SEKIGAWA, K.
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NOTES ON SOME 3- AND 4-DIMENSIONAL RIEMANNIAN MANIFOLDS

BY KOUEI SEKIGAWA

1. Introduction. The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M, g) satisfies

$$(*) \qquad R(X, Y)\cdot R=0 \qquad \text{for all tangent vectors } X \text{ and } Y,$$

where $R(X, Y)$ operates on R as a derivation operator of the tensor algebra at each point of M. Conversely, does this algebraic condition (*) on the curvature tensor field R imply that $\nabla R=0$? Let R_1 be the Ricci tensor of (M, g). Then (*) implies in particular

$$(**) \qquad R(X, Y)\cdot R_1=0 \qquad \text{for all tangent vectors } X \text{ and } Y.$$

In the present paper, we shall show that if the covariant derivative of the curvature tensor satisfies some algebraic conditions at each point, then the Riemannian manifold is locally symmetric.

In general, according to [4], we have

PROPOSITION A. *Let (M, g) be an $m(\geqq 3)$-dimensional real analytic Riemannian manifold. Assume that*

(1.1) *the restricted holonomy group is irreducible,*

$$(1.2) \qquad R(X, Y)\cdot R=0, \qquad \textit{that is } (*),$$

$$(1.3) \qquad R(X, Y)\cdot \nabla^k R=0, \qquad \textit{for } k=1, 2, \cdots$$

Then (M, g) is locally symmetric.

In this note, we shall prove

THEOREM B. *Let (M, g) be a 3-dimensional real analytic Riemannian manifold. Assume* (1.1), (1.2) *and*

$$(1.4) \qquad R(X, Y)\cdot \nabla R=0 \qquad (\textit{or } R(X, Y)\cdot \nabla_Z R=0).$$

Then (M, g) is a space of constant curvature.

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THEOREM C. *Let (M, g) be a 4-dimensional real analytic Riemannian manifold. Assume* (1. 1), (1. 2) *and* (1. 4). *Then (M, g) is locally symmetric.*

2. 3-dimensional cases. Let (M, g) be a 3-dimensional real analytic Riemannian manifold. R (resp. R_1) denotes the curvature tensor (resp. the Ricci tensor) of (M, g). R^1 denotes a field of symmetric endomorphism satisfying $R_1(X, Y) = g(R^1 X, Y)$. It is known that the curvature tensor R of (M, g) is given by

$$R(X, Y) = R^1 X \wedge Y + X \wedge R^1 Y - \frac{\operatorname{trace} R^1}{2} X \wedge Y \tag{2. 1}$$

for all tangent vectors X and Y.

At each point $p \in M$, we may choose an orthonormal basis $\{e_i\}$ such that $R^1 e_i = \lambda_i e_i$, $1 \leqq i, j, k, \cdots \leqq 3$. Then, from (*) (or equivalently (**)) and (2. 1), we see that essentially only the following cases are possible:

(I) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda, \quad \lambda \neq 0,$

(II) $\lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = 0, \quad \lambda \neq 0,$

(III) $\lambda_1 = \lambda_2 = \lambda_3 = 0.$

For (I), according to [3], we have

PROPOSITION 2. 1. *If the rank of the Ricci tensor R_1 is 3 at least at one point of M, then (M, g) is a space of constant curvature.*

Next, we assume that the rank of R_1 is at most 2 on M. Then (II) or (III) is valid on M. If the rank of R_1 is 2 at some point of M, then the rank of R_1 is also 2 near the point. Thus, let $W = \{p \in M$; the rank of R_1 is 2 at $p\}$, which is an open set of M. For any $p_0 \in W$, let W_0 be the connected component of p_0 in W. Then non-zero eigenvalue of R^1, say, λ, is a real analytic function on W_0 and furthermore, we may take two real analytic distributions T_1 and T_0 corresponding to λ and 0 respectively on W_0. Thus, for any $p \in W_0$, we may choose a real analytic field of orthonormal basis $\{E_i\}$ near p in such a way that $\{E_a\}$ and $\{E_3\}$ are bases for T_1 and T_0 respectively. Here $a, b, c, \cdots = 1, 2$. From (2. 1) and (II), we have

LEMMA 2. 2. *With respect to the above basis $\{E_i\}$,*

$$R(E_1, E_2) = \lambda E_1 \wedge E_2, \tag{2. 2}$$

all the others being zero.

In general, for a local real analytic field of orthonormal basis $\{E_i\}$ on an open set U in a real analytic Riemannian manifold (M, g), we may put

$$\nabla_{E_i} E_j = \sum_{k=1}^{m} B_{ijk} E_k, \tag{2. 3}$$

where $m=\dim M$ and B_{ijk} $(i, j, k, =1, 2, \cdots, m)$ are real analytic functions on U satisfying $B_{ijk}=-B_{ikj}$.

