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ON THE NUMBER OF AUTOMORPHISMS OF A COMPACT BORDERED RIEMANN SURFACE

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1. Introduction. For nonnegative integers g and k $(2g+k-1\geqq 2)$, let $N(g, k)$ be the order of the largest group of conformal selfmappings (automorphisms) which a compact bordered Riemann surface of genus g and with k boundary components can admit. (If $k=0$ we understand the number $N(g, k)$ for a compact Riemann surface of genus g.) Hurwitz [4] proved that $N(g, 0)\leqq 84(g-1)$. Accola [1] and Maclachlan [7] proved independently that $N(g, 0)\geqq 8(g+1)$ for all g's. Furthermore, Macbeath [6] showed that $N(g, 0)=84(g-1)$ for infinitely many values of g, Accola and Maclachlan showed independently that $N(g, 0)=8(g+1)$ for infinitely many values of g, and many other exact estimations for $N(g, 0)$ were given by Accola [1], Maclachlan [7] and Kiley [5]. The problem seems, however, to remain still open for nfinitelyi many values of g.

On the other hand, for $k\geqq 1$, Oikawa [8, 9] gave a general estimation such that $N(g, k)\leqq 12(g-1)+6k$, and he determined $N(1, k)$ completely. Earlier than he, Heins [3] had determined $N(0, k)$ (in this case naturally $k\geqq 3$) completely. Tsuji [10] treated hyperelliptic Riemann surfaces, and determined $N(2, k)$ exactly.

In this paper we shall prove the following results.

THEOREM 1. $N(g, 1)=4g+2$, *for all* $g\geqq 1$.

THEOREM 2. $N(g, 2)=8g$, *for all* $g\geqq 1$.

THEOREM 3. $N(g, 3)=12g+6$, *if* $g=0$ *or* $g=1$,

$N(g, 3)=6g+3$, *if* $g\neq 0$, $g\neq 1$ *and* $j^2+j+1\equiv 0$ (mod $2g+1$) *has a solution,*

$N(g, 3)=4g+14$, *if* $g\equiv 1$ (mod 9) *and* $j^2+j+1\equiv 0$ (mod $2g+1$) *does not have a solution,*

$N(g, 3)=4g+6$, *if* $g\equiv 0$ (mod 3) *and* $j^2+j+1\equiv 0$ (mod $2g+1$) *does not have a solution,*

$$N(g, 3)=\frac{24g+12}{5}, \quad \textit{if } g=2 \textit{ or } g=7,$$

and

$N(g, 3)=4g+2$, *otherwise.*

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Let $N'(g, k)$ be the order of the largest group of automorphisms of a k-times punctured compact Riemann surface of genus g. Oikawa [8, 9] has proved that $N(g, k)=N'(g, k)$, therefore, it is sufficient to prove the theorems for $N'(g, k)$.

2. Before proving these theorems we shall state some preparatory results. Let W be a Riemann surface and let G be a properly discontinuous group of automorphisms of W. For any subgroup H of G, we can regard W/H as a Riemann surface having a conformal structure which is induced from the conformal structure of W [2]. Let π be the natural projection of W onto W/H. Then we have

LEMMA. *If H is a normal subgroup of G, then for each element f in G there is an automorphism h of W/H satisfying $\pi\circ f=h\circ\pi$.*

Let W be a compact Riemann surface of genus g. We project all the branch points of W with respect to π into W/H and denote them by $\hat{p}_1, \cdots, \hat{p}_r$. Noting that the ramification indices of all the points over $\hat{p}_i$, $i=1, \cdots, r$, are the same, respectively, we denote the corresponding indices by $\nu_1-1, \cdots, \nu_r-1$. Then from the Riemann-Hurwitz relation [4] we have

$$\frac{2g-2}{\mathrm{ord}\,(H)}=2g_0-2+\sum_{i=1}^{r}\left(1-\frac{1}{\nu_i}\right), \tag{1}$$

where $\mathrm{ord}\,(H)$ denotes the order of H and g_0 denotes the genus of W/H. We shall also use the notation $\langle f_1, f_2, \cdots\rangle$ to denote the group generated by the elements $f_1, f_2, \cdots$.

