KŌDAI MATH. SEM. REP.
24 (1972), 251—258

# ON BASIC DOMAINS OF EXTREMAL FUNCTIONS

BY MAKOTO SAKAI

## 1. Introduction.

Let $P_{1W}(z; \zeta)$ (or $P_{0W}(z; \zeta)$) be the unique function with the smallest (or largest, resp.) real part of the coefficient $a=a(F)$ among all univalent functions $F(z)$ on a plane domain $W$ which are normalized by

$$(1) \qquad F(z)=\begin{cases} \dfrac{1}{z-\zeta}+a(z-\zeta)+\cdots & \text{about } \zeta\neq\infty, \\ z+\dfrac{a}{z}+\cdots & \text{about } \zeta=\infty. \end{cases}$$

The extremal properties of the functions $M_W(z; \zeta)=(1/2)(P_{0W}(z; \zeta)-P_{1W}(z; \zeta))$ and $N_W(z; \zeta)=(1/2)(P_{0W}(z; \zeta)+P_{1W}(z; \zeta))$ are well-known. For instance, $N_W(z; \zeta)$ is the unique function with the largest outer area among all univalent functions $F(z)$ on a plane domain which are normalized by (1), and $M_W^*(z; \zeta)=M_W(z; \zeta)/M_W'(\zeta; \zeta)$ is the unique function with the smallest inner area among all analytic functions $f(z)$ on a plane domain $W\notin O_{AD}$ for which $f'(\zeta)=1$ (cf. Sario-Oikawa [3]).

In the present paper we shall show:

THEOREM 1. *The following conditions are equivalent*:

(a) *$M_W(z; \zeta)$ is univalent for some $\zeta\in W$.*

(b) *For some $\zeta\in W$, the complement of the image domain of $N_W(z; \zeta)$ consists of a closed disk of positive radius and a relatively closed set with zero area.*

(c) *$W$ is conformally equivalent to $\{|z|<1\}-E$, where $E$ is a set satisfying $E\cap K\in N_D$ for every compact subset $K$ of $\{|z|<1\}$.*

THEOREM 2. *The following conditions are equivalent*:

(a) *$M_W(z; \zeta)$ is linear for some $\zeta\in W$.*

(b) *$N_W(z; \zeta)$ is linear for every $\zeta\in W$. (i.e.*

$$N_W(z; \zeta)=\begin{cases} \dfrac{1}{z-\zeta} & (\zeta\neq\infty), \\ z & (\zeta=\infty) \end{cases}$$

Received May 17, 1971.

*for every* $\zeta \in W$.)

(c) *W is either*

(i) *a disk* $\Delta$ (*in the wider sense*) *less a relatively closed set E satisfying* $E \cap K \in N_D$ *for every compact subset K of* $\Delta$,

*or*

(ii) *of class* $O_{AD}$.

Weaker versions of theorem 2 were previously proved by Ozawa [2] and Suita [7]. Let $W$ be a plane domain and let $L^2(W)$ be the class of analytic functions on $W$ possessing a single-valued indefinite integral with a finite Dirichlet integral. We denote by $K_W(z;\zeta)$ Bergman's kernel function on a plane domain $W$ and set

$$\Gamma_W(z;\zeta)=\frac{1}{\pi^2}\iint_{W^c}\frac{1}{(t-z)^2\overline{(t-\zeta)^2}}\,dudv \qquad (t=u+iv)$$

The problem of seeking the eigen values of the integral equation

$$f(z)=\lambda\iint_W (K_W(z;\zeta)-\Gamma_W(z;\zeta))\overline{f(\zeta)}\,d\xi d\eta \qquad (f\in L^2(W),\ \zeta=\xi+i\eta)$$

is called the Fredholm eigen value problem for $W$ (cf. Schiffer [5], [6] and Ozawa [2]). Ozawa [2] showed that if all spectra of the Fredholm eigen value problem for $W$ are equal to zero and the area of the complement of $W$ is equal to zero, then $W$ is of class $O_{AD}$. Suita [7] showed that $N_W(z;\zeta)$ is linear for all $\zeta\in W$ if and only if all spectra of the Fredholm eigen value problem for $W$ are equal to zero, and showed that if all spectra of the Fredholm eigen value problem for $W$ are equal to zero and there exists a continuum as an isolated boundary, then this continuum is a circle and the remaining components are of class $N_D$. A necessary and sufficient condition for all spectra of the Fredholm eigen value problem for $W$ to be equal to zero is given by theorem 2.