From (2. 2) and (2. 3), we have

$$(\nabla_{E_1}R)(E_1, E_2)=(E_1\lambda)E_1\wedge E_2+\lambda B_{1\,23}E_1\wedge E_3+\lambda B_{1\,31}E_2\wedge E_3,$$

$$(\nabla_{E_2}R)(E_1, E_2)=(E_2\lambda)E_1\wedge E_2-\lambda B_{2\,13}E_2\wedge E_3-\lambda B_{2\,32}E_1\wedge E_3,$$

$$(\nabla_{E_3}R)(E_1, E_2)=(E_3\lambda)E_1\wedge E_2+\lambda B_{3\,31}E_2\wedge E_3+\lambda B_{3\,23}E_1\wedge E_3,$$

$$(\nabla_{E_1}R)(E_2, E_3)=\lambda B_{1\,31}E_1\wedge E_2,$$

$$(\nabla_{E_2}R)(E_3, E_1)=\lambda B_{2\,32}E_1\wedge E_2.$$

From above equations, we have the following:

$$E_3\lambda+\lambda(B_{1\,31}+B_{2\,32})=0, \tag{2. 4}$$

$$B_{3\,13}=B_{3\,23}=0. \tag{2. 5}$$

Furthermore, we have

$$\begin{aligned}&(R(E_1, E_2)\cdot\nabla_{E_1}R)(E_1, E_2)\\ &=[R(E_1, E_2), (\nabla_{E_1}R)(E_1, E_2)]-(\nabla_{E_1}R)(R(E_1, E_2)E_1, E_2)-(\nabla_{E_1}R)(E_1, R(E_1, E_2)E_2)\\ &=\lambda^2B_{1\,31}E_1\wedge E_3+\lambda^2B_{1\,32}E_2\wedge E_3,\end{aligned} \tag{2. 6}$$

and similarly

$$(R(E_1, E_2)\cdot\nabla_{E_2}R)(E_1, E_2)=\lambda^2B_{2\,32}E_2\wedge E_3+\lambda^2B_{2\,31}E_1\wedge E_3.$$

Thus, from (1. 4) and (2. 6), we have

$$B_{1\,31}=B_{1\,32}=B_{2\,31}=B_{2\,32}=0. \tag{2. 7}$$

From (2. 7), we see that T_1 and T_0 are parallel on W_0 and hence the open subspace $(W_0, g|_{W_0})$ is reducible. Since (M, g) is real analytic, we can conclude that (M, g) is reducible. Therefore, we have theorem B.

3. 4-dimensional cases. Let (M, g) be a 4-dimensional real analytic Riemannian manifold satisfying the condition (*). At each point $p\in M$, we may choose an orthonormal basis $\{e_i\}$ such that $R^1e_i=\lambda_ie_i$, $1\leqq i, j, k, \cdots \leqq 4$. From (**), by the similar arguments as in §2, we see that essentially only the following cases are possible:

(I) $\lambda_1=\lambda_2=\lambda_3=\lambda_3=\lambda, \quad \lambda\neq 0,$

(II) $\lambda_1=\lambda_2=\lambda, \quad \lambda_3=\lambda_4=\mu, \quad \lambda, \mu\neq 0, \quad \lambda\neq\mu,$

(III) $\lambda_1=\lambda_2=\lambda_3=\lambda, \quad \lambda_4=0, \quad \lambda\neq 0,$

(IV) $\lambda_1=\lambda_2=\lambda, \qquad \lambda_3=\lambda_4=0, \qquad \lambda\neq 0,$

(V) $\lambda_1=\lambda_2=\lambda_3=\lambda_4=0.$

First, for (I), according to [2], we have

PROPOSITION 3. 1. *If (M, g) is a 4-dimensional Einstein space satisfying the condition (*), then it is locally symmetric.*

Secondly, we assume that (II) is valid at some point of M. Then, (II) is also valid near the point. Thus, let $W=\{p\in M$; (II) is valid at $p\}$, which is an open set of M. For any $p_0\in W$, let W_0 be the connected component of p_0 in W. Then non-zero eigenvalues of R^1, say, λ and μ, are real analytic functions on W_0 and we may take two real analytic distributions T_1 and T_2 corresponding to λ and μ respectively on W_0. Thus, for any $p\in W_0$, we may choose a real analytic field of orthonormal basis $\{E_i\}$ near p in such a way that $\{E_a\}$ and $\{E_u\}$ are bases for T_1 and T_2 respectively. Here $a, b, c, \cdots =1, 2$ and $u, v, w, \cdots =3, 4$. From (*) and (II), we have

LEMMA 3. 2. *With respect to the above basis $\{E_i\}$,*

(3. 1) $$R(E_1, E_2)=\lambda E_1\wedge E_2, \qquad R(E_3, E_4)=\mu E_3\wedge E_4,$$

all the other components being zero.