3. Proof of theorem 1. Wiman [11][1)] proved the following: $4g+2$ is the order of the largest cyclic group of automorphisms which a compact Riemann surface of genus g can admit. From this fact we can easily conclude theorem 1. We shall, however, give a proof for the sake of completeness.

Let W be a compact Riemann surface of genus g ($\geqq 1$). We take a point p on W and let G be the group of automorphisms of $W-\{p\}$. It is obvious that G is a cyclic group of finite order. Then from the formula (1) we have

$$\frac{2g-2}{\mathrm{ord}\,(G)}=2g_0-2+\sum_{i=1}^{r}\left(1-\frac{1}{\nu_i}\right)$$

where g_0 denotes the genus of W/G, and $\nu_1, \cdots, \nu_r$ are as in paragraph 2. Without loss of generality we may assume that ν_1 corresponds to p and is equal to $\mathrm{ord}\,(G)$.

If $g_0\geqq 1$, then we have $\mathrm{ord}\,(G)\leqq 2g-1$.

Assume that $g_0=0$ and $r\geqq 4$, then we have

$$\frac{2g-2}{\mathrm{ord}\,(G)}=-2+1-\frac{1}{\mathrm{ord}\,(G)}+\sum_{i=2}^{r}\left(1-\frac{1}{\nu_i}\right)$$
$$\geqq -1-\frac{1}{\mathrm{ord}\,(G)}+\frac{3}{2}.$$

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1) Unfortunately, the anthor could not see directly his paper.

This implies that ord $(G)\leqq 4g-2$.

Assume that $g_0=0$ and $r=3$, then we have

$$\frac{2g-2}{\mathrm{ord}\,(G)}=1-\frac{1}{\mathrm{ord}\,(G)}-\frac{1}{\nu_2}-\frac{1}{\nu_3}.$$

Noting that ord (G) is the least common multiple of ν_2 and ν_3, we have ord $(G)\leqq 4g+2$.

Summing up these estimations we obtain $N(g, 1)\leqq 4g+2$.

To show that $N(g, 1)=4g+2$ we shall give an example of a once-punctured compact Riemann surface of genus g which admits $4g+2$ automorphisms. Let W be the compact Riemann surface of genus g defined by the algebraic equation

$$y^2=x(x^{2g+1}-1).$$

Let p be the point on W which corresponds to $x=0$. Then

$$f\colon (x, y)\longrightarrow (e^{2\pi i/(2g+1)}x,\ e^{\pi i/(2g+1)}y)$$

is an automorphism of $W-\{p\}$. We conclude that ord $(\langle f\rangle)=4g+2$. Therefore, we have $N(g, 1)=4g+2$.

4. Proof of theorem 2. Let W be a compact Riemann surface of genus g $(\geqq 1)$. We distiguish two points p_1 and p_2 on W. Let G be the group of automorphisms of $W-\{p_1, p_2\}$, and let H be the group of automorphisms of W each of which fixes the points p_1 and p_2. Obviously we have ord $(G)\leqq 2$ ord (H). From the formula (1) we have

$$\frac{2g-2}{\mathrm{ord}\,(H)}=2g_0-2+\sum_{i=1}^{r}\left(1-\frac{1}{\nu_i}\right)$$

where g_0 denotes the genus of W/H. Hence H is a cyclic group, we may assume that $\nu_1=\nu_2=$ ord (H) which correspond to p_1 and p_2 respectively.

If $g_0\geqq 1$ then ord $(H)\leqq g$.

If $g_0=0$ and $r=2$, then we have

$$\frac{2g-2}{\mathrm{ord}\,(H)}=-2+2\left(1-\frac{1}{\mathrm{ord}\,(H)}\right).$$

This implies that $g=0$ which is a contradiction.

Therefore, if $g_0=0$, then $r\geqq 3$. In this case we have

$$\frac{2g-2}{\mathrm{ord}\,(H)}=-2+2\left(1-\frac{1}{\mathrm{ord}\,(H)}\right)+\sum_{i=3}^{r}\left(1-\frac{1}{\nu_i}\right)$$

$$\geqq -\frac{2}{\mathrm{ord}\,(H)}+\frac{1}{2}.$$

This implies that ord $(H)\leqq 4g$. Therefore, we have ord $(G)\leqq 2$ ord $(H)\leqq 8g$. Consequently, we conclude that $N(g, 2)\leqq 8g$.

