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KŌDAI MATH. SEM. REP.
24 (1972), 410–421

THE KERNEL FUNCTIONS OF SZEGÖ TYPE ON RIEMANN SURFACES

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§ 1. Introduction.

1. The Bergman kernel and the Szegö kernel are typical kernel functions in complex analysis. The Bergman kernel is considered on an arbitrary Riemann surface, but in most cases the Szegö kernel is only considered on a regular region in the plane. Ozawa [9] considered the Szegö kernel on some infinitely connected plane region, and Hawly-Schiffer [6] considered it on a compact bordered Riemann surface by regarding it as a half-order differential. However, it is difficult to consider it on an arbitrary plane region and the latter method does not lead to study of kernel functions, but to study of half-order differentials. Under these circumstances, a kernel function called the Rudin kernel whose existence was pointed out by Rudin [11] is very interesting.

Let S be an open Riemann surface and let $H_2(S)$ be the class of analytic functions on S for which $|f|^2$ has a harmonic majorant on S. For fixed $x, t \in S$, there exists the Rudin kernel $R_t(\tau, x)$ $(\in H_2(S))$ which is characterized by the following reproducing property:

$$f(x) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial S_n} f(\tau) \overline{R_t(\tau, x)} \frac{\partial g_n(\tau, t)}{\partial \nu} ds_\tau,$$

for all $f \in H_2(S)$. Here $\{S_n\}_{n=0}^{\infty}$ $(x, t \in S_n)$ is an arbitrary regular exhaustion of S, ∂S_n is the relative boundary of S_n, $g_n(\tau, t)$ is the Green function of S_n with pole at t and the derivative is taken along the inner normal. The limit exists and is independent of the choice of regular exhaustions $\{S_n\}_{n=0}^{\infty}$.

The object of this paper is to give some fundamental properties of the Rudin kernel and some related reproducing kernels on a compact bordered Riemann surface.

§ 2. Preliminaries.

2. For any Riemann surface S, let $H_p(S)$ $(p>0)$ be the class of analytic functions on S such that $|f|^p$ has harmonic majorants u on S. Let u_f denote the least harmonic majorant of $|f|^p$. Then we define H_p-norm with respect to t by

Received August 6, 1971.

$||f||_p^t = u_f(t)^{1/p}$ for any fixed point t on S. Here $u_f(t)$ is represented by any regular exhaustion $\{S_n\}_{n=0}^{\infty}$ as follows:

$$u_f(t) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{\partial S_n} |f(\tau)|^p \frac{\partial g_n(\tau, t)}{\partial \nu} ds_\tau.$$

$H_p(S)$ is analytically invariant (cf. [1]) and the norm is conformally invariant. Let $H_p^t(S)$ be the linear space of all $f \in H_p(S)$ with finite norm at $t \in S$. The normed space $H_p^t(S)$ is a complete separable linear Hausdorff space and if $p \geqq 1$, $H_p^t(S)$ is a Banach space; if $p>1$, it is uniformly convex. Let $H_p^{t1}(S)$ be the unit ball $||f||_p^t \leqq 1$. The functions $f \in H_p^{t1}(S)$ are locally uniformly bounded and therefore $H_p^{t1}(S)$ is a normal family.

§ 3. Extremal problems in $H_p^{t1}(S)$ ($p \geqq 1$) and constructions of kernels.

The existence of the Rudin kernel is easily assured by a general theory of reproducing kernels [2]. An important problem in the theory of kernel functions is to construct the so-called adjoint L-kernel ([12], pp. 40–46). Hence, we consider extremal problems in $H_p^{t1}(S)$ under the duality relation ([5], [8]), which gives basic meanings of the relation between K-kernel and L-kernel. Here we assume that S is a compact bordered Riemann surface with contours $\{C_m\}_{m=1}^k$ and of genus l.

3. We consider the problem of maximizing $|f(x)|$ in $H_p^{t1}(S)$. The existence of the extremal functions $f_p^*(\tau; x, t)$ is assured by the compactness of $H_p^{t1}(S)$. If $p>1$, the uniform convexity of $H_p^t(S)$ guarantees the uniqueness of the extremal functions except for a rotation [3]. Let T be the linear functional on $H_p(S)$ defined by $Tf = f(x)$. This functional is represented by the Cauchy integral [10]:

$$Tf = f(x) = \frac{1}{2\pi} \int_{\partial S} f(\tau) i dW(\tau, x), \tag{3.1}$$

where $f(\tau)$ means the Fatou boundary values of f at $\tau \in \partial S$. Here $W(\tau, x)$ is defined as follows: $W(\tau, x) = g(\tau, x) + ig^*(\tau, x)$, where $g^*(\tau, x)$ is the harmonic conjugate of $g(\tau, x)$. Since $g(\tau, x) = 0$ on ∂S, we have

