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ON THE BEHAVIOUR OF A SERIES ASSOCIATED WITH THE ALLIED SERIES OF A FOURIER SERIES

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1. Let $0\leq\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_n\to\infty$ as $n\to\infty$. The series $\sum a_n$ is said to be summable $|R, \lambda_n, 1|$ if

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\lambda_n}-\frac{1}{\lambda_{n+1}}\right\}\left|\sum_{\nu=1}^{n}\lambda_\nu a_\nu\right|<\infty.$$

Also, the series $\sum a_n$ is said to be summable $|N, 1/(n+1)|$ if

$$\sum_{n=1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=0}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)a_{n-\nu}\right|<\infty,$$

where

$$P_n=\sum_{\nu=0}^{n}\frac{1}{\nu+1}\sim\log n.$$

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable in $(-\pi, \pi)$ and let

$$f(t)\sim\frac{1}{2}a_0+\sum(a_n\cos nt+b_n\sin nt),$$

where the coefficients a_n and b_n are given by the usual Euler-Fourier formulae. The allied series of the above series is

$$\sum(b_n\cos nt-a_n\sin nt)\equiv\sum B_n(t).$$

We write

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}.$$

2. In an attempt to show that the behaviour of the series $\sum B_n(x)/\log(n+1)$ is more or less like that of the allied series $\sum B_n(x)$, Mohanty and Ray [4] recently established the following

THEOREM A. *If $\psi(t)$ is of bounded variation in $(0, \pi)$ and $|\psi(t)|/t\log(k/t)$ $(k>\pi)$*

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is integrable in $(0, \pi)$, *then the series* $\sum B_n(x)/\log(n+1)$ *is summable* $|R, e^{n^\alpha}, 1|$, *where* $0<\alpha<1$.

It has been recently [1] established that every series summable by the method $|N, 1/(n+1)|$ is also summable by the method $|R, e^{n^\alpha}, 1|$ but the converse is, in general, *false*. In this paper we establish the following

THEOREM. *If* $\psi(t)$ *is of bounded variation in* $(0, \pi)$ *and* $|\psi(t)|/t \log(k/t)$ $(k>\pi)$ *is integrable in* $(0, \pi)$, *then the series* $\sum B_n(x)/\log(n+1)$ *is summable* $|N, 1/(n+1)|$.

It is interesting to note that although the series $\sum B_n(x)/\log(n+1)$ behaves like the allied series $\sum B_n(x)$ as far as summability $|R, e^{w^\alpha}, 1|$ is concerned as is shown by Theorem A and Theorem 5 in [2], the function $\psi(t)/\log(k/t)$ in Theorem A playing the role of $\psi(t)$ in the corresponding one for the series $\sum B_n(x)$, our theorem in view of a known result (see [3], Theorem 1) clearly shows that the *same is not true* if instead of the summability $|R, e^{w^\alpha}, 1|$ the summability $|N, 1/(n+1)|$ is considered.

3. The following lemma is pertinent to the proof of our theorem.

LEMMA.

(3. 1)
$$\sum_{\nu=0}^{[n/2]-2}\left|\Delta\left(\frac{(n+1)P_n-(\nu+1)P_\nu}{(n-\nu)\log(n-\nu+1)}\right)\right|=O(1),$$

and

(3. 2)
$$\sum_{n=[1/t]+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=[n/2]}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\cos(n-\nu)t}{(n-\nu)\log(n-\nu+1)}\right|=O(1).$$

The estimate in (3. 1) is known (see Lemma 4 in [5]). The estimate in (3. 2) of the lemma can be obtained similarly as the estimation of $\sum_3$ in [5].

4. Proof of the theorem. Before proceeding to prove the theorem we note that (see Lemma 1 in [4]) the assumption of the theorem is equivalent to

(4. 1)
$$\int_0^\pi \log\frac{k}{t}\left|d\left\{\frac{\psi(t)}{\log(k/t)}\right\}\right|<\infty,$$

and

(4. 2)
$$\lim_{t\to 0+}\frac{\psi(t)}{\log(k/t)}=0,$$

Using the condition (4. 2) and proceeding as in [4] we have

$$\frac{B_n(x)}{\log(n+1)}=\frac{2}{\pi}\int_0^\pi d\left\{\frac{\psi(t)}{\log(k/t)}\right\}\int_t^\pi \log\frac{k}{u}\,\frac{\sin nu}{\log(n+1)}\,du$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \frac{B_{n-\nu}(x)}{\log (n-\nu+1)} \right|$$

$$\leqq \frac{2}{\pi} \int_0^\pi \left| d\left\{ \frac{\psi(t)}{\log (k/t)} \right\} \right| \left| \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \int_t^\pi \log \frac{k}{u} \frac{\sin (n-\nu)u}{\log (n-\nu+1)} du \right| \right|.$$

Hence by virtue of the condition (4. 1), in order to establish the theorem, it is sufficient to show that uniformly in $0 < t \leqq \pi$,

$$\Sigma \equiv \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \int_t^\pi \log \frac{k}{u} \frac{\sin (n-\nu)u}{\log (n-\nu+1)} du \right| = O\left(\log \frac{k}{t}\right). \tag{4. 3}$$

