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THE LINEAR OPERATOR METHOD AND LINEAR $\mathfrak{S}$ TOPOLOGIES

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In 1955, Sario [7] published his basic paper on the Linear Operator Method. The objective there was the construction of harmonic functions p with the behavior of a prescribed singularity function s near the ideal boundary. This objective was described in terms of constructing a harmonic function p for which $p-s$ has a regular, or normal behavior near the boundary; that is, $p-s$ is itself the image of an operator L which is reminiscent of a Dirichlet operator and is called normal [1], [6], and [7]. In this sense, the harmonic function p is thought of as an extension of the singularity function s, modulo a regular singularity function defined on a regular boundary neighborhood W' of the Riemann surface W. It is shown in [7] that except for constants, this extension is unique, and furthermore, that if the difference of two singularity functions is regular, then each will have the same harmonic extension, except for a constant.

In [5], Rodin and Sario have placed this extension problem in the natural setting of quotient spaces, wherein they have given an elegant solution phrased in terms of establishing that the natural mapping $p \to s = p|_{W'}$ will induce an algebraic isomorphism. The purpose of the present effort is to seek linear topologies under which this basic natural mapping will in fact induce a topological isomorphism.

1. Notation. We consider a regular boundary neighborhood W' of an open Riemann surface W. The boundary of W' is denoted by α and the closure of W', a union of bordered Riemann surface with border α, is denoted $\bar{W}'$. We call the linear space ot all harmonic functions on W by $H(W)$, and with a slight abuse of consistency we define $H(W')$ to be the linear space consisting of all functions which are harmonic on W', continuous on $\bar{W}'$, and have vanishing flux on the ideal boundary β of W.

When $C(\alpha)$ denotes the linear space of continuous functions on α, we let L: $C(\alpha) \to H(W')$ be a normal operator in the usual sense ([1], [5], [6], or [7]). The main existence theorem of Sario [7] phrased in algebraic language, is

THEOREM. [Rodin-Sario, 5]. *The natural mapping Φ: $H(W) \to H(W')$ defined by*

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$p \to p|_{W'}$ induces an isomorphism

$$\bar{\Phi}\colon H(W)/K \to H(W')LC(\alpha)$$

in the respective quotient spaces. Here, K is the set of all constant functions of $H(W)$.

Since we wish to study the continuity of $\bar{\Phi}$ and $\bar{\Phi}^{-1}$ in terms of topologies that will be given in no. 2, we shall construct $\Psi\colon H(W') \to H(W)$, such that $\bar{\Psi} = \bar{\Phi}^{-1}$, and study its continuity. This construction is carried out in detail in the manner of [5] with the exception that Ψ here will be defined on all of $H(W')$, rather than only on a set of representatives mod $L[C(\alpha)]$. That is, in terms of a regular region $\Omega \supset W - W'$, we define, for $s \in H(W')$

$$\Psi(s) = \begin{cases} Df & \text{on } \Omega, \\ LDf + s - Ls & \text{on } W - W', \end{cases}$$

where D is the Dirichlet operator applied to $\partial\Omega$ and f satisfies $(I-LD)f = s - Ls$ on $\partial\Omega$ in a Banach space of $C(\alpha)$ where $||LD|| < 1$, again as in [5]. Certainly the mapping Ψ here and that of [5] will agree when $s \equiv 0$ on α, and hence these will induce the same quotient homomorphism.

2. The linear topologies. In this investigation, we are interested in two topologies and their quotients as well as the associated weak topologies. In the notation of [3] and [4], we let $\mathfrak{S}$ be a collection of sets of W, and for each $S \in \mathfrak{S}$, we define $M(S, [-\varepsilon, \varepsilon])$ as $\{p \in H(W);\ |p(z)| \leq \varepsilon \text{ for all } z \in S\}$. As a neighborhood system of θ in $H(W)$, we consider $\{M(S, [-\varepsilon, \varepsilon]);\ S \in \mathfrak{S} \text{ and } \varepsilon > 0\}$. A necessary and sufficient condition that this system generate a system of neighboorhoods of θ for a linear topology is that $p(S)$ be bounded in $\boldsymbol{R}$ for each $p \in H(W)$ and each $S \in \mathfrak{S}$ ([4] and [8]). Important examples of such linear topologies are obtained by considering $\mathfrak{S}$ as

(a) the collection of all compact sets of W,

(b) the collection of all finite sets of W.