$$i dW(\tau, x) = \frac{\partial g(\tau, x)}{\partial \nu} ds_\tau > 0, \qquad (\tau \in \partial S)$$

which implies that $dW(\tau, x)$ has precisely $2l+k-1$ zeros in S. For $\{f(\tau) \in L_p(\partial S) | f \in H_p(S)\}$ is a closed subspace of $L_p(\partial S)$, we apply the Hahn-Banach theorem to extend T from $\{f(\tau) \in L_p(\partial S) | f \in H_p(S)\}$ to $L_p(\partial S)$, preserving its norm, so that T is represented as follows, using the representation theorem of Riesz,

$$T\hat{f} = \frac{1}{2\pi} \int_{\partial S} \hat{f}(\tau) h_p(\tau) i dW(\tau, t), \qquad \text{for all } \hat{f} \in L_p(\partial S), \tag{3.2}$$

where if $p>1$, $h_p \in L_q(\partial S)$, $1/p + 1/q = 1$, and

$$\left(\frac{1}{2\pi}\int_{\partial S}|h_p(\tau)|^q i dW(\tau, t)\right)^{1/q} = |f_p^*(x; x, t)|,$$

and if $p=1$, h_1 is mesurable on ∂S and $|h_1(\tau)|=|f_1^*(x; x, t)|$ a. e. on ∂S. From (3.1) and (3.2), we have

$$\int_{\partial S} f(\tau)\left(\frac{idW(\tau, x)}{idW(\tau, t)} - h_p(\tau)\right) idW(\tau, t) = 0 \qquad \text{for all} \quad f \in H_p(S).$$

It is valid for functions f analytic in S and continuous on $\bar{S}$. Hence from the F. and M. Riesz theorem [14], there exists a function $F \in H_1(S)$ such that

$$\left(\frac{dW(\tau, x)}{dW(\tau, t)} - h_p(\tau)\right) idW(\tau, t) = F(\tau)\omega_0^*(\tau) \qquad \text{a. e. along} \quad \partial S,$$

i.e.

$$h_p(\tau) = \frac{1}{idW(\tau, t)}[idW(\tau, x) - F(\tau)\omega_0^*(\tau)] \qquad \text{a. e. on} \quad \partial S.$$

Here, ω_0^* is defined as follows. For $m=1, 2, \cdots, k$, let ϕ_m be a 1-1 conformal mapping from an annulus $R=\{z|r<|z|<1\}$ onto a neighborhood of C_m such that $\phi_m(\{z|\ |z|=r\})=C_m$, $\phi_m(R)\subset S$ and $\phi_m(\{z|\ |z|=1\})\cap C_m=\phi$. We fix an ω_0 such that ω_0 is a nonvanishing analytic differential on $\bar{S}$. Then in terms of uniformizer $re^{i\theta}=\phi_m^{-1}(\tau)$ in the neighborhood of C_m, ω_0 has the form $a_m(re^{i\theta})dr+b_m(re^{i\theta})d\theta$, we define ω_0^* as $b_m(e^{i\theta})d\theta$. Substitution of f_p^* into (3.2) yields

$$\begin{aligned}
|f_p^*(x; x, t)| &= \left|\frac{1}{2\pi}\int_{\partial S} f_p^*(\tau; x, t) h_p(\tau) idW(\tau, t)\right| \\
&\leqq \frac{1}{2\pi}\int_{\partial S}|f_p^*(\tau; x, t)|\ |h_p(\tau)| idW(\tau, t) \\
&\leqq \|f_p^*(\tau; x, t)\|_p^t \left(\frac{1}{2\pi}\int_{\partial S}|h_p(\tau)|^q idW(\tau, t)\right)^{1/q} \\
&\leqq |f_p^*(x; x, t)| \qquad \text{for} \quad p>1.
\end{aligned}$$

Hence we have for some real θ

$$\begin{aligned}
&f_p^*(\tau; x, t)h_p(\tau) = e^{i\theta}|f_p^*(x; x, t)|\ |f_p^*(\tau; x, t)|^p \qquad \text{a. e. on} \quad \partial S, \\
&\|f_p^*(\tau; x, t)\|_p^t = 1.
\end{aligned} \tag{3.3}$$

If $p=1$, we obtain for some real θ

$$\begin{aligned}
&f_1^*(\tau; x, t)h_1(\tau) = e^{i\theta}|f_1^*(x; x, t)|\ |f_1^*(\tau; x, t)| \qquad \text{a. e. on} \quad \partial S, \\
&\|f_1^*(\tau; x, t)\|_1^t = 1.
\end{aligned} \tag{3.4}$$

Here we note that f_p^* and h_p can be continued analytically across ∂S.