From (2. 3) and (3. 1), we have

$$(\nabla_{E_u}R)(E_1, E_2)=(E_u\lambda)E_1\wedge E_2+\lambda\sum_{v=3}^{4} B_{u1v}E_v\wedge E_2+\lambda\sum_{v=3}^{4} B_{u2v}E_1\wedge E_v,$$

$$(\nabla_{E_1}R)(E_2, E_u)=-\mu\sum_{v=3}^{4} B_{12v}E_v\wedge E_u+\lambda B_{1u1}E_1\wedge E_2,$$

$$(\nabla_{E_2}R)(E_u, E_1)=-\mu\sum_{v=3}^{4} B_{21v}E_u\wedge E_v+\lambda B_{2u2}E_1\wedge E_2.$$

Thus, by the second Bianchi identity, we have

(3. 2) $$B_{uav}=0, \qquad a=1, 2, \qquad u, v=3, 4.$$

Similarly we have

(3. 3) $$B_{aub}=0, \qquad a, b=1, 2, \qquad u=3, 4.$$

From (3. 2) and (3. 3), we see that T_1 and T_2 are parallel on W_0 and hence the open subspace $(W_0, g|_{W_0})$ is reducible. Since (M, g) is real analytic, we can concludee that (M, g) is reducible. Thus we have

PROPOSITION 3. 3. *If (II) is valid at some point of M, then (M, g) is a local product space of two 2-dimensional Riemannian manifolds.*

Thirdly, we assume that the rank of R_1 is at most 3 on M and 3 at some point of M. Then (III) is valid at the point and furthermore (III) is also valid near the point. Thus, let $W=\{p\in M;$ (III) is valid at $p\}$, which is an open set of M. For any $p_0\in W$, let W_0 be the connected component of p_0 in W. Then non-zero eigenvalue of R^1, say, λ, is a real analytic function on W_0 and we may take two real analytic distributions T_1 and T_0 corresponding to λ and 0 respectively on W_0. Thus, for any $p\in W_0$, we may choose a real analytic field of orthonormal basis $\{E_i\}$ near p in such a way that $\{E_a\}$ and $\{E_4\}$ are bases for T_1 and T_0 respectively. Here $a, b, c, \cdots =1, 2, 3$. From (*), (2. 1) and (III), we have

LEMMA 3. 4. *With respect to the above basis* $\{E_i\}$,

(3. 4) $$R(E_a, E_b)=KE_a\wedge E_b,$$

all the others being zero, where $K=\lambda/2$.

From (2. 3) and (3. 4), we have

(3. 5) $$(\nabla_{E_a}R)(E_b, E_c)=(E_aK)E_b\wedge E_c+KB_{ab4}E_4\wedge E_c+KB_{ac4}E_b\wedge E_4,$$
$$(\nabla_{E_b}R)(E_c, E_a)=(E_bK)E_c\wedge E_a+KB_{bc4}E_4\wedge E_a+KB_{ba4}E_c\wedge E_4,$$
$$(\nabla_{E_c}R)(E_a, E_b)=(E_cK)E_a\wedge E_b+KB_{ca4}E_4\wedge E_b+KB_{cb4}E_a\wedge E_4,$$

(3. 6) $$(\nabla_{E_4}R)(E_a, E_b)=(E_4K)E_a\wedge E_b+KB_{44a}E_b\wedge E_4+KB_{4b4}E_a\wedge E_4,$$
$$(\nabla_{E_a}R)(E_b, E_4)=K\sum_{c=1}^{3}B_{a4c}E_c\wedge E_b,$$
$$(\nabla_{E_b}R)(E_4, E_a)=K\sum_{c=1}^{3}B_{b4c}E_a\wedge E_c.$$

From (3. 5) and (3. 6), we have the following:

(3. 7) $$E_aK=0, \qquad a=1, 2, 3,$$

(3. 8) $$B_{a4b}=0, \quad a\neq b, \quad \text{and} \quad B_{44a}=0, \quad a=1, 2, 3,$$

(3. 9) $$E_4K+K(B_{a4a}+B_{b4b})=0, \qquad a\neq b.$$

And furthermore, we have

(3. 10) $$(R(E_a, E_b)\cdot\nabla_{E_c}R)(E_a, E_b)=K^2B_{c4a}E_a\wedge E_4+K^2B_{c4b}E_b\wedge E_4.$$

Thus, from (1. 4) and (3. 10), we have

(3. 11) $$B_{ab4}=0, \qquad a, b=1, 2, 3.$$

From (3. 7), (3. 8), (3. 9) and (3. 11), we have

PROPOSITION 3. 5. *Assume that the rank of the Ricci tensor* R_1 *of* (M, g) *is at* *most* 3 *on* M *and actually* 3 *at least at one point of* M. *If* (M, g) *satisfies* (*) *and* (1. 4), *then* (M, g) *is a local product space of a* 3-*dimensional space of constant curvature and a* 1-*dimensional space.*

