = 0, \qquad \nabla_k\nabla_j\nabla_i\beta = 0, \cdots$$

hold at the point p. Thus, if M^{2n} is a real analytic submanifold, then $\beta=0$ or $\beta=-1$ at every point of M_1. Then we have

LEMMA 2. 2. *If M^{2n} is a real analytic submanifold and $\beta=0$ (resp. $\beta=-1$) at a point of $M_0 \cap M_1$, then $\beta=0$ (resp. $\beta=-1$) holds at every point of M_1, provided $n>1$.*

From equations (1. 4) of Gauss, we have

$$K_{ji} = (2n-1)g_{ji} - k_{jt}k_i^t \tag{2. 17}$$

by virtue of (2. 15) and hence

$$k = 2n(2n-1) - k_{ji}k^{ji}, \tag{2. 18}$$

where K_{ji} and k are respectively the Ricci tensor and the curvature scalar of M^{2n}. On the other hand, multiplying (2. 15) by $(1-\lambda^2)$ and using (2. 16), we have

$$2\beta(\beta+1)\lambda(u_jv_i - u_iv_j) - 2(1-\lambda^2)f_{ts}k_j^t k_i^s + 2\beta(1-\lambda^2)f_{ji} = 0,$$

from which, transvecting with f_h^i,

(2. 19) $$(1-\lambda^2)k_{it}k_h{}^t=\beta(\beta+1)(u_iu_h+v_iv_h)-\beta(1-\lambda^2)g_{ih},$$

from which,

(2. 20) $$k_{ji}k^{ji}=2\beta(\beta-n+1).$$

We now consider the following equation:

(2. 21) $$K_{ji}=\frac{1}{2n}kg_{ji}$$

at a point p of $M_0\cap M_1$. Then, from (2. 17) and (2. 18), we see that (2. 21) is equivalent to the condition

(2. 22) $$k_{jt}k_i{}^t=cg_{ji},$$

c being a certain constant. Subsituting (2. 22) into (2. 19), we find

$$(1-\lambda^2)(\beta+c)g_{ih}=\beta(\beta+1)(u_iu_h+v_iv_h),$$

which implies $\beta=0$ or $\beta=-1$ at the point p, provided $n>1$. Conversely, if we suppose that $\beta=0$ or $\beta=-1$ at a point p, then we have (2. 21) at the point p, by virtue of (2. 17), (2. 18), (2. 19) and (2. 20). Thus we have

LEMMA 2. 3. *The equation* (2. 21) *holds at a point p of M_1, provided $n>1$, if and only if $\beta=0$ or $\beta=-1$ at the point p.*

It has been proved in [3]

LEMMA 2. 4. *Let M^{2n} $(n>1)$ be complete, $\lambda\neq$ constant and $\lambda(1-\lambda^2)\neq 0$ almost everywhere in M^{2n}. If $\beta=0$ at every point of M_1, then M^{2n} is a great sphere $S^{2n}(1)$ in the unit sphere $S^{2n+1}(1)$. If $\beta=-1$ at every point of M_1, then M^{2n} is the product of two n-dimensional spheres $S^n(1/\sqrt{2})$ of radius $1/\sqrt{2}$.*

Therefore, from Lemmas 2. 2, 2. 3 and 2. 4, we have

THEOREM. *Suppose that a complete orientable $2n$-dimensional manifold M^{2n} is embedded in a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$, $\lambda(1-\lambda^2)\neq 0$ almost everywhere in M^{2n} and the structure tensor $f_i{}^h$ and the second fundamental tensor $k_i{}^h$ of M^{2n} anticommute. If M^{2n} is a real analytic hypersurface in $S^{2n+1}(1)$ and*

$$K_{ji}=\frac{1}{2n}kg_{ji}$$

holds at a point of M^{2n} at which $1-\lambda^2\neq 0$, then M^{2n} is, provided $n>1$, either a great sphere $S^{2n}(1)$ of $S^{2n+1}(1)$ or the product of two n-dimensional spheres $S^n(1/\sqrt{2})$ of radius $1/\sqrt{2}$.

BIBLIOGRAPHY

[1] BLAIR, D. E., AND G. D. LUDDEN, On intrinsic structures similar to those induced on S^{2n}. Kōdai Math. Sem. Rep. **23** (1971), 262–266.

[2] BLAIR, D. E., G. D. LUDDEN, AND K. YANO, Induced structures on submanifolds. Kōdai Math. Sem. Rep., **22** (1970), 188–198.

[3] BLAIR, D. E., G. D. LUDDEN, AND K. YANO, Hypersurfaces of an odd-dimensional sphere. J. of Differential Geometry **5** (1971), 479–486.

[4] BLAIR, D. E., G. D. LUDDEN, AND K. YANO, On the intrinsic geometry of $S^n \times S^n$. Math. Ann. **194** (1971), 68–77.

[5] BLAIR, D. E., G. D. LUDDEN, AND K. YANO, Hypersurfaces with a parallel vector field in an odd-dimensional sphere. Differential Geometry, in honor of Kentaro Yano, Kinokuniya, Tokyo, (1972), 33–39.

[6] YANO, K., AND U-HANG KI, On quasi-normal (f, g, u, v, λ)-structures. Kōdai Math. Sem. Rep. **24** (1972), 106–120.

[7] YANO, K., AND M. OKUMURA, On (f, g, u, v, λ)-structures. Kōdai Math. Sem. Rep. **22** (1970), 401–423.

[8] YANO, K., AND M. OKUMURA, On normal (f, g, u, v, λ)-structures on submanifolds of codimension 2 in a Euclidean space. Kōdai Math. Sem. Rep. **23** (1971), 172–197.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.