The topology of (a) is the usual k-topology of uniform convergence on compact sets, and the topology of (b) is called the topology p of simple convergence. Obvious analogies occur when W is replaced by W'.

We record some elementary properties of the topological linear spaces $H(W)_T$ and $H(W')_{T'}$ which result from furnishing the linear spaces $H(W)$ and $H(W')$ with the linear topologies T and T', when T (or T') is taken as either the k- or p-topology.

(i) The spaces $H(W)_T$ and $H(W')_{T'}$ are Hausdorff locally convex linear spaces.

(ii) The linear spaces $H(W)_k$ and $H(W')_{k'}$ have a countable neighborhood base at the origin because the neighborhood base at θ for this topology may be replaced by the countable collection of neighborhoods $\{M(\overline{W}_n, [-r, r])\}$, where $(\overline{W}_n)$ exhausts W and r is rational.

(iii) The quotient space $H(W)/K$ furnished with the quotient topology $q(T)$ is again a locally convex topological linear space. The same is true for $H(W')/LC(\alpha)$ equipped with $q(T')$.

(iv) The linear mapping Φ: $H(W)_T \rightarrow H(W')_{T'}$ of no. 1 is continuous. The elementary proof follows from the observation that, when $H(W)$ and $H(W')$ have the same $\mathfrak{S}$ topology, $H(W)_T$ is a subspace (topologically) of $H(W')_{T'}$, and Φ is only the identity.

(v) The quotient space $H(W)_k/K$ is Hausdorff, for according to [8, p. 20], it suffices to show that K is k-closed. For this, let $h(x)>h(y)$, and observe that the neighborhood $h+M(\{x, y\}, [-\varepsilon, \varepsilon])$ contains no constant functions when $\varepsilon<(1/3)(h(x)-h(y))$.

(vi) The quotient space $H(W')_{k'}/LC(\alpha)$ is a Hausdorff space, because by virtue of (ii), each singularity function $s \in \overline{LC(\alpha)}$ is obtained as $s=\lim_n s_n$, uniformly on compact sets of $\overline{W}'$, where $s_n=Lg_n$, with $g_n \in C(\alpha)$. If we call $s|\alpha=g$, we have that $g_n \rightarrow g$ uniformly on α, and by virtue of the definition of the normal operator L, this means that $Lg_n \rightarrow Lg$ uniformly on $\overline{W}'$. Hence it follows that $Lg=s$. This means that $LC(\alpha)$ is k-closed, and the quotient space $H(W')_{k'}/LC(\alpha)$ is Hausdorff.

(vii) Since quotient topologies are final topologies, it follows that $\bar{\Phi}$

$$\begin{array}{ccc} H(W)_T & \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} & H(W')_{T'} \\ \phi \downarrow & & \downarrow \phi' \\ H(W)_T/K & \underset{\bar{\Phi}}{\overset{\bar{\Psi}}{\leftrightarrows}} & H(W')_{T'}/LC(\alpha) \end{array}$$

is continuous if and only if $\bar{\Phi}\circ\phi=\phi'\circ\Phi$ is continuous. An analogous condition will determine the continuity of $\bar{\Psi}$.

3. The main theorem. In order to establish that $\bar{\Phi}$ is a topological isomorphism, it suffices to establish that $\bar{\Phi}^{-1}$ is continuous, since we have just observed that the continuity of $\bar{\Phi}$ follows from the continuity of Φ. Now each of the linear spaces $H(W)$ and $H(W')$ is complete as well as metric, and the same is true for their quotients. Hence, according to the Banach Homomorphism Theorem, the continuity of $\bar{\Phi}$ will imply that $\bar{\Phi}$ is open [8, p. 77], and the main result stated in Theorem 1 is established. However, it is possible to give a direct proof which is simpler than the proof of the Theorem cited, and this is now carried out.