For fixed $x_0 \in \partial S$, let Δ be a neighborhood of x_0 which satisfies the following

(1) Δ is conformally equivalent to $U=\{|z|<1\}$, i.e. $\psi(\Delta)=U$.

(2) $x, \{t_j\} \notin \bar{\Delta}$. Here $\{t_j\}_{j=1}^{2l+k-1}$ are critical points of $g(\tau, t)$.

(3) $\partial\Delta \cap S$ is an analytic curve.

Let γ be the part of ∂U such that $\psi(\partial\Delta \cap \bar{S})-\{\text{end points}\}$. From (3.3) or (3.4) we have

$$A(z) \equiv f_p^*(\psi^{-1}(z)) h_p(\psi^{-1}(z)) = e^{i\theta}|f_p^*(x)| \; |f_p^*(\psi^{-1}(z))|^p \qquad \text{a. e. on } \gamma. \tag{3.5}$$

From $A(z) \in H_1(U)$ and $\arg A(z)=\theta$ a.e. on γ, $A(z)$ can be continued analytically across γ [11]. Hence $A(z)$ is analytic on $\bar{U}$ except for end points of γ and continuous on $\bar{U}$. $A(z)$ can be represented by the Poisson integral as follows:

$$A(z)=e^{i\lambda} \prod_j \left(\frac{z-z_j}{1-\bar{z}_j z}\right) \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|A(\psi^{-1}(e^{i\varphi}))| \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\varphi\right). \tag{3.6}$$

Here $\{z_j\} (\in U)$ are zero points of $A(z)$. From $f_p^*, h_p \in H_1(U)$, they can be represented uniquely as follows [7]:

$$\begin{aligned} f^*(\psi^{-1}(z)) &= e^{i\lambda_1} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_p^*(\psi^{-1}(e^{i\varphi}))| \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\varphi\right) \\ &\quad \times \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\mu_1(\varphi)\right) \times \text{some factors of } \prod_j \left(\frac{z-z_j}{1-\bar{z}_j z}\right), \end{aligned} \tag{3.7}$$

$$\begin{aligned} h_p(\psi^{-1}(z)) &= e^{i\lambda_2} \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|h_p(\psi^{-1}(e^{i\varphi}))| \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\varphi\right) \\ &\quad \times \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\mu_2(\varphi)\right) \times \text{some factors of } \prod_j \left(\frac{z-z_j}{1-\bar{z}_j z}\right). \end{aligned} \tag{3.8}$$

Here $\mu_j(\varphi)$ are non-increasing functions. Now from (3.5), (3.6), (3.7) and (3.8) we have

$$\lambda=\lambda_1+\lambda_2; \quad 1=\exp\left(\frac{1}{2\pi}\int_0^{2\pi} d\mu_1(\varphi)+d\mu_2(\varphi)\right).$$

Hence μ_1, μ_2 are constants. From (3.5), we have

$$\log|f_p^*(\psi^{-1}(z))| = \frac{1}{p}[\log|A(z)| - \log|f_p^*(x)|] \qquad \text{a. e. on } \gamma.$$

Therefore the factor

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f_p^*(\psi^{-1}(e^{i\varphi}))| \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\varphi$$

is analytic on γ. i.e. f_p^* and also h_p are analytic on γ.

Let $H_{x,t}(S)$ be the class of functions such that

$$\left\{\frac{1}{idW(\tau, t)}(idW(\tau, x)-F(\tau)\omega_0^*(\tau)) \text{ for all } F\in H_1(S)\right\}.$$

Then we are aware that, conversely if a function of $H_p^{t_1}(S)$ and a function of $H_{x,t}(S)$ satisfy (3.3) or (3.4), they are extremal functions. Now we have the following theorem:

THEOREM 3.1 (cf. [5], [11]) *For fixed $x, t\in S$, there exist the extremal functions $f_p^*(\tau; x, t)$ and $h_p(\tau)$ uniquely, which maximizes $f(x)$ in $H_p^{t_1}(S)(p>1)\cap\{f(x)>0\}$ and minimizes H_q-norm in $H_{x,t}(S)$ respectively. These extremal functions are characterized by the following properties:*

$$f_p^*(\tau; x, t)h_p(\tau)=f_p^*(x; x, t)|f_p^*(\tau; x, t)|^p \text{ on } \partial S \text{ and}$$

$$\|f_p^*(\tau; x, t)\|_p^t=1.$$

If $p=1$, there exist the extremal functions $f_1^*(\tau; x, t)$ and $h_1(\tau)$ which maximizes $f(x)$ in $H_1^{t_1}(S)\cap\{f(x)>0)$ and minimizes ess sup $|h(\tau)|(\tau\in\partial S)$ in $H_{x,t}(S)$ respectively. They are characterized by the following properties:

$$f_1^*(\tau; x, t)h_1(\tau)=f_p^*(x; x, t)|f_1^*(\tau; x, t)| \text{ on } \partial S \text{ and}$$

$$\|f_1^*(\tau; x, t)\|_1^t=1.$$

Furthermore $f_p^*(\tau; x, t)$ and $h_p(\tau)$ can be continued analytically across ∂S and $h_1(\tau)$ is determined uniquely.