## 2. Proof of theorem 1.

A compact set $E$ in the plane is called an extremal set of vertical (or horizontal) slits if $E^c$ is a domain such that

$$P_{1_{E^c}}(z;\infty)\equiv z$$

(or

$$P_{0_{E^c}}(z;\infty)\equiv z,$$

resp.). The complement of the image of $P_{1_W}(z;\zeta)$ (or $P_{0_W}(z;\zeta)$) is an extremal set of vertical (or horizontal, resp.) slits.

LEMMA 1. *Let E be an extremal set of vertical slits in the* $z(=x+iy)$*-plane* *andlet $U$ be a simply connected domain such that $E\subset U$. If a* 1-1 *mapping $\varphi$ of $U$ into the $w(=u+iv)$-plane is of class $C^2(U)$ and satisfies $\partial u/\partial y=0$ on $U$, then the image $\varphi(E)$ of $E$ is an extremal set of vertical slits.*

*Proof.* Let $V\supset E$ be a simply connected domain whose boundary $\partial V$ is a simple closed smooth curve. Oikawa [1] showed that $E$ is an extremal set of vertical slits if and only if

$$\iint_{V-E}\frac{\partial h}{\partial y}dxdy=0 \tag{2}$$

for every $h\in C^1BD(\bar{V}-E)$ which vanishes identically on $\partial V$, where $C^1BD(\Omega)$ is the class of bounded $C^1$-functions with a finite Dirichet integral on $\Omega$. Since $E$ is an extremal set of vertical slits, (2) is valid for some $V$ such that $\bar{V}\subset U$. It is sufficient to show that

$$\iint_{\varphi(V)-\varphi(E)}\frac{\partial H}{\partial v}dudv=0$$

for every $H\in C^1BD(\varphi(V)-\varphi(E))$ which vanishes identically on $\partial(\varphi(V))$. Since $\partial u/\partial y=0$ on $U$ we have $\partial x/\partial v=0$ on $\varphi(U)$. Hence we have

$$\frac{D(u,v)}{D(x,y)}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y},$$

$$\frac{D(x,y)}{D(u,v)}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v},$$

and

$$\left(\frac{\partial x}{\partial u}\circ\varphi\right)\left(\frac{\partial y}{\partial v}\circ\varphi\right)\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}=1.$$

Therefore we have $\partial\{(\partial y/\partial v\circ\varphi)(\partial u/\partial x)(\partial v/\partial y)\}/\partial y=0$ on $U$. The function $(H\circ\varphi)\{(\partial y/\partial v\circ\varphi)\cdot(\partial u/\partial x)(\partial v/\partial y)\}$ is of class $C^1BD(\bar{V}-E)$ and vanishes identically on $\partial V$. We assume without loss of generality that $D(u,v)/D(x,y)>0$ on $U$, and we have

$$\begin{aligned}0&=\iint_{V-E}\frac{\partial}{\partial y}\left[(H\circ\varphi)\left\{\left(\frac{\partial y}{\partial v}\circ\varphi\right)\frac{\partial u}{\partial v}\frac{\partial v}{\partial y}\right\}\right]dxdy\\&=\iint_{V-E}\left\{\frac{\partial(H\circ\varphi)}{\partial x}\left(\frac{\partial x}{\partial x}\circ\varphi\right)+\frac{\partial(H\circ\varphi)}{\partial y}\left(\frac{\partial y}{\partial v}\circ\varphi\right)\right\}\frac{D(u,v)}{D(x,y)}dxdy\\&=\iint_{\varphi(V)-\varphi(E)}\frac{\partial H}{\partial v}dudv.\end{aligned}$$

q.e.d.