THEOREM 1. *The isomorphism $\bar{\Phi}$: $H(W)/K \rightarrow H(W')/LC(\alpha)$ is topological when the domain and range are equipped with the quotient topologies $q(k)$ and $q(k')$ induced by uniform convergence on compact sets.*

Proof. In order to show that Ψ is continuous, we consider the neighborhood $M(K, [-1, 1])$ of θ in $H(W)$ and suppose first that $K \subset \bar{\Omega}$. It follows from the definition of f in no. 1 and [9] that $||f||_{\partial\Omega} \leqq ||(I-LD)^{-1}||\ ||s-Ls||_{\partial\Omega} \leqq (1/\delta)(||s||_{\partial\Omega}+||Ls||_{\partial\Omega})$, where $\delta=(1-||LD||)$ and, as usual, $||s||_{\partial\Omega}=\sup_{z\in\partial\Omega}|s(z)|$. Since L is a normal operator and $\partial\Omega\subset W'$, it follows that $||Ls||_{\partial\Omega}\leqq||Ls||_{\overline{W}'}=||s||_{\alpha}$. Then from the definition of Ψ, we have $||\Psi(s)||_K=||Df||_K\leqq||Df||_{\bar{\Omega}}=||f||_{\partial\Omega}\leqq(1/\delta)(||s||_{\partial\Omega}+||s||_{\alpha})$. It follows that

$$(1)\qquad \Psi\colon M\left(\partial\Omega\cup\alpha,\left[-\frac{\delta}{2},\frac{\delta}{2}\right]\right)\to M(K,[-1,1]).$$

If, on the other hand, $K\subset\overline{W}'$, then $\Psi(s)=LDf+s-Ls$, where, for the last term, $||Ls||_K\leqq||Ls||_{\overline{W}'}=||Ls||_{\alpha}=||s||_{\alpha}$. Since Df is defined on α, this means that $||LDf||_K\leqq||Df||_{\alpha}$. The inclusion $\alpha\subset\Omega$, implies that $||Df||_{\alpha}\leqq||f||_{\partial\Omega}$, but $||f||_{\partial\Omega}\leqq(1/\delta)(||s||_{\partial\Omega}+||s||_{\alpha})$ is already known. The result is that $||LDf||_K\leqq(1/\delta)(||s||_{\partial\Omega}+||s||_{\alpha})$ and we have

$$(2)\qquad \Psi\colon M\left(\partial\Omega\cup\alpha,\left[-\frac{\delta}{6},\frac{\delta}{6}\right]\right)\cap M\left(K,\left[-\frac{1}{3},\frac{1}{3}\right]\right)\cap M\left(\alpha,\left[-\frac{1}{3},\frac{1}{3}\right]\right)\to M(K,[-1,1]).$$

To complete the proof for the arbitrary compact set $K\subset W$, we need only observe that $K=(K\cap\bar{\Omega})\cup(K\cap\overline{W}')$, a pair of compact sets in $\bar{\Omega}$ and $\overline{W}'$ respectively, from which it follows that $M(K\cap\bar{\Omega},[-1,1])\cap M(K\cap\overline{W}',[-1,1])=M(K,[-1,1])$. Of course (1) applies to the first of these neighborhoods and (2) applies to the second. Since $0<\delta<1$, this means that

$$\Psi\colon M\left(\partial\Omega\cup\alpha\cup K,\left[-\frac{\delta}{6},\frac{\delta}{6}\right]\right)\to M(K,[-1,1])$$

and Ψ is continuous at θ, that is, continuous.

The proof is completed with the observation from (vii) of no. 2, that $\phi\circ\Psi$ is continuous.

4. The weakened topology. In terms of the notation E' for the topological dual of $E=H(W)_k/K$ and F' for the topological dual of $F=H(W')_{k'}/LC(\alpha)$, we place the pairs E and E', as well as F and F', in duality [3] by means of the bilinear form $(\bar{h},f)\to\langle\bar{h},f\rangle=f(\bar{h})$. The resulting dualities are written $\langle E,E'\rangle$ and $\langle F,F'\rangle$ respectively. The weakest linear topology on E for which each $f\in E'$ is continuous is denoted by $\sigma(E,E')$.

THEOREM 2. *If the topologies $q(k)$ and $q(k')$ on E and F of Theorem* 1 *are weakened to $\sigma(E,E')$ and $\sigma(F,F')$ respectively, the mapping $\bar{\Phi}$ remains a topological isomorphism.*

The continuity of $\bar{\Phi}$ and $\bar{\Psi}$ follows immediately from Theorem 1 and Proposition 6, p. 103 of [4]. But Theorem 2 is only a restatement of Theorem 1 if the weakened topologies σ are identical with the original quotient topologies. The following proposition suggested by [3, Chap. IV, §2, ex. 4(b)] will help in establishing that these weakened topologies are in fact properly weaker than the original quotient topologies.

