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ON ESSENTIALLY ISOMETRIC CONFORMAL TRANSFORMATION GROUPS

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A conformal transformation group G of a Riemannian manifold $(\mathfrak{B}, g)$, $\mathfrak{B}$ being a manifold and g a Riemannian metric tensor field on $\mathfrak{B}$, is said to be *essentially isometric*, if there exists a positive-valued function f on $\mathfrak{B}$ such that G is an isometry group of another Riemannian manifold $(\mathfrak{B}, fg)$. Throughout this paper, we assume that manifolds, functions, vector fields and tensor fields are differentiable and of class C^∞, Riemannian manifolds are connected and conformal transformation groups are connected and effective. The purpose of the present paper is to prove some theorems concerning essentially isometric conformal transformation groups.

For a Riemannian manifold $(\mathfrak{B}, g)$ of dimension n, we denote by K, S and k respectively the curvature tensor, the Ricci tensor and the curvature scalar. We consider two tensor fields C and $'C$ defined respectively by

$$C(X, Y)Z=K(X, Y)Z+\frac{1}{n-2}[L(X, Z)Y-L(Y, Z)X+g(X, Z)l(Y)-g(Y, Z)l(X)],$$

$$'C(X, Y, Z)=(\nabla_X S)(Y, Z)-(\nabla_Y S)(X, Z),$$

X, Y and Z being arbitrary vector fields, where L and l are tensor fields defined respectively by

$$L(X, Y)=S(X, Y)-\frac{k}{2(n-1)}g(X, Y),$$

$$g(l(X), Y)=L(X, Y).$$

When $n>3$ (resp. when $n=3$), we call C (resp. $'C$) the *Weyl conformal curvature tensor* of $(\mathfrak{B}, g)$.

The following theorems concerning essentially isometric conformal transformation groups are well-known.

THEOREM A (Hlavatý [3]). *If the Weyl conformal curvature tensor field of a Riemannian manifold $(\mathfrak{B}, g)$, $\mathfrak{B}$ being of dimension >2, is not zero at every point of $\mathfrak{B}$, then any conformal transformation group of $(\mathfrak{B}, g)$ is essentially isometric.*

THEOREM B (Ishihara [4]). *If a conformal transformation group G of a Riemannian manifold is compact, then G is essentially isometric.*

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We now consider a connected and effective Lie transformation group G of a manifold $\mathfrak{V}$ and denote by $\mathfrak{G}$ the Lie algebra of G. To an element X of $\mathfrak{G}$ there corresponds canonically a vector field $\tilde{X}$ on $\mathfrak{V}$ and the correspondence is an isomorphism of $\mathfrak{G}$ onto a Lie algebra $\tilde{\mathfrak{G}}$ consisting of $\tilde{X}$ corresponding to X of $\mathfrak{G}$. We denote by $\mathcal{L}_{\tilde{X}}$ the Lie derivation with respect to $\tilde{X}$. The group G is a conformal transformation group of a Riemannian manifold $(\mathfrak{V}, g)$ if and only if

$$\mathcal{L}_{\tilde{X}}g = 2\phi_{\tilde{X}}g \tag{1}$$

for any element X of $\mathfrak{G}$, $\phi_{\tilde{X}}$ being a function on $\mathfrak{V}$ which is called the function associated with $\tilde{X}$. The group G is an isometry group of $(\mathfrak{V}, g)$ if and only if

$$\mathcal{L}_{\tilde{X}}g = 0$$

for any element X of $\mathfrak{G}$.

We now prove the following

LEMMA 1. *A necessary and sufficient condition that a conformal transformation group G of a Riemannian manifold $(\mathfrak{V}, g)$ be essentially isometric is that there exists a positive-valued function f on $\mathfrak{V}$ satisfying*

$$\mathcal{L}_{\tilde{X}}f = -2\phi_{\tilde{X}}f \tag{2}$$

for any element X of $\mathfrak{G}$, where $\mathfrak{G}$ is the Lie algebra of G and $\phi_{\tilde{X}}$ is the function associated with $\tilde{X}$.

Proof. Suppose that G is essentially isometric. Then there exists a positive-valued function f on $\mathfrak{V}$ such that

$$\mathcal{L}_{\tilde{X}}(fg) = 0 \tag{3}$$

for any element X of $\mathfrak{G}$, from which we have (2) by virtue of (1). Conversely, suppose that the relation (2) holds for some positive-valued function f on $\mathfrak{V}$ and for any element X of $\mathfrak{G}$. Then, from (1) and (2), we have (3) which means that G is essentially isometric.

LEMMA 2. *Assume that a conformal transformation group G of a Riemannian manifold $(\mathfrak{V}, g)$ is essentially isometric. If G contains a Lie subgroup M which is a transitive isometry group of $(\mathfrak{V}, g)$, then G is necessarily an isometry group of $(\mathfrak{V}, g)$.*

Proof. By Lemma 1, there exists a positive-valued function f on $\mathfrak{V}$ satisfying (2). If we denote by $\mathfrak{M}$ the Lie algebra of the Lie subgroup M which is transitive and isometric, then we have

$$\mathcal{L}_{\tilde{X}}f = 0 \tag{4}$$

for any element X of $\mathfrak{M}$ because $\phi_{\tilde{X}} = 0$. On the other hand, since $\mathfrak{V}$ is connected and M is transitive, it follows from (4) that f is a positive constant on $\mathfrak{V}$. Consequently we have the relation (4) for any element X of $\mathfrak{G}$, $\mathfrak{G}$ being the Lie algebra of G. Therefore, from (2), we have $\phi_{\tilde{X}}=0$ which means that G is an isometry group.