An example shows that $N(g, 2)=8g$. Let W be the compact Riemann surface of genus g which is defined by the algebraic equation

$$y^2=x(x^{2g}-1).$$

Let p_1 and p_2 be the points on W which correspond to $x=0$ and $x=\infty$ respectively. Then

$$f_1\colon (x, y) \longrightarrow (e^{\pi i/g}x, e^{\pi i/2g}y)$$

and

$$f_2\colon (x, y) \longrightarrow (1/x, iy/x^{g+1})$$

are automorphisms of $W-\{p_1, p_2\}$, and we see that $\operatorname{ord}(\langle f_1, f_2\rangle)\geqq 8g$. Therefore, we conclude that $N(g, 2)=8g$.

5. Proof of theorem 3. In the first place for each g ($\geqq 1$) we shall show an example which assures that $N(g, 3)$ is greater than or equal to $4g+2$. Let W be the compact Riemann surface of genus g defined by the equation

$$y^2=x(x^{2g+1}-1).$$

Let p_1 be the point on W which corresponds to $x=0$, and p_2, p_3 the points corresponding to $x=\infty$. From the proof of theorem 1 we conclude that $\operatorname{ord}(\langle f\rangle)=4g+2$ which assures that $N(g, 3)\geqq 4g+2$. For $g=0$ Heins [3] showed that $N(0, 3)=6$. Henceforth, we shall omit the case $g=0$ from our consideration.

6. Let W be a compact Riemann surface of genus g and we distinguish three points p_1, p_2 and p_3 on W. Let G be the group of automorphisms of $W-\{p_1, p_2, p_3\}$. Hence, every member of G can be extended to an automorphism of W, we also denote the group which consists of them by G. Let f_1 denote a generator of the cyclic subgroup of G which consists of all the elements of G that fix the points p_1, p_2 and p_3. For simplicity's sake we shall denote $\operatorname{ord}(\langle f_1\rangle)$ by n. Let f_2 denote an element of G such that $f_2(p_1)=p_2, f_2(p_2)=p_3$ and $f_2(p_3)=p_1$ and let f_3 denote an element of G such that $f_3(p_1)=p_1, f_3(p_2)=p_3$ and $f_3(p_3)=p_2$.

It is easy to see that $\operatorname{ord}(G)$ does not exceed $6n$ regardless of the existence of f_2 or f_3. More precisely, $\operatorname{ord}(G)\leqq 6n$ if $G=\langle f_1, f_2, f_3\rangle$, $\operatorname{ord}(G)\leqq 3n$ if $G=\langle f_1, f_2\rangle$, $\operatorname{ord}(G)\leqq 2n$ if $G=\langle f_1, f_3\rangle$, $\operatorname{ord}(G)=n$ if $G=\langle f_1\rangle$ and $\operatorname{ord}(G)\leqq 6$ otherwise.

If the genus of $W/\langle f_1\rangle$, denoted by g_0, is positive, then by the formula (1) we have

$$\operatorname{ord}(G)\leqq 6n\leqq 4g+2.$$

Indeed, without loss of generality we may assume that $\nu_1=\nu_2=\nu_3=n$, and therefore we have

$$\frac{2g-2}{n}\geqq 3\left(1-\frac{1}{n}\right).$$

Therefore, we may assume that g_0 is equal to zero. In this case from the formula (1) we have

$$n \leqq 2g+1.$$

Hence, $2n \leqq 4g+2$, it is to be observed only when $G=\langle f_1, f_2, f_3\rangle$ and $G=\langle f_1, f_2\rangle$.

7. We shall observe the following five cases. During the discussion of these cases we assume that $\nu_1=\nu_2=\nu_3=n$ in the formula (1).

Case (A): $r=3$ in the formula (1).

In this case we have

$$\frac{2g-2}{n}=-2+3\left(1-\frac{1}{n}\right).$$

Then we see that $\operatorname{ord}(G) \leqq 12g+6$ if $G=\langle f_1, f_2, f_3\rangle$ and $\operatorname{ord}(G) \leqq 6g+3$ if $G=\langle f_1, f_2\rangle$. We shall discuss this case in detail in the following paragraph.