4. Now a Rudin kernel $R_t(\tau, x)$ and the adjoint L-kernel $\mathcal{L}_t(\tau, x)$, which is an analytic differential on $\bar{S}$, except for a simple pole at x with residue 1 can be defined as follows:

$$R_t(\tau, x)=f_2^*(x; x, t)f_2^*(\tau; x, t),$$

$$\mathcal{L}_t(\tau, x)=ih_2(\tau)idW(\tau, t).$$

From the relation between f_2^* and h_2 on ∂S, we have

$$\overline{R_t(\tau, x)}idW(\tau, t)=\frac{1}{i}\mathcal{L}_t(\tau, x) \quad \text{along } \partial S. \tag{3.9}$$

That the Rudin kernel is a function and the adjoint L-kernel is a differential is a remarkable fact, because both the Bergman kernel and the Szegö kernel are in fact differentials. It is easy to see that the Rudin kernel has a reproducing property as follows and is characterized by it:

$$f(x)=\frac{1}{2\pi}\int_{\partial S}f(\tau)\overline{R_t(\tau, x)}\frac{\partial g(\tau, t)}{\partial\nu}ds_\tau \quad \text{for all } f\in H_2(S).$$

We list up some elementary properties.

(I) $$R_t(\tau, x)=\overline{R_t(x, \tau)}.$$

(II) $$R_t(\tau, t)\equiv 1 \qquad \text{(Rudin [11])}.$$

(III) $R_t(\tau, x)/R_t(x, x)$ is the unique extremal function minimizing H_2-norms among the functions $f\in H_2(S)$ with $f(x)=1$. Especially we have $R_t(x, x)\geqq 1$.

(IV) Let S_1, S_2 be regular subregions on an arbitrary Riemann surface $\hat{S}$ such that $S_2\supset S_1\ni x, t$. Then we have $R_t^{(1)}(x, x)\geqq R_t^{(2)}(x, x)$. Here $R_t^{(j)}(\tau, x)$ $(j=1, 2)$ is the Rudin kernel of S_j with respect to x, t.

We consider the differential class $H_p^D(S)$ which consists of analytic differentials ω on S such that $\omega/\omega_0=f\in H_p(S)$. It is defined independently of the choice of ω_0 which is introduced in Theorem 3.1. Let $\{\alpha_x(\tau)\}$ be the class of analytic differentials with a simple pole at x, residue 1, which satisfy

$$\int_{\partial S}|\alpha_x(\tau)|^2\frac{1}{i\,dW(\tau, t)}<\infty.$$

(V) In $\{\alpha_x\}$, $\mathcal{L}_t(\tau, x)$ is characterized by the following:

$$\int_{\partial S}\frac{\omega(\tau)\overline{\mathcal{L}_t(\tau, x)}}{i\,dW(\tau, t)}=0 \qquad \text{for all} \quad \omega\in H_2^D(S).$$

On an arbitrary open Riemann surface, we shall state some further properties of the Rudin kernel.

5. Let $N(\tau; x, t)$ be a Neumann function on S with poles at x and t. i.e. $N(\tau; x, t)$ is harmonic on S except for x, t, where $N(z(\tau); x, t)+\log|z(\tau)-z(x)|$ and $N(z_0(\tau); x, t)-\log|z_0(\tau)-z_0(t)|$ are harmonic, in terms of local parameters z and z_0 around x and t, respectively. $N(\tau; x, t)$ is continuously differentiable on $\bar{S}$ and $\partial N(\tau; x, t)/\partial\nu=0$ on ∂S. We set $V(\tau; x, t)=N(\tau; x, t)+iN^*(\tau; x, t)$ and define meromorphic differentials as follows:

$$\tilde{p}(\tau; x, t)=\frac{1}{2}[\quad dV(\tau; x, t)-dW(\tau, x)-dW(\tau, t)],$$

$$\tilde{q}(\tau; x, t)=\frac{1}{2}[-dV(\tau; x, t)-dW(\tau, x)-dW(\tau, t)],$$

$$p(\tau; x, t)=\frac{1}{2}[\quad dV(\tau; x, t)-dW(\tau, x)+dW(\tau, t)],$$

$$q(\tau; x, t)=\frac{1}{2}[-dV(\tau; x, t)-dW(\tau, x)+dW(\tau, t)].$$

Since $dV(\tau; x, t)$ and $i\,dW(\tau, x)$ are real along ∂S, we have

$$\tilde{p}(\tau; x, t) = -\overline{\tilde{q}(\tau; x, t)}, \tag{3.10}$$

$$p(\tau; x, t) = -\overline{q(\tau; x, t)} \qquad \text{along} \quad \partial S.$$