LEMMA 2. *Let $W$ be a subdomain of a domain $W'$. Equality $M'_W(\zeta;\zeta)=M'_{W'}(\zeta;\zeta)$ holds for some $\zeta\in W$ if and only if $W'-W$ is a set satisfying $(W'-W)\cap K\in N_D$ for every compact subset $K$ of $W'$.*

*Proof.* If $M'_W(\zeta;\zeta)=M'_{W'}(\zeta;\zeta)=0$ for some $\zeta\in W$, then the assertion is evident.

Assume $M'_W(\zeta;\zeta)=M'_{W'}(\zeta;\zeta)>0$ for some $\zeta\in W$. To see that $W'-W$ is a set mentioned above, it is sufficient to show that for every $a\in W'-W$ there exists a neighborhood $V$ of $a$ such that $E=\bar{V}\cap(W'-W)$ is of class $N_D$. We set $w_\nu=u_\nu+iv_\nu=P_{\nu W'}(z;\zeta)$, $b_\nu=P_{\nu W'}(a;\zeta)$ $(\nu=0,1)$ and $\varphi=P_{1W'}\circ P_{1W'}^{-1}$. Then we have $\partial u_1(u_0,v_0)/\partial u_0=\mathrm{Re}\,\varphi'(w_0)=\mathrm{Re}\,P'_{1W'}(z;\zeta)/P'_{0W'}(z;\zeta)>0$ for every $w_0\in P_{0W'}(W')$. Hence a mapping

$$\psi:(u_0,v_0)\longmapsto(u,v)=(u_1(u_0,v_0),v_1(\mathrm{Re}\,b_0,v_0))$$

is a 1-1 mapping of class $C^2$ on some neighborhood $U_0$ of $b_0$. Let $V_0$ be a neighborhood of $b_0$ such that $\bar{V}_0\subset U_0$. We set $V=P_{0W'}^{-1}(V_0)$ and $E=\bar{V}\cap W'-W)$. From the assumption we have $S_{W'-E}(\zeta)=S_{W'}(\zeta)$, where $S_{W'}(\zeta)$ is the span of $W'$ at $\zeta$. Hence $P_{\nu W'-E}(z;\zeta)=P_{\nu W'}(z;\zeta)$ $(\nu=0,1)$. This shows that $P_{0W'}(E)$ (or $P_{1W'}(E)$) is an extremal set of horizontal (or vertical, resp.) slits. Since $\partial v/\partial u_0=0$ on $U_0$ and $\partial u/\partial v_1=0$ on $\varphi(U_0)$, from lemma 1 we have that $\psi(P_{0W'}(E))=(\psi\circ\varphi^{-1})(P_{1W'}(E))$ is an extremal set of horizontal and vertical slits. This implies that $(\psi\circ P_{0W'})(E)$ is of class $N_D$. Since $\psi\circ P_{0W'}$ is a 1-1 mapping of class $C^2$ on $P_{0W'}^{-1}(U_0)$, $E$ is also of class $N_D$. The converse is evident. q.e.d.

(a) implies (c). Let $W_1$ be the image domain of $M_W^*(z;\zeta)$, and set $M_1^*=M_{W_1}^*(w;0)$. Assume that $M_W^*$ is univalent. Then the composite function $M_1^*\circ M_W^*$ has the smallest inner area among all analytic functions $f(z)$ on $W$ for which $f'(\zeta)=1$, and hence it is equal to $M_W^*$. Therefore $M_1^*$ is the identity function on $W$. Let $f_{SB}$ be a function with the smallest supremum of absolute modulus among all bounded univalent functions $f(z)$ on $W$ for which $f'(\zeta)=1$. Then the outer boundary of the image domain of $f_{SB}$ is a circle whose center is the origin, and $f_{SB}$ is a function with the smallest inner area among all univalent functions $f(z)$ on $W$ for which $f'(\zeta)=1$. Since $M_W^*$ is univalent, $f_{SB}$ is equal to $M_W^*$. By lemma 2, we have $W_1=\Delta-E$ where $\Delta$ is a disk and $E$ is a set satisfying $E\cap K\in N_D$ for every compact subset $K$ of $\Delta$. q.e.d.

