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ON UNIVALENT ENTIRE FUNCTIONS

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§1. Shah and Trimble [2] proved the following result. Let $f(z)$ be a transcendental entire function such that

$$f'(z)=ce^{\beta z}\prod_{n=1}^{N}\left(1-\frac{z}{z_n}\right), \tag{1.1}$$

where $0\leqq N\leqq\infty$, and c, β, z_n are all complex numbers such that $c\neq 0$, $|\beta|\leqq 1$ and $|z_n|>2$. Then f maps $D=\{z: |z|<1\}$ univalently onto a convex domain if

$$|\beta|+\prod_{n=1}^{N}\frac{1}{|z_n|-1}\leqq 1. \tag{1.2}$$

In this paper we shall improve the condition (1.2) to conclude only the univalence of $f(z)$.

THEOREM 1. *Suppose $f(z)$ is a transcendental entire function such that $f'(z)$ is given by* (1.1), *where c, β, z_n are all complex numbers such that $c\neq 0$, $|\beta|\leqq 2$ and $|z_n|>1$. Let*

$$\gamma_n=|z_n|-\sqrt{|z_n|^2-1}.$$

Then $f(z)$ is univalent in D if

$$\left(\frac{|\beta|}{2}+\sum_{n=1}^{N}\gamma_n\right)^2+2\sum_{n=1}^{N}\gamma_n^2\leqq 1. \tag{1.3}$$

Proof. Denote the Schwarzian derivative of $f(z)$ by $\{f, z\}$, i.e.,

$$\{f, z\}=\left(\frac{f''}{f'}\right)'-\frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Nehari [1] proved that for an analytic function f to be univalent in D it is necessary that

$$|\{f, z\}|\leqq\frac{6}{(1-|z|^2)^2}, \qquad z\in D \tag{1.4}$$

and sufficient that

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$$|\{f, z\}| \leqq \frac{2}{(1-|z|^2)^2}, \qquad z \in D. \tag{1.5}$$

Now we have for $z \in D$

$$(1-|z|^2)^2 |\{f, z\}| \leqq \sum_{n=1}^{N} \left(\frac{1-|z|^2}{|z_n|-|z|} \right)^2 + \frac{1}{2} \left(|\beta| + \sum_{n=1}^{N} \frac{1-|z|^2}{|z_n|-|z|} \right)^2. \tag{1.6}$$

Define $h(x)$ by

$$h(x) = \frac{1-x^2}{a-x} \qquad (a>1)$$

for $0 \leqq x \leqq 1$. Then

$$\max_{0 \leqq x \leqq 1} h(x) = h(a-\sqrt{a^2-1}) = 2(a-\sqrt{a^2-1}).$$

Hence for every $z \in D$

$$\frac{1-|z|^2}{|z_n|-|z|} \leqq 2\gamma_n. \tag{1.7}$$

Now (1.6), (1.7) and (1.3) yield (1.5), and hence $f(z)$ is univalent in D.

§ **2.** Let $f_p(z)$ ($p=0$ or $p=1$) be a transcendental entire function defined by

$$f_p(z) = c^{1-p} z^p e^{\beta z} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right), \tag{2.1}$$

where $c \neq 0$ is real, $\beta \leqq 0$, $0 \leqq N \leqq \infty$ and $\{a_n\}_{n=1}^{N}$ are all real numbers such that $a_{n+1} \geqq a_n > 1$. Let $a_0 = 0$ and $\{a_j^{(k)}\}$ denote the sequence of zeros of $f_p^{(k)}(z)$, where $|a_j^{(k)}| \leqq |a_{j+1}^{(k)}|$, and it is understood that j starts from 0 when $p=1$ and j starts from 1 when $p=0$. $a_j^{(0)}$ denotes a_j.

Under these definitions and notations, we have

Lemma. *For all $k \geqq 1$, $f_p^{(k)}(z)$ has exactly $N+p$ number of zeros which are all real and positive, and*

$$a_n^{(k-1)} \leqq a_n^{(k)} \leqq a_{n+1}^{(k-1)} \leqq a_{n+1}^{(k)}. \tag{2.2}$$

Further,

$$f_p^{(k)}(z) = f_p^{(k)}(0) e^{\beta z} \sum_{n=\delta}^{N} \left(1 - \frac{z}{a_n^{(k)}}\right), \tag{2.3}$$

where $\delta = 0$ if $p=1$, and $\delta = 1$ if $p=0$.