From Theorem A and Lemma 2, we have

THEOREM 1. *Let $(\mathfrak{V}, g)$ be a Riemannian manifold of dimension >2, whose Weyl conformal curvature tensor field vanishes nowhere. If a conformal transformation group G of $(\mathfrak{V}, g)$ contains a Lie subgroup which is a transitive isometry group of $(\mathfrak{V}, g)$, then G is necessarily an isometry group.*

We next prove the following

LEMMA 3. *If a conformal transformation group G of a Riemannian manifold $(\mathfrak{V}, g)$ is transitive and the center of G is not discrete, then G is essentially isometric.*

Proof. Since the center Z of G is not discrete, the Lie algebra $\mathfrak{Z}$ of Z is of positive dimension. Therefore we can take a non-zero element Y of $\mathfrak{Z}$. The vector field $\tilde{Y}$ on $\mathfrak{V}$, which corresponds canonically to Y, is left invariant by the action of G because Z is the center of G. Since G is assumed to be transitive and effctive, the vector field $\tilde{Y}$ is not zero at every point of $\mathfrak{V}$. Consequently, a function f on $\mathfrak{V}$ defined by

$$f=g(\tilde{Y}, \tilde{Y})^{-1}$$

is positive-valued. We can prove that

$$\mathcal{L}_{\tilde{X}} f=-2\phi_{\tilde{X}} f$$

holds for any element X of $\mathfrak{G}$, $\mathfrak{G}$ being the Lie algebra of G. Hence, by Lemma 1, G is essentially isometric.

We shall prove the following

THEOREM 2. *Let G be a transitive conformal transformation group of a Riemannian manifold. If G is isomorphic onto the direct product of a vector group and a compact Lie group, then G is essentially isometric.*

Proof. Assume that G is isomorphic onto the direct product of a vector group R^l of dimension l and a compact Lie group K. First suppose $l=0$. Then G is compact and hence it follows from Theorem B that G is essentially isometric. Next suppose $l>0$. Then, since the center Z of G contains the vector group R^l, Z is not discrete. Thus, by Lemma 3, G is essentially isometric.

To get Theorem 3, we need the following

LEMMA 4. *Let G be a connected and effective isometry group of a compact and orientable Riemannian manifold $(\mathfrak{V}, g)$. Then G is isomorphic onto the direct* *product of a vector group and a compact Lie group.*

Proof. Consider an arbitrary function F on $\mathfrak{B}$. Then we have

$$\int_{\mathfrak{B}} \mathcal{L}_{\tilde{X}} F dv = 0$$

for any element X of $\mathfrak{G}$, where dv is the volume element of $(\mathfrak{B}, g)$ and $\mathfrak{G}$ the Lie algebra of G. Therefore, for any X, Y and V of $\mathfrak{G}$, we have

$$\int_{\mathfrak{B}} \mathcal{L}_{\tilde{X}}(g(\tilde{Y}, \tilde{V}))dv = 0,$$

which reduces to

$$\int_{\mathfrak{B}} g([\tilde{X}, \tilde{Y}], \tilde{V})dv + \int_{\mathfrak{B}} g(\tilde{Y}, [\tilde{X}, \tilde{V}])dv = 0. \tag{5}$$

If we put

$$Q(X, Y) = \int_{\mathfrak{B}} g(\tilde{X}, \tilde{Y})dv$$

for any elements X and Y of $\mathfrak{G}$, then Q is a symmetric positive definite bilinear form on $\mathfrak{G} \times \mathfrak{G}$ because G is assumed to be effective. We have, from (5),

$$Q([X, Y], V) + Q(Y, [X, V]) = 0,$$

which shows that Q is an invariant form of $\mathfrak{G}$. Hence, $\mathfrak{G}$ is decomposed into the direct sum

$$\mathfrak{G} = \mathfrak{Z} + [\mathfrak{G}, \mathfrak{G}],$$

where $\mathfrak{Z}$ is the center of $\mathfrak{G}$ and the ideal $[\mathfrak{G}, \mathfrak{G}]$ is semisimple and compact (see [2] for instance). Consequently, the group G is isomorphic onto the direct product of a vector group and a compact Lie group.

The above proof of Lemma 4 is quite similar to the proof of the following theorem due to Barbance [1].

THEOREM C. *If $(\mathfrak{B}, g)$ is a compact and orientable Riemannian manifold of dimension >2 and the Weyl conformal curvature tensor field is not zero at every point of $\mathfrak{B}$, then the largest connected group of conformal transformations of $(\mathfrak{B}, g)$ is isomorphic onto the direct product of a vector group and a compact Lie group.*

From Lemma 4, we have immediately the following

THEOREM 3. *Let G be a conformal transformation group of a compact Riemannian manifold $(\mathfrak{B}, g)$. If G is essentially isometric then G is isomorphic onto a vector group and a compact Lie group.*

In fact, in the case where $\mathfrak{V}$ in Theorem 3 is not orientable, by considering a double covering of $(\mathfrak{V}, g)$, which is a compact and orientable Riemannian manifold, we can get the result of Theorem 3.

From Theorems A and 3, we have Theorem C, while Theorem 3 holds for any conformal transformation group of a compact Riemannian manifold.

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