PROPOSITION. *Let $\langle F, G\rangle$ be an algebraic duality between the linear spaces F and G, and suppose that the weak topology $\sigma(F, G)$ is locally convex with a countable base at θ. Then G has a countable Hamel base.*

Proof. Let $(p_n)_{n\in N}$ be a countable collection of continuous semi-norms describing the locally convex topology σ of F. Since $\sigma(F, G)$ is the weakest topology for which $x\to\langle x, y\rangle$ is continuous for each $y\in G$, it follows that the collection of all sets of the form $\{x;\ |\langle x, y\rangle|\leqq 1,\ y\in H\}$, H a finite set, is a base at θ for this topology. But each semi-norm is continuous for $\sigma(F, G)$, and it follows that for each natural number n, there is a finite subset $G_n\subset G$ such that

$$\{x;\ |\langle x, y\rangle|\leqq 1,\ y\in G_n\}\subset\{x;\ p_n(x)\leqq 1\}.$$

That is, the countable base for the locally convex topology σ on F can be replaced by the collection of all finite intersections of the sets of the form $\{x;\ |\langle x, y\rangle|\leqq 1\}$, where $y\in\bigcup G_n$, again a countable collection.

We label this countable collection $\bigcup G_n$ of G by $(y_k)_{k\in N}$, and establish that this collection is in fact a Hamel Base for G. For an arbitrary $y_0\in G$, and weak neighborhood $\{x;\ |\langle x, y\rangle|\leqq 1\}$ there is a finite set $H\subset(y_k)_{k\in N}$ such that $\{x;\ |\langle x, y\rangle|\leqq 1,\ y\in H\}$ $\subset\{x;\ |\langle x, y_0\rangle|\leqq 1\}$. Hence, if $\langle x, y\rangle=0$ for each $y\in H$, it follows that $\langle x, y_0\rangle=0$, and we conclude that $y_0=\sum_{y_t\in H}\lambda_t y_t$ ([4] p. 50 or [8], p. 124); that is, the collection $(y_k)_{k\in N}$ is a Hamel base, and the proof is complete.

Alternatively, the proposition follows from a direct application of the exercise already cited. For suppose that $F_{\sigma(F,G)}$ is metrizable. This means that $\sigma(F_\sigma, F'_\sigma)$ $=\sigma(F, G)$ is metrizable as well, and in fact that $\sigma(F_\sigma, F'_\sigma)=\tau(F_\sigma, F'_\sigma)$, the associated Mackey topology. Hence, according to the exercise, we have that F'_σ has a countable Hamel Base, and the same must hold for G.

Since the topological linear space $H(W)_k$ is already known to be locally convex with a countable base at θ, the same is true for the quotient space $H(W)_k/K$, and we can proceed to use the proposition to establish the existence of a new topology for which the mapping $\bar{\Phi}$ is again a topological isomorphism.

COROLLARY. *On the linear space $E=H(W)_k/K$, the weakened quotient topology $\sigma(E, E')$ is properly weaker than the quotient topology $q(k)$.*

Proof. As a natural bilinear functional for $H(W)/K$ and its topological dual $(H(W)_k/K)'$, we consider $\langle\bar{h}, f\rangle=f(\bar{h})$. Certainly if $\langle\bar{h}, f\rangle=0$ for all $\bar{h}\in H(W)/K$, then f is 0, and if $\langle\bar{h}, f\rangle=0$ for all $f\in(H(W)_k/K)'$, we show that $\bar{h}=\bar{\theta}$ by considering the special family of functionals defined for each $x\in W$ by

$$l_x\colon \bar{h}\to h(x)-h(x_0).$$

Such l_x is well-defined, for if $\bar{h}_1=\bar{h}_2$, then $l_x(\bar{h}_1)=h_1(x)-h_1(x_0)=h_2(x)-h_2(x_0)=l_x(\bar{h}_2)$. To see that each l_x belongs to the topological dual, it suffices to show that the composition $l_x\circ\phi\colon h\to h(x)-h(x_0)$ is a continuous mapping of $H(W)_k\to\boldsymbol{R}$. But this mapping is essentially an evaluation and is continuous for the topology of simple convergence and hence is continuous for the topology of compact convergence as well. The condition $\langle \bar{h}, f\rangle = 0$ for all f in the dual then means that $0 = l_x(h) = h(x) - h(x_0)$ for all $x \in W$; that is, $h(x)$ is identically constant. The bilinear functional $(\bar{h}, f) \to \langle \bar{h}, f\rangle$ is then a duality (algebraic) between $E = H(W)_k/K$ and E'.