Now

$$\Sigma \leqq \sum_{n=1}^{[1/t]} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \int_0^\pi \log \frac{k}{u} \frac{\sin (n-\nu)u}{\log (n-\nu+1)} du \right|$$

$$+ \sum_{n=1}^{[1/t]} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \int_0^t \log \frac{k}{u} \frac{\sin (n-\nu)u}{\log (n-\nu+1)} du \right|$$

$$+ \sum_{n=[1/t]+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{[n/2]-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \int_t^\pi \log \frac{k}{u} \frac{\sin (n-\nu)u}{\log (n-\nu+1)} du \right| \tag{4. 4}$$

$$+ \sum_{n=[1/t]+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=[n/2]}^{n-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1} \right) \int_t^\pi \log \frac{k}{u} \frac{\sin (n-\nu)u}{\log (n-\nu+1)} du \right|$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \quad \text{say.}$$

Using the estimate (see proof of (2. 2. 10) in [4])

$$\int_0^\pi \log \frac{k}{u} \sin nu \, du = O\left(\frac{\log n}{n} \right),$$

we have

$$\begin{aligned} \Sigma_1 &= O(1) \sum_{n=1}^{[1/t]} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_n \log (n-\nu)}{(\nu+1)(n-\nu) \log (n-\nu+1)} \\ &= O(1) \sum_{n=1}^{[1/t]} \frac{1}{(n+1) P_{n-1}} \sum_{\nu=0}^{n-1} \frac{1}{n-\nu} + O(1) \sum_{n=1}^{[1/t]} \frac{1}{(n+1) P_{n-1}} \sum_{\nu=0}^{n-1} \frac{1}{\nu+1} \\ &= O(1) \sum_{n=1}^{[1/t]} \frac{1}{n+1} \\ &= O\left(\log \frac{k}{t} \right). \end{aligned} \tag{4. 5}$$

And

$$\sum\nolimits_2 = O\left(t \log \frac{k}{t}\right) \sum_{n=1}^{[1/t]} \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} \frac{P_n}{(\nu+1)\log(n-\nu+1)}$$

(4.6)
$$= O\left(t \log \frac{k}{t}\right) \sum_{n=1}^{[1/t]} \frac{1}{P_{n-1}} \sum_{\nu=0}^{n-1} \frac{1}{\nu+1}$$

$$= O\left(\log \frac{k}{t}\right).$$

By the application of the second mean value theorem we have

$$\sum\nolimits_3 = \sum_{n=[1/t]+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{[n/2]-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1}\right) \frac{\log (k/t)}{\log (n-\nu+1)} \int_t^\mu \sin (n-\nu)u \, du \right| \quad (t \leqq \mu \leqq \pi)$$

$$\leqq \left(\log \frac{k}{t}\right) \sum_{n=[1/t]+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{[n/2]-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1}\right) \frac{\cos (n-\nu)t}{(n-\nu)\log (n-\nu+1)} \right|$$

$$+ \left(\log \frac{k}{t}\right) \sum_{n=[1/t]+1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{\nu=0}^{[n/2]-1} \left(\frac{P_n}{\nu+1} - \frac{P_\nu}{n+1}\right) \frac{\cos (n-\nu)\mu}{(n-\nu)\log (n-\nu+1)} \right|$$

$$= \left(\log \frac{k}{t}\right) \left(\sum\nolimits_{3,1} + \sum\nolimits_{3,2}\right), \quad \text{say.}$$

Applying Abel's transformation to the inner sum in $\sum_{3,1}$ and using the estimate

$$\sum \frac{\cos (n-\nu)t}{\nu+1} = O\left(\log \frac{k}{t}\right),$$

we get

$$\sum\nolimits_{3,1} = O\left(\log \frac{k}{t}\right) \sum_{n=[1/t]+1}^{\infty} \frac{1}{(n+1)P_n P_{n-1}} \sum_{\nu=0}^{[n/2]-2} \left| \Delta \frac{(n+1)P_n - (\nu+1)P_\nu}{(n-\nu)\log (n-\nu+1)} \right|$$

$$+ O\left(\log \frac{k}{t}\right) \sum_{n=[1/t]+1}^{\infty} \frac{1}{(n+1)P_n P_{n-1}} \frac{(n+1)P_n - [n/2]P_{[n/2]-1}}{(n-[n/2]+1)\log (n-[n/2]+2)}$$

$$= O\left(\log \frac{k}{t}\right) \sum_{n=[1/t]+1}^{\infty} \frac{1}{(n+1)\log^2 (n+1)}$$

$$= O(1).$$

Similarly we can show that $\sum_{3,2} = O(1)$, and then

(4.7)
$$\sum\nolimits_3 = O\left(\log \frac{k}{t}\right).$$

Again, proceeding similarly as in $\sum_3$, we get

$$\Sigma_4 \leqq \left(\log\frac{k}{t}\right)\sum_{n=[1/t]+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=[n/2]}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\cos(n-\nu)t}{(n-\nu)\log(n-\nu+1)}\right|$$

$$+\left(\log\frac{k}{t}\right)\sum_{n=[1/t]+1}^{\infty}\frac{1}{P_nP_{n-1}}\left|\sum_{\nu=[n/2]}^{n-1}\left(\frac{P_n}{\nu+1}-\frac{P_\nu}{n+1}\right)\frac{\cos(n-\nu)\mu}{(n-\nu)\log(n-\nu+1)}\right| \quad (t\leqq\mu\leqq\pi)$$

so that by the application of the estimate in (3. 2) of the lemma we get

$$\Sigma_4 = O\left(\log\frac{k}{t}\right). \tag{4. 8}$$

Combining the estimates in (4. 4) through (4. 8) we get the estimate in (4. 3). This completes the proof of the theorem.

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