Now $(1/i)\mathcal{L}_t(\tau, x) - (1/i)\tilde{q}(\tau; x, t)$ and $R_t(\tau, x) i dW(\tau, x) + i\tilde{p}(\tau; x, t)$ are analytic on $\bar{S}$ (note $R_t(t, x) = 1$) and we have

$$\overline{R_t(\tau, x) i dW(\tau, t) + i\tilde{p}(\tau; x, t)} = \frac{1}{i}\mathcal{L}_t(\tau, x) - \frac{1}{i}\tilde{q}(\tau; x, t) \qquad \text{along} \quad \partial S.$$

Therefore we have the following representations (cf. [4]):

$$\frac{1}{i}\mathcal{L}_t(\tau, x) = \frac{1}{i}\tilde{q}(\tau; x, t) + \sum_{\mu=1}^{2l+k-1} \alpha_\mu(x, t)\omega_\mu(\tau),$$

$$R_t(\tau, x) i dW(\tau, t) = -i\tilde{p}(\tau; x, t) + \sum_{\mu=1}^{2l+k-1} \overline{\alpha_\mu(x, t)}\omega_\mu(\tau). \tag{3.11}$$

Here $\{\alpha_\mu(x, t)\}$ are constants and $\{\omega_\mu(\tau)\}$ is a base of the analytic differentials which are real along ∂S. For the local parameter z around τ, we set $\omega_\mu(\tau) = W_\mu(z)dz$, $\tilde{p}(\tau; x, t) = \tilde{P}(z; x, t)dz$ and so on. Then we obtain the following identities by putting in (3.11) $\tau = t_j$:

$$\sum_{\mu=1}^{2l+k-1} \overline{\alpha_\mu(x, t)} W_\mu(t_j) = i\tilde{P}(t_j; x, t) \qquad (j = 1, 2, \cdots, 2l+k-1). \tag{3.12}$$

For simplicity, we stand t_j itself for the local parameter around t_j in the sequel. Further we assume that $\{t_j\}$ are all simple. In the other cases, we modify (3.12) according to the multiplicity of $\{t_j\}$ in the usual fashion. Then $\{\overline{\alpha_\mu(x, t)}\}$ is regarded as a solution of $(2l+k-1)$-equations (3.12). Furthermore, we are aware that the solution of (3.12) is determined uniquely. Finally let θ_t be a differential analytic on $\bar{S}$ except for a simple pole at t with residue $-i$ and positive along ∂S. We note the fact that the result of Theorem 3.1 is valid in a reduced form by replacing $idW(\tau, t)$ with θ_t. The corresponding kernels are constructed similarly, and they have a representation in the form (3.11). The representation is determined uniquely again. Hence we have the following

LEMMA 3.1. *Let $\{\hat{t}_j\}_{j=1}^{2l+k-1}$ be zero points of a differential θ_t. Then we have*

$$\det [W_\mu(\hat{t}_j)] \neq 0.$$

Here we assume that $\{\hat{t}_j\}_{j=1}^{2l+k-1}$ are all simple. In the other cases, we obtain modified forms.

6. Using this lemma, we are ready to construct the reproducing kernels which correspond to the Rudin kernel. Let $\delta(x, t)$ be a constant such that $\delta(x, t) \cdot q(\tau; x, t)/dW(\tau, t)$ has residue 1 at x. We consider $(2l+k-1)$-equations which correspond to (3.12):

$$\sum_{\mu=1}^{2l+k-1} \beta_\mu(x,t) W_\mu(t_j) = -\delta(x,t) Q(t_j; x, t) \qquad (j=1,2,\cdots,2l+k-1).$$