Case (B): $r=4$ in (1).

In this case we have

$$\frac{2g-2}{n}=-2+3\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{\nu_4}\right)$$

$$\geqq 1-\frac{3}{n}+\frac{1}{2}.$$

Therefore, we obtain $n \leqq (4g+2)/3$. In this case $G=\langle f_1, f_2, f_3\rangle$ cannot occur by virtue of lemma. If $G=\langle f_1, f_2\rangle$, we have $\operatorname{ord}(G) \leqq 4g+2$. In the case (B) there is nothing more to do.

Case (C): $r=5$ in (1).

In this case we have

$$\frac{2g-2}{n}=-2+3\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{\nu_4}\right)+\left(1-\frac{1}{\nu_5}\right).$$

If $G=\langle f_1, f_2\rangle$, we obtain $\operatorname{ord}(G) \leqq 3(2g+1)/2<4g+2$. This may be omitted. If $G=\langle f_1, f_2, f_3\rangle$ occurs, we have $\nu_4=\nu_5=m$ by lemma, and m divides n. Then we obtain

$$\operatorname{ord}(G) \leqq 6n=4g+2+\frac{4n}{m}.$$

This case shall be treated in detail later on.

Case (D): $r=6$ in (1).

In this case we have

$$\frac{2g-2}{n}=-2+3\left(1-\frac{1}{n}\right)+\sum_{i=4}^{6}\left(1-\frac{1}{\nu_i}\right).$$

If $G=\langle f_1, f_2\rangle$, we have $\operatorname{ord}(G)\leqq 6(2g+1)/5<4g+2$. There is nothing more to do. If $G=\langle f_1, f_2, f_3\rangle$ occurs, by lemma we have $\nu_4=\nu_5=\nu_6=m$, and m divides n. If $m=2$, we have $n=(4g+2)/5$. Therefore, g must satisfy $2g+1\equiv 0 \pmod 5$ and $\operatorname{ord}(G)\leqq(24g+12)/5$. This is to be treated later on. If $m\geqq 3$, we have $\operatorname{ord}(G)\leqq 6n\leqq 4g+2$. This may be omitted.

Case (E): $r\geqq 7$ in (1).

In this case we have

$$\frac{2g-2}{n}=-2+3\left(1-\frac{1}{n}\right)+\sum_{i=4}^{r}\left(1-\frac{1}{\nu_i}\right)$$

$$\geqq 1-\frac{3}{n}+2.$$

Therefore, we obtain $\operatorname{ord}(G)\leqq 6n\leqq 4g+2$. In this case there is nothing to do.

8. **The case** (**A**). In this case we may assume that f_2^3 is equal to the identity and that f_3^2 is equal to the identity, where f^j denotes the j-th iteration of f. Furthermore, we may assume that f_2 has a fixed point which we denote by q_1. Let π be the natural projection mapping of W onto $W/\langle f_1\rangle$ and let γ_1 be a simple curve starting and ending at $q=\pi(q_1)$, which is freely homotopic in $W/\langle f_1\rangle-\{\pi(p_1), \pi(p_2), \pi(p_3)\}$, to an arbitrary small circle centered at $\pi(p_1)$. Let $\gamma_2=\pi\circ f_2\circ\pi^{-1}(\gamma_1)$ and let $\gamma_3=\pi\circ f_2^2\circ\pi^{-1}(\gamma_1)$. By lemma these are uniquely determined regardless of a choice of a branch of π^{-1}. Let q_{i+1} be the terminal point of the lift of γ_1 starting at q_i $(i=1, \cdots, 2g+1)$. Henceforth, we consider the suffixes of q's by $\operatorname{mod} 2g+1$. If we set q_{1+j} the terminal point of the lift of γ_2 starting at q_1, then by the monodromy theorem we establish that q_{1+ij} is the terminal point of the lift of γ_2 starting at $q_{1+(i-1)j}$ $(i=1, \cdots, 2g+1)$. This assures that $f_2(q_{1+i})=q_{1+ij}$. Then we have