(c) implies (b). Let $\Delta$ be an open disk. Then $N_\Delta(z;\zeta)$ is linear for every $\zeta\in\Delta$. So the assertion is evident.

(b) implies (a). Assume that (b) holds and let $c$ be the center of the closed disk. Then $F(z)=1/\{N_W(z;\zeta)-c\}$ is univalent on $W$ and satisfies $F'(\zeta)=1$. Let $A_i(f)$ be the inner area associated with an analytic function $f(z)$ on $W$, and let $A_e(g)$ be the outer area associated with a univalent function $g(z)$ on $W$ which are normalized by (1). Then we have $A_i(F)\cdot A_e(N_W)=\pi^2$. From the well-known identity $A_i(M_W^*)\cdot A_e(N_W)=\pi^2$, we have $A_i(F)=A_i(M_W^*)$. Hence $M_W^*$ is equal to $F$ and is univalent. q.e.d.

## 3. Proof of theorem 2.

We show first the following lemmas:

LEMMA 3. *Let $W$ be a subdomain of a domain $W'$. If the area of $W'-W$ is* *equal to zero and* $N_W(z;\zeta)$ *(or* $M_W(z;\zeta)$*) can be extended analytically onto* $W'$*, then* $M_{W'}(z;\zeta)=M_W(z;\zeta)$, $N_{W'}(z;\zeta)=N_W(z;\zeta)$ *on* $W$*, and* $W'-W$ *is a set satisfying* $(W'-W)\cap K\in N_D$ *for every compact subset* $K$ *of* $W'$.

*Proof.* If $W$ is of class $O_{AD}$, then the assertion is evident. We a ssume that $W$ is not of class $O_{AD}$. If $W\subset W'$, then we have $A_i(M_W^*)\leqq A_i(M_{W'}^*)$ and $A_e(N_W)\geqq A_e(N_{W'})$. Assume the area of $W'-W$ is equal to zero and $N_W$ (or $M_W$) can be extended analytically onto $W'$. Then we have $A_e(N_W)=A_e(N_{W'})$ (or $A_i(M_W^*)=A_i(M_{W'}^*)$, resp.). From the identity $A_i(M^*)\cdot A_e(N)=\pi^2$, we have $A_e(N_W)=A_e(N_{W'})$ and $A_i(M_W^*)=A_i(M_{W'}^*)$. The assertion follows from lemma 2. q.e.d.

LEMMA 4. *Let* $\mu$ *be a finite complex Baire measure with a compact support* $K$ *such that the interior* $K^\circ$ *of* $K$ *is empty and the complement* $K^c$ *of* $K$ *is connected, and let* $n$ *be a natural number. If the integral*

$$\int_K \frac{1}{(z-\zeta)^n}\, d\mu(z)$$

*is identically zero for every* $\zeta\in K^c$*, then* $\mu=0$.

*Proof.* Let $r$ be a positive number such that $K\subset\{|z|<r\}$, and let $f$ be a polynomial. Then we have

$$\begin{aligned}\int_K f^{(n-1)}(z)\, d\mu(z) &= \int_K \left\{ \frac{(n-1)!}{2\pi i} \int_{\{|\zeta|=r\}} \frac{f(\zeta)}{(\zeta-z)^n}\, d\zeta \right\} d\mu(z) \\ &= \frac{(-1)^n(n-1)!}{2\pi i} \int_{\{|\zeta|=r\}} \left\{ \int_K \frac{1}{(z-\zeta)^n}\, d\mu(z) \right\} f(\zeta)\, d\zeta \\ &= 0.\end{aligned}$$

Since $K^\circ$ is empty and $K^c$ is connected, every continuous function on $K$ can be approximated uniformly on $K$ by polynomials. Hence for every continuous function $g$ on $K$ we have

$$\int_K g(z)\, d\mu(z)=0.$$

By the Riesz representation theorem we have $\mu=0$. q.e.d.