Forthly, we assume that the rank of R_1 is at most 2 on M. Then (IV) or (V) is valid on M. If the rank of R_1 is 2 at some point of M, then the rank of R_1 is also 2 near the point. Thus, let $W=\{p\in M$; the rank of R_1 is 2 at $p\}$, which is an open set of M. For any $p_0\in W$, let W_0 be the connected component of p_0 in W. Then non-zero eigenvalue of R^1, say, λ, is a real analytic function on W_0 and we may take two real analytic distributions T_1 and T_0 corresponding to λ and 0 respectively on W_0. Thus, for any $p\in W_0$, we may choose a real analytic field of orthonormal basis $\{E_i\}$ near p in such a way that $\{E_a\}$ and $\{E_u\}$ are bases for T_1 and T_0 respectively. Here $a, b, c, \cdots =1, 2$ and $u, v, w, \cdots =3, 4$. Then, we have

LEMMA 3. 6. *With respect to the above basis* $\{E_i\}$,

$$R(E_1, E_2)=\lambda E_1\wedge E_2, \tag{3.12}$$

all the others being zero.

From (2. 3) and (3. 12), we have

$$(\nabla_{E_u}R)(E_1, E_2)=(E_u\lambda)E_1\wedge E_2+\lambda\sum_{v=3}^{4}B_{uv1}E_2\wedge E_v+\lambda\sum_{v=3}^{4}B_{u2v}E_1\wedge E_v, \tag{3.13}$$

$$(\nabla_{E_1}R)(E_2, E_u)=\lambda B_{1u1}E_1\wedge E_2,$$

$$(\nabla_{E_2}R)(E_u, E_1)=\lambda B_{2u2}E_1\wedge E_2,$$

$$(\nabla_{E_1}R)(E_1, E_2)=(E_1\lambda)E_1\wedge E_2+\lambda\sum_{v=3}^{4}B_{1v1}E_2\wedge E_v+\lambda\sum_{v=3}^{4}B_{12v}E_1\wedge E_v, \tag{3.14}$$

$$(\nabla_{E_2}R)(E_1, E_2)=(E_2\lambda)E_1\wedge E_2+\lambda\sum_{v=3}^{4}B_{2v1}E_2\wedge E_v+\lambda\sum_{v=3}^{4}B_{22v}E_1\wedge E_v.$$

From (3. 13), we have

$$B_{uva}=0, \qquad u, v=3, 4, \tag{3.15}$$

$$E_u\lambda+\lambda(B_{1u1}+B_{2u2})=0, \qquad u=3, 4. \tag{3.16}$$

From (3. 15), we see that T_0 is involutive and from lemma 3. 6, each maximal integral submanifold of T_0 in W_0 is locally flat with respect to the induced metric. And, from (1. 4), (3. 14) and (3. 15), considering $(R(E_1, E_2)\cdot\nabla_{E_1}R)(E_1, E_2)=0$ and $(R(E_1, E_2)\cdot\nabla_{E_2}R)(E_1, E_2)=0$, we have

PROPOSITION 3. 7. *Assume that the rank of the Ricci tensor* R_1 *of* (M, g) *is at most* 2 *on* M *and actually* 2 *at least at one point of* M. *If* (M, g) *satisfies* (*) *and* (1. 4), *then* (M, g) *is a local product space of* 2-*dimensional Riemannian manifold* *and a 2-dimensional locally flat space.*

Thus, from Propositions 3. 1, 3. 3, 3. 5, 3. 7, we have theorem C. Furthermore, from (3. 7), (3. 8) and (3. 9), by the similar arguments as in [3], we have

PROPOSITION 3. 8. *Assume that the rank of the Ricci tensor R_1 of (M, g) is at most 3 on M and actually 3 at least at one point of M. If (M, g) satisfies (*) and is complete, then (M, g) is a local product space of a 3-dimensional space of of constant curvature and a 1-dimensional space.*

Thus, from Propositions 3. 1, 3. 3, 3. 8, we have

THEOREM 3. 9. *Let (M, g) be a 4-dimensional complete and irreducible real analytic Riemannian manifold and the rank of the Ricci tensor R_1 of (M, g) is 3 or 4 on M. If (M, g) satisfies (*), then (M, g) is locally symmetric.*

REMARK. In 3-dimensional cases, we see that the condition (*) is equivalent to (**) and the condition (1. 4) is equivalent to

$$R(X, Y)\cdot\nabla_Z R_1=0. \tag{1. 4)'$$

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FACULTY OF SCIENCE,
NIIGATA UNIVERSITY.