Proof. When $p=1$, the proof was given in [2]. We omit the proof for the case when $p=0$, since it is essentially same as given in [2].

Shah and Trimble [2] also proved that if $f_1(z)$ is defined by (2.1) then $f_1(z)$

and all its derivatives map D univalently onto convex domains if and only if

$$|\beta|+\sum_{n=1}^{N}\frac{1}{|a_k^{(1)}|-1}\leqq 1.$$

We prove

THEOREM 2. *Suppose $f_0(z)$ is a transcendental entire function defined by* (2. 1). *Then $f_0(z)$ and all its derivatives are univalent in D if*

$$\left(\frac{|\beta|}{2}+\sum_{n=1}^{N}\gamma_n^{(1)}\right)^2+2\sum_{n=1}^{N}(\gamma_n^{(1)})^2\leqq 1, \tag{2. 4}$$

where

$$\gamma_n^{(1)}=a_n^{(1)}-\sqrt{(a_n^{(1)})^2-1}.$$

Proof. Define $\gamma_n^{(k)}=a_n^{(k)}-\sqrt{(a_n^{(k)})^2-1}$ for $k\geqq 1$. By (2. 2) we have for $k\geqq 1$

$$\sum_{n=1}^{N}\gamma_n^{(k)}\leqq\sum_{n=1}^{N}\gamma_n^{(k-1)}\leqq\sum_{n=1}^{N}\gamma_n^{(1)}.$$

Hence from (2. 4)

$$\left(\frac{|\beta|}{2}+\sum_{n=1}^{N}\gamma_n^{(k)}\right)^2+2\sum_{n=1}^{N}(\gamma_n^{(k)})^2\leqq 1.$$

By Theorem 1 $f^{(k-1)}(z)$ is univalent in D for every $k\geqq 1$.

§3. REMARK. (i) In fact, the condition (1. 2) implies the condition (1. 3). To verify this we first notice that

$$1\geqq\sum_{n=1}^{N}\frac{1}{|z_n|-1}=\sum_{n=1}^{N}\frac{2\gamma_n}{(1-\gamma_n)^2}>2\sum_{n=1}^{N}\gamma_n.$$

Now

$$\left(|\beta|+\sum_{n=1}^{N}\frac{1}{|z_n|-1}\right)-\left\{\left(\frac{|\beta|}{2}+\sum_{n=1}^{N}\gamma_n\right)^2+2\sum_{n=1}^{N}\gamma_n^2\right\}$$

$$\geqq\left(|\beta|+2\sum_{n=1}^{N}\gamma_n\right)-\left\{\frac{|\beta|^2}{4}+|\beta|\sum_{n=1}^{N}\gamma_n+3\left(\sum_{n=1}^{N}\gamma_n\right)^2\right\}$$

$$\geqq\frac{|\beta|}{2}\left(1-\frac{|\beta|}{2}\right)+\frac{1}{2}\Sigma\gamma_n>0.$$

Hence (1. 2) implies (1. 3).

(ii) In Theorem 1, the condition (1. 3) cannot be replaced by any condition which is sharper than

$$\frac{1}{2}\left(|\beta|+\sum_{n=1}^{N}\frac{1}{|z_n|}\right)^2+\sum_{n=1}^{N}\frac{1}{|z_n|^2}\leqq 6.$$

This can be easily seen from the fact that for $f_0(z)$ defined in § 2 we have

$$f_0'(z)=f_0'(0)e^{\beta z}\sum_{n=1}^{N}\left(1-\frac{z}{a_n^{(1)}}\right),$$

and hence by (1. 4)

$$\frac{1}{2}\left(|\beta|+\Sigma\frac{1}{a_n^{(1)}}\right)^2+\Sigma\frac{1}{(a_n^{(1)})^2}=|\{f, z\}|_{z=0}\leqq 6.$$

References

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