(a) implies (c). If $M_W$ is a constant, then the constant is equal to zero and $W$ is of class $O_{AD}$. Therefore the assertion follows from theorem 1.

(c) implies (b). Let $\Delta$ be an open disk. Then $N_\Delta(z;\zeta)$ is linear for every $\zeta\in\Delta$. Hence the assertion is evident.

(b) implies (a). Assume that $N_W(z;\zeta)$ is linear for every $\zeta\in W$. Then for every linear transformation $L(z)$, $N_{L(W)}(z;\zeta)$ is linear for every $\zeta\in L(W)$. Therefore in the following we assume that

(i) $\overline{W}$ is compact if $W$ has an exterior point, and

(ii) $W^c$ is compact if $W$ has no exterior point.

Proof of the case (i). $(-1/\pi)N'_W(z;\zeta)$ is the unique function analytic on $W-\{\zeta\}$ and satisfying

(iii) $$\lim_{r\to 0}\iint_{W-\{|z-\zeta|\leq r\}}\left\{F'(z)\cdot\overline{-\frac{1}{\pi}N'_W(z;\zeta)}\right\}dxdy=0$$

for every $F\in AD(W)$, having a pole at $\zeta$ with the principal part $1/\{\pi(z-\zeta)^2\}$ if $\zeta\neq\infty$ and $z^2/\pi$ if $\zeta=\infty$, and possessing a single-valued indefinite integral with a finite Dirichlet integral outside of a neighborhood of $\zeta$. (In the case $\infty\in W$, the integral of (3) is to be taken over $W-\{\infty\}$.) (cf. Schiffer [4] and Sario-Oikawa [3]). Since $\overline{W}$ is compact, the formula (3) is valid for $F(z)=z$. Hence we have

$$\lim_{r\to 0}\iint_{W-\{|z-\zeta|\leq r\}}\frac{1}{(z-\zeta)^2}dxdy=0$$

for every $\zeta\in W$. For every open disk $\Delta$ and $\Delta'$ such that $\bar{\Delta}\subset\Delta'$, we have

$$\iint_{\Delta'-\bar{\Delta}}\frac{1}{(z-\zeta)^2}dxdy=0$$

for every $\zeta\in\Delta$. Hence for a fixed open disk $\Delta_0$ such that $\bar{\Delta}_0\subset W$ we have

$$\iint_{W-\bar{\Delta}_0}\frac{1}{(z-\zeta)^2}dxdy=0$$

for every $\zeta\in\Delta_0$. Let $\Delta_1$ be a bounded open disk such that $W\subset\Delta_1$, then we have

(4) $$\iint_{\bar{\Delta}_1-W}\frac{1}{(z-\zeta)^2}dxdy=0$$

for every $\zeta\in\Delta_0$. We denote by $\Gamma$ the connected component of $\bar{\Delta}_1-W$ which contains the boundary of $\Delta_1$, and set $E=\bar{\Delta}_1-W-\Gamma$ and $W'=W\cup E$. Let $\{W_n\}$ be a canonical exhaustion of $W$ and denote by $\Omega_n$ the simply connected bounded domain whose boundary is the outer boundary of $W_n$. Set $E_n=E\cap\Omega_n$. Then $E_n$ and $\bar{\Delta}_1-W-E_n$ are compact. From (4) we have