To use the Proposition, we need only observe that for $f = l_x$ in the dual, the mapping $\bar{h} \to \langle \bar{h}, f\rangle$ is continuous. Certainly the weakened topology σ is no finer than the original quotient topology q. Furthermore, if these were equal, then the dual E' would have a countable Hamel Base. To see that this is a contradiction, we choose x_0 outside the disc Δ, and observe that the subset $S = \{l_x \colon x \in \Delta\} \subset E'$ contains uncountably many elements, and is a linearly independent set, since it is possible to interpolate on each finite set of points of W [2]. Hence the set S can have no countable Hamel Base, and this contradiction completes the proof of the corollary.

COROLLARY. *The weakened topology $\sigma(F, F')$ is properly weaker than the quotient $q(k')$ induced on $F = H(W')_{k'}/LC(\alpha)$.*

5. The p-topology of simple convergence. Since the conclusion of Theorem 2 is that the isomorphism $\bar{\Phi}$ remains topological when the quotient for the k-topology is weakened in both the domain and range, it is certainly reasonable to ask about a further weakening of each of these topologies to the quotients induced by the p-topology of simple convergence. We assert that the answer to this question is

THEOREM 3. *Let $\mathfrak{S}$ be the collection of all finite sets of an open Riemann surface W, and $\mathfrak{S}'$ be the reduction of $\mathfrak{S}$ to W'. The mapping $\bar{\Phi}$: $H(W)/K \to H(W')/L[C(\alpha)]$ fails to be topological in the quotients of the resulting p-topologies.*

Proof. To show first that Ψ fails to be continuous, it is sufficient to find $\{z_0, z_0'\} \subset W$ so that Ψ: $M(\{z_1, \cdots, z_n\}, [-\varepsilon, \varepsilon]) \to M(\{z_0, z_0'\}, [-1, 1])$ for no pair $\varepsilon > 0$ and finite set $\{z_1, \cdots, z_n\} \subset \overline{W}'$. With $\{z_0, z_0'\}$ taken as fixed in $W - \overline{W}'$, we let $\{z_1, \cdots, z_n\}$ be an arbitrary finite set of $\overline{W}'$, and choose, with [2], $h(z) \in H(W)$ so that $h(z_0) = h(z_1) = \cdots = h(z_n) = 0$ and $h(z_0') = 1$. If we take $h|_{W'}$ as the singularity function $s(z)$, then we will have the relation $p - s = L(p - s)$, where $p = \Psi(s)$. But s is defined on all of W, as is p, and it then follows that s and $\Psi(s)$ differ by a constant possibly depending on s itself, that is $\Psi(s) = s + k$.

It follows from the construction of s that $\lambda s \in M(\{z_1, \cdots, z_n\}, [-\varepsilon, \varepsilon])$ for each $\varepsilon > 0$ and all λ. Now, if $p(z_0) = h(z_0)$, then $p(z_0') \neq 0$, and $\lambda p = \Psi(\lambda s) \notin M(\{z_0, z_0'\}, [-1, 1])$ for sufficiently large $|\lambda|$. On the other hand, the assumption $p(z_0) \neq h(z_0)$ yields the same result, for then $p(z_0) \neq 0$. Hence the mapping Ψ fails to be continuous in the p-topologies, and in particular, $\Psi^{-1}M(\{z_0, z_0'\}, [-1, 1])$ fails to be a neighborhood in $H(W')$. Furthermore, $\Psi^{-1}[M(\{z_0, z_0'\}, [-1, 1]) + K]$ fails to be a neighborhood, as well.

We complete the proof by observing that the continuity of $\bar{\Psi}$ will lead to a contradiction. For according to the diagram of no. 2, the continuity of $\bar{\Psi}$ will imply the same for $\phi \circ \Psi$. Since ϕ is open, it then follows that $\bar{\Psi}^{-1} \circ \phi^{-1}(\phi M) = \Psi^{-1}(M + K)$ is a neighborhood in $H(W')$ for each neighborhood M of $H(W)$.

This is a contradiction when the neighborhood M is taken as $M(\{z_0, z'_0\}, [-1, 1])$ as in the paragraph above.

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