The existence of a solution of this equation is assured by Lemma 3.1. We set this solution $\{\beta_\mu(x,t)\}$ again. Now we construct a function $\hat{L}_t(\tau,x)$ which is analytic on $\bar{S}$, except for a simple pole at x with residue 1, and an analytic differential $\hat{\mathfrak{R}}_t(\tau,x)$ on $\bar{S}$ as follows:

$$(3.13)\qquad \begin{aligned} \hat{L}_t(\tau,x) &= \frac{\delta(x,t)q(\tau;x,t) + \sum_{\mu=1}^{2l+k-1} \beta_\mu(x,t)\omega_\mu(\tau)}{dW(\tau,t)}, \\ \hat{\mathfrak{R}}_t(\tau,x) &= -\overline{\delta(x,t)}p(\tau;x,t) + \sum_{\mu=1}^{2l+k-1} \overline{\beta_\mu(x,t)}\omega_\mu(\tau). \end{aligned}$$

From (3.10), we have

$$(3.14)\qquad \overline{\hat{\mathfrak{R}}_t(\tau,x)} = \frac{1}{i}\hat{L}_t(\tau,x)idW(\tau,t), \qquad \text{along } \partial S.$$

In $H_2^p(S)$, we introduce an inner product (ω_1,ω_2) as follows:

$$(\omega_1,\omega_2) = \frac{1}{2\pi}\int_{\partial S} \omega_1 \cdot \overline{\omega_2} \frac{1}{idW(\tau,t)}.$$

Under this inner product, $H_2^p(S)$ becomes a Hilbert space. In terms of local parameters z and ξ, around τ and x, respectively, we set $\hat{\mathfrak{R}}_t(\tau,x) = \hat{R}_t(z,\xi)dz$. Then (3.14) implies that $\hat{R}_t(z,\xi)dz$ has the reproducing property in $H_2^p(S)$ as follows:

$$(3.15)\qquad a(\xi) = \frac{1}{2\pi}\int_{\partial S} a(z)dz\overline{\hat{R}_t(z,\xi)dz}\frac{1}{idW(\tau,t)} \quad \text{for all } \omega(\tau)=a(z)dz \in H_2^p(S).$$

We call the analytic differential $\hat{\mathfrak{R}}_t(\tau,x)$ the conjugate Rudin kernel. In this case, the adjoint L-kernel $\hat{L}_t(\tau,x)$ is not a differential but a function. Summing up, we obtain the following theorem:

THEOREM 3.2. *For fixed $x, t \in S$, an analytic differential $\hat{\mathfrak{R}}_t(\tau,x)$ on $\bar{S}$ and a function $\hat{L}_t(\tau,x)$, analytic on $\bar{S}$ except for a simple pole at x with residue* 1, *exist uniquely and satisfy the following*

$$\overline{\hat{\mathfrak{R}}_t(\tau,x)} = \frac{1}{i}\hat{L}_t(\tau,x)idW(\tau,t) \qquad \textit{along } \partial S.$$

Here the conjugate Rudin kernel $\hat{\mathfrak{R}}_t(\tau,x)$ is characterized by the reproducing property (3.15). *On the other hand, the adjoint L-kernel $\hat{L}_t(\tau,x)$ is characterized by the following extremal property*:

$$\min_{\{h(\tau,x)\}} \frac{1}{2\pi}\int_{\partial S} |h(\tau,x)|^2 \frac{\partial g(\tau,t)}{\partial\nu} ds_\tau = \frac{1}{2\pi}\int_{\partial S} |\hat{L}_t(\tau,x)|^2 \frac{\partial g(\tau,t)}{\partial \nu} ds_\tau = \hat{R}_t(\xi,\xi),$$

where $\{h(\tau, x)\}$ is the class of $L_2(\partial S)$-functions which are analytic on S, except for a simple pole at x with residue 1. Here we assume that a local parameter ξ around x is fixed arbitrary.

We list up some preliminary properties.

(I) $\hat{R}_t(z_1, z_2)=\overline{\hat{R}_t(z_2, z_1)}$.

(II) $\hat{R}_t(z_1, z_2)dz_1\overline{dz_2}$ is an invariant form with respect to change of both variables.

From (3.9) and (3.14), we have $\int_{\partial S}\hat{L}_t(\bar{\tau}, \tau_1)\mathcal{L}_t(\bar{\tau}, \tau_2)=0$. Here if we fix local parameters z_1 and z_2 around τ_1 and τ_2 respectively, such that $z_1=\psi_1(\tau_1)$ and $z_2=\psi_2(\tau_2)$, we have $\hat{L}_t(\tau_2, \psi_1^{-1}(z_1))dz_1=-\mathcal{L}_t(\tau_1, \psi_2^{-1}(z_2))$.

(III) If we take the local parameter z around τ, $\hat{L}_t(\bar{\tau}, \psi^{-1}(z))dz$ is an analytic differential with a simple pole at $\bar{\tau}$. On the other hand, $\mathcal{L}_t(\bar{\tau}, \tau)$ is an analytic function with respect to τ except for a simple pole at $\bar{\tau}$. In particular, the residue of $\hat{L}_t(\tau, x)\mathcal{L}_t(\tau, x)$ at x is zero.