(5) $$\iint_{E_n}\frac{1}{(z-\zeta)^2}dxdy=-\iint_{\bar{\Delta}_1-W-E_n}\frac{1}{(z-\zeta)^2}dxdy$$

for every $\zeta\in\Delta_0$. The function

$$h_n(\zeta)=\iint_{E_n}\frac{1}{(z-\zeta)^2}dxdy$$

is analytic outside of $E_n$ and satisfies $h_n(\infty)=0$. By (5) $h_n(\zeta)$ can be extended analytically onto $E_n$. Therefore $h_n(\zeta)$ is identically zero, and consequently the area of $E$ is equal to zero. From lemma 3 $N_{W'}(z;\zeta)$ is also linear and $E$ is a set satisfying $E\cap K\in N_D$ for every compact subset $K$ of $W'$. Since $W'$ is simply connected, from theorem 1 we know that the image domain of $N_{W'}(z;\zeta)$ is a disk (in the wider sense), and hence $W'$ is a disk. Therefore $M_W(z;\zeta)=M_{W'}(z;\zeta)$ is linear.

Proof of the case (ii). We show that if $W$ has no exterior point and if $N_W(z;\zeta)$ is linear for every $\zeta\in W$, then $W^c$ is of class $N_D$. Assume that $W^c$ is not of class $N_D$. Then there exist two bounded open disks $\varDelta_1$ and $\varDelta_2$ such that $\bar{\varDelta}_1\cap\bar{\varDelta}_2=\phi$, $\bar{\varDelta}_1\cap W^c\notin N_D$ and $\bar{\varDelta}_2\cap W^c\notin N_D$. Let $\varDelta_3$ be a bounded open disk such that $\bar{\varDelta}_1\subset\varDelta_3$ and $\bar{\varDelta}_2\subset(\bar{\varDelta}_3)^c$. Let $\{W_n\}$ be a canonical exhaustion of $\varDelta_3-W^c$ and denote by $\Omega_n$ the simply connected bounded domain whose boundary is the outer boudary of $W_n$. Set $E_n=W^c\cap\Omega_n$. Using the same argument in the case (i) with respect to a nonconstant function $F\in AD((\bar{\varDelta}_2\cap W^c)^c)$, we have

$$\iint_{E_n}\frac{\overline{F'(z)}}{(z-\zeta)^2}\,dxdy=-\iint_{W-\bar{\varDelta}_3}\frac{\overline{F'(z)}}{(z-\zeta)^2}\,dxdy-\iint_{\varDelta_3\cap(W^c-E_n)}\frac{\overline{F'(z)}}{(z-\zeta)^2}\,dxdy \tag{6}$$

for every $\zeta\in\varDelta_3\cap W$, and hence

$$\iint_{E_n}\frac{\overline{F'(z)}}{(z-\zeta)^2}\,dxdy=0$$

for every $\zeta\in E_n^c$. From lemma 4 the area of $E_n$ is equal to zero. Therefore the area of $\varDelta_3\cap W^c$ is equal to zero and from lemma 3 we know that $\bar{\varDelta}_1\cap W^c$ is of class $N_D$. This is a contradiction. q.e.d.

REMARK. Let $F$ be a subset of $W$ which has at least one accumulating point in $W$. Assume that $N_W(z;\zeta)$ is linear for every $\zeta\in F$. Then (5) and (6) are valid for every $\zeta\in F$, and hence the same consequence follows from the unicity theorem.

## REFERENCES


[1] OIKAWA, K., Minimal slit regions and linear operator method. Kōdai Math. Sem. Rep. **17** (1965), 187–190.

[2] OZAWA, M., Fredholm eigen value problem for general domains. Kōdai Math. Sem. Rep. **12** (1960), 38–44.

[3] SARIO, L., AND K. OIKAWA, Capacity functions. Springer-Verlag, Berlin (1969).

[4] SCHIFFER, M., The kernel function of an orthonormal system. Duke Math. J. **13** (1946), 529–539.

[5] Schiffer, M., The Fredholm eigen values of plane domains. Pacific J. Math. **7** (1957), 1187–1225.

[6] ———, Fredholm eigen values of multiply-connected domains. Pacific J. Math. **9** (1959), 211–269.

[7] Suita, N., On Fredholm eigen value problem for plane domains. Kōdai Math. Sem. Rep. **13** (1961), 109–112.

Department of Mathematics,
Tokyo Institute of Technology.