$$q_2=f_2^3(q_2)=f_2^2(q_{1+j})=f_2(q_{1+j^2})=q_{1+j^3}.$$

Therefore we have

$$j^3-1\equiv 0 \qquad \pmod{2g+1}.$$

If there exists f_3, we may also assume that f_3 has a fixed point which is different from p_1, and we denote it by q_1'. Let γ_2' be a simple curve starting and ending at $q'=\pi(q_1')$, which is freely homotopic in $W/\langle f_1\rangle-\{\pi(p_1), \pi(p_2), \pi(p_3)\}$, to a small circle centered at $\pi(p_2)$, and let $\gamma_3'=\pi\circ f_3\circ\pi^{-1}(\gamma_2')$. Let q_{i+1}' be the terminal point of the lift of γ_2' starting at q_i' $(i=1, \cdots, 2g+1)$, and set q_{1+j}' the terminal point of the lift of γ_3' starting at q_1'. Then we have

$$q_2'=f_3^2(q_2')=f_3(q_{1+j}')=q_{1+j^2}'.$$

Therefore, we have

$$j^2-1\equiv 0 \qquad (\mathrm{mod}\ 2g+1).$$

Consequently, if the case $G=\langle f_1, f_2, f_3\rangle$ occurs, $j^3-1\equiv 0$ (mod $2g+1$) and $j^2-1 \equiv 0$ (mod $2g+1$) has a common solution, i.e. $j=1$. Then we continue a branch of $\pi^{-1}(q)$ along γ_1, γ_2 and γ_3 successively. Hence, there is no branch point in W but p_1, p_2 and p_3, we have $2g+1=3$. Therefore, this case does not occur except for $g=1$.

If the case $G=\langle f_1, f_2\rangle$ occurs, $j^2+j+1\equiv 0$ (mod $2g+1$) has a solution.

9. The case (C) and the case (D). In these cases we consider an intermediate covering surface $W/\langle f_1^{n/m}\rangle$ of $W/\langle f_1\rangle$. The natural projection mapping $W/\langle f_1^{n/m}\rangle$ onto $W/\langle f_1\rangle$ does not ramify but $\pi(p_1)$, $\pi(p_2)$ and $\pi(p_3)$. Hence, we can apply the discussion in paragraph 8 to $W/\langle f_1^{n/m}\rangle$, we may conclude that if the case $G=\langle f_1, f_2, f_3\rangle$ occurs, $n=m$ or $n=3m$. The former corresponds to $g'=0$ and the latter to $g'=1$, where g' denotes the genus of $W/\langle f_1^{n/m}\rangle$.

In the case (C), if the case $n=m$ occurs, we establish that $3n=2g+3$ which implies that $g\equiv 0$ (mod 3) and if the case $n=3m$ occurs, we establish that $9m=2g+7$ which implies that $g\equiv 1$ (mod 9).

In the case (D), if $n=m=2$ then $g=2$ and if $n=3m=6$ then $g=7$.

10. Examples. To show the exactness it is sufficient to construct some examples.

EXAMPLE 1. For $g=1$, let W be the Riemann surface defined by the equation

$$y^3=x^3-1.$$

Let p_1, p_2 and p_3 be the points corresponding to $x=1$, $e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively. Set

$$f_1\colon (x, y) \longrightarrow (x, e^{2\pi i/3}y),$$

$$f_2\colon (x, y) \longrightarrow (e^{2\pi i/3}x, y)$$

and

$$f_3\colon (x, y) \longrightarrow (1/x, -y/x).$$

Then we have

$$N(g, 3)=12g+6.$$

EXAMPLE 2. For g such that $j^2+j+1\equiv 0$ (mod $2g+1$) has a solution and $g\neq 1$, let W be the Riemann surface defined by the equation

$$y^{2g+1}=(x-1)(x-e^{2\pi i/3})^j(x-e^{4\pi i/3})^{j^2}$$

where j is a solution of $j^2+j+1\equiv 0$ (mod $2g+1$). Let p_1, p_2 and p_3 be the points corresponding to $x=1$, $e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively. Set

$$f_1\colon (x, y) \longrightarrow (x, e^{2\pi i/(2g+1)}y)$$

and

$$f_2\colon (x, y) \longrightarrow \left(e^{2\pi i/3}x, \frac{e^{2\pi(1+j+j^2)i/3(2g+1)}y^j}{(x-e^{4\pi i/3})^{(j^3-1)/(2g+1)}}\right).$$