(IV) In $\{h(\tau, x)\}$, $\hat{L}_t(\tau, x)$ is characterized by the following:

$$\int_{\partial S} f(\tau)\overline{\hat{L}_t(\tau, x)}\frac{\partial g(\tau, t)}{\partial \nu}ds_\tau=0 \text{ for all } f\in H_2(S).$$

We note that the property (IV) of the Rudin kernel is not valid in general with respect to the conjugate Rudin kernel.

7. Next, we consider a reproducing kernel $\hat{\mathfrak{R}}_t^E(\tau, x)$ in the closed subspace $H_2^{DE}(S)$ of $H_2^D(S)$ which consists of exact differentials. The existence of the kernel $\hat{\mathfrak{R}}_t^E(\tau, x)$ is assured by the general theory of the reproducing kernels [2]. Hence we shall state some properties of the adjoint L-kernel. At first, we consider the Bergman kernel $K(z, \bar{\xi})$ in the Hilbert space which consists of exact differentials df, analytic in S and $df\in L_2(S)$. Here z and ξ are fixed local parameters around τ and x respectively. $K(z, \bar{\xi})dz\overline{d\xi}$ is invariant with respect to change of both variables, and $K(z, \bar{\xi})$ satisfies

$$K(z, \bar{\xi})=\overline{K(\xi, \bar{z})}. \tag{3.16}$$

Let $L(z, \xi)dz$ be the adjoint L-kernel. $L(z, \xi)dz$ is an exact differential, analytic on $\bar{S}$ except for x where it has a double pole:

$$L(z, \xi)dz=\left(\frac{1}{\pi}\frac{1}{(z-\xi)^2}+\text{regular terms}\right)dz,$$

and satisfies

$$-K(z, \bar{\xi})dz=\overline{L(z, \xi)dz} \quad \text{along } \partial S \tag{3.17}$$

and

$$L(z,\xi)=L(\xi,z) \qquad \text{(cf. [13], pp. 135–137).} \tag{3.18}$$

Since $K(z,\bar{\xi})dz$ belongs to $H_2^{DE}(S)$, we have

$$K(\eta,\bar{\xi})=\frac{1}{2\pi}\int_{\partial S}K(z,\bar{\xi})dz\overline{\hat{R}_t^E(z,\eta)dz}\frac{1}{idW(\tau,t)}.$$

Hence

$$K(\xi,\bar{\eta})=-\frac{1}{2\pi}\int_{\partial S}L(\xi,z)dz\hat{R}_t^E(z,\eta)dz\frac{1}{idW(\tau,t)}. \tag{3.19}$$

Because $K(z,\bar{\eta})dz$ and $L(z,\eta)dz$ are exact differentials we may define $\tilde{K}(x,\bar{\eta})$ and $\tilde{L}(x,\eta)$ as follows:

$$\tilde{K}(x,\bar{\eta})=\int^x K(z,\bar{\eta})dz, \qquad \tilde{L}(x,\eta)=\int^x L(z,\eta)dz.$$

We integrate (3.19) from t to x, then we have

$$\tilde{K}(x,\bar{\eta})-\tilde{K}(t,\bar{\eta})=-\frac{1}{2\pi}\int_{\partial S}(\tilde{L}(x,z)-\tilde{L}(t,z))dz\hat{R}_t^E(z,\eta)dz\frac{1}{idW(\tau,t)}.$$

Here we assume that $\{t_j\}$ are all simple. In the other cases, we can modify the following arguments slightly. Then we assume that $dW(\tau,t)$ has the following expansion around t_j:

$$[W''(t_j,t)\,(z(\tau)-t_j)+\text{regular terms}]dz, \qquad (W''(t_j,t)\neq 0).$$

Using the Cauchy integral formula, and substitute τ for x, we have

$$\frac{i\hat{R}_t^E(z,\eta)dz}{idW(\tau,t)}=-\pi[\tilde{K}(\tau,\bar{\eta})-\tilde{K}(t,\bar{\eta})]-\pi\sum_{j=1}^{2l+k-1}\frac{\hat{R}_t^E(t_j,\eta)}{W''(t_j,t)}[\tilde{L}(\tau,t_j)-\tilde{L}(t,t_j)]. \tag{3.20}$$

Here $d\tilde{K}(\tau,\bar{\eta})+d\tilde{L}(\tau,\eta)$ and $d\tilde{K}(\tau,\bar{\eta})-d\tilde{L}(\tau,\eta)$ are a pure imaginary and a real analytic differential along ∂S, respectively. Consequently, we can take constants $\{\alpha_m(\eta)\}_{m=1}^k$ such that

$$\overline{\tilde{K}(\tau,\bar{\eta})}=-\tilde{L}(\tau,\eta)+\alpha_m(\eta) \qquad \text{on} \quad C_m. \tag{3.21}$$

From (3.20) and (3.21), we have

$$\begin{aligned}\frac{-i\overline{\hat{R}_t^E(z,\eta)dz}}{idW(\tau,t)}&=-\pi[-\tilde{L}(\tau,\eta)+\alpha_m(\eta)-\overline{\tilde{K}(t,\bar{\eta})}]\\ &\quad-\pi\sum_{j=1}^{2l+k-1}\left(\overline{\frac{\hat{R}_t^E(t_j,\eta)}{W''(t_j,t)}}\right)[-\tilde{K}(\tau,\bar{t}_j)+\overline{\alpha_m(t_j)}-\overline{L(t,t_j)}] \qquad \text{on} \quad C_m.\end{aligned} \tag{3.22}$$

Finally we define the adjoint L-kernel $\hat{L}_t^E(\tau,\eta)$ which is analytic on $\bar{S}$ except for a simple pole at η with residue 1 and define constants $\{A_m(\eta)\}_{m=1}^k$ as follows:

$$\hat{L}_t^E(\tau,\eta)=-\pi\tilde{L}(\tau,\eta)-\pi\sum_{j=1}^{2l+k-1}\frac{\hat{R}_t^E(\eta,t_j)}{W''(t_j,t)}\tilde{K}(\tau,\bar{t}_j), \tag{3.23}$$

$$A_m(\eta)=\pi\alpha_m(\eta)-\pi\overline{\tilde{K}(t,\bar{\eta})}-\pi\sum_{j=1}^{2l+k-1}\frac{\overline{\hat{R}_t^E(t_j,\eta)}}{\overline{W''(t_j,t)}}[-\overline{\alpha_m(t_j)}+\overline{\tilde{L}(t,t_j)}]. \tag{3.24}$$

From (3.22), (3.23) and (3.24), we have

$$\overline{\hat{R}_t^E(z,\eta)dz}=\frac{1}{i}[\hat{L}_t^E(\tau,\eta)+A_m(\eta)]\frac{\partial g(\tau,t)}{\partial\nu}ds_\tau \qquad \text{on} \quad C_m. \tag{3.25}$$

Here we can normalize that $A_k(\eta)=0$. We integrate (3.25) along C_m, and we have

$$A_m(\eta)=\frac{-1}{\dfrac{1}{2\pi}\displaystyle\int_{C_m}\frac{\partial g(\tau,t)}{\partial\nu}ds_\tau}\cdot\frac{1}{2\pi}\int_{C_m}\hat{L}_t^E(\tau,\eta)\frac{\partial g(\tau,t)}{\partial\nu}ds_\tau \qquad \text{on} \quad C_m. \tag{3.26}$$

From (3.25), we have the following theorem:

THEOREM 3.3. *We fix points x,t on S and a local parameter ξ around x. Then the extremal function $\hat{L}_t^E(\tau,\eta)$ and k-tuple $(A_1,A_2,\cdots,A_{k-1},0)$ of complex numbers which minimize*

$$\sum_{m=1}^{k}\frac{1}{2\pi}\int_{C_m}|h(\tau,\xi)+a_m|^2\frac{\partial g(\tau,t)}{\partial\nu}ds_\tau$$

in $\{h(\tau,\xi)\}$ and $\{(a_1,a_2,\cdots,a_{k-1},0)|a_m\in\boldsymbol{C}\}$ are uniquely determined. Moreover, they satisfy (3.25). *Conversely they are characterized by* (3.25).

Finally we state the interrelation of kernels. Using the reproducing property we have

$$\hat{R}_t^E(\eta,\xi)=\hat{R}_t(\eta,\xi)-\frac{1}{2\pi i}\sum_{m=1}^{k-1}\overline{A_m(\xi)}\int_{C_m}\overline{\hat{R}_t(z,\eta)dz}. \tag{3.27}$$

Next, from

$$\int_{\partial S}R_t(\tau,x)\hat{R}_t^E(z,\eta)dz=0,$$

we have

$$\begin{aligned}\hat{L}_t^E(\tau,\eta)&=-L_t(\eta,\tau)-\frac{1}{2\pi i}\sum_{m=1}^{k-1}A_m(\eta)\int_{C_m}L_t(z,\tau)dz\\&=\hat{L}_t(\tau,\eta)+\frac{1}{2\pi i}\sum_{m=1}^{k-1}A_m(\eta)\int_{C_m}\hat{L}_t(\tau,z)dz.\end{aligned} \tag{3.28}$$

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