Then we have

$$N(g, 3)=6g+3.$$

EXAMPLE 3. For g such that $j^2+j+1\equiv 0 \pmod{2g+1}$ does not have a solution and $g\equiv 1 \pmod 9$, let W be the Riemann surface defined by the equation

$$y^{(2g+7)/3}=x^{(g-1)/3}(x^3-1).$$

Let p_1, p_2 and p_3 be the points corresponding to $x=1, e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively. Set

$$f_1\colon (x, y) \longrightarrow (x, e^{6\pi i/(2g+7)}y),$$

$$f_2\colon (x, y) \longrightarrow (e^{2\pi i/3}x, y)$$

and

$$f_3\colon (x, y) \longrightarrow (1/x, -y/x)$$

Then we have

$$N(g, 3)=4g+14.$$

EXAMPLE 4. For g such that $j^2+j+1\equiv 0 \pmod{2g+1}$ does not have a solution and $g\equiv 0 \pmod 3$, let W be the Riemann surface defined by the equation

$$y^{(2g+3)/3}=x^{(g-3)/3}(x^3-1).$$

Let p_1, p_2 and p_3 be the points corresponding to $x=1, e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively. Set

$$f_1\colon (x, y) \longrightarrow (x, e^{6\pi i/(2g+3)}y),$$

$$f_2\colon (x, y) \longrightarrow (e^{2\pi i/3}x, e^{4\pi i/3}y)$$

and

$$f_3\colon (x, y) \longrightarrow (1/x, -y/x).$$

Then we have

$$N(g, 3)=4g+6.$$

EXAMPLE 5. For $g=2$ or 7, let W be the Riemann surface defined by the equation

$$y^{(4g+2)/5}=(x^3-1)(x^3+1)^{(2g+1)/5}.$$

Let p_1, p_2 and p_3 be the points corresponding to $x=1, e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively. Set

$$f_1\colon (x, y) \longrightarrow (x, e^{5\pi i/(2g+1)}y),$$

$$f_2\colon (x, y) \longrightarrow (e^{2\pi i/3}x, y)$$

and

$$f_3\colon (x, y) \longrightarrow (1/x, e^{\pi i/2}y/x^{3(g+3)/(2g+1)}).$$

Then we have

$$N(g, 3) = \frac{24g+12}{5}.$$

Summing up, we have concluded our theorem 3.

11. Some criteria for the solubility of the congruence $j^2+j+1\equiv 0 \pmod{2g+1}$. If p is a prime number, then the following congruence holds for every integer j (Fermat's theorem):

$$j^p - j \equiv 0 \qquad (\text{mod } p).$$

Suppose that $g\equiv 0$ (mod 3) and that $2g+1$ is prime, we have

$$j^{2g+1} - j = (j^2+j+1)P(j)$$

where $P(j)$ is a polynomial of degree $2g-1$ with integral coefficients. The congruence $P(j)\equiv 0$ (mod $2g+1$) has at most $2g-1$ solutions while the congruence $j^{2g+1}-j\equiv 0$ (mod $2g+1$) has $2g+1$ solutions, and consequently, the congruence $j^2+j+1 \equiv 0$ (mod $2g+1$) has two solutions.

Suppose that $g\equiv 2$ (mod 3) and that $2g+1$ is prime, we have

$$j^{2g+1} - j = (j^2+j+1)P(j) - (2j+1)$$

where $P(j)$ is a polynomial of degree $2g-1$ with integral coefficients. If the congruence $j^2+j+1\equiv 0$ (mod $2g+1$) has a solution, then the congruence $2j+1\equiv 0$ (mod $2g+1$) must have the same solution. This is impossible.

It is obvious that if the congruence $j^2+j+1\equiv 0$ (mod p) is unsoluble then for every multiple of p, denoted by q, the congruence $j^2+j+1\equiv 0$ (mod q) is unsoluble, and it is easily seen that every number of the form $6m+5$ is divisible by a prime number of the form $6m'+5$.

Thus we conclude that if $g\equiv 2$ (mod 3) then the congruence $j^2+j+1\equiv 0$ (mod $2g+1$) is unsoluble.

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