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# ON QUASI-NORMAL $(f, g, u, v, \lambda)$-STRUCTURES

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## § 0. Introduction.

Let $M$ be a $C^\infty$ differentiable manifold and assume that $M$ admits a tensor field $f$ of type (1, 1), two vector fields $U$, $V$, two 1-forms $u$, $v$ and a function $\lambda$ satisfying

$$
(0.1)\qquad
\begin{aligned}
&f^2X=-X+u(X)U+v(X)V,\\
&fU=-\lambda V, \qquad u(fX)=+\lambda v(X),\\
&fV=+\lambda U, \qquad v(fX)=-\lambda u(X),\\
&u(U)=1-\lambda^2, \qquad u(V)=0,\\
&v(U)=0, \qquad v(V)=1-\lambda^2
\end{aligned}
$$

for any vector field $X$. Such a manifold $M$ is said to have an $(f, U, V, u, v, \lambda)$-structure [1], [2]. A manifold $M$ with $(f, U, V, u, v, \lambda)$-structure is even-dimensional [2].

An $(f, U, V, u, v, \lambda)$-structure is said to be *normal* if the tensor field $S$ of type (1, 2) defined by

$$
(0.2)\qquad S(X, Y)=N(X, Y)+(du)(X, Y)U+(dv)(X, Y)V
$$

vanishes, where $N$ is the Nijenhuis tensor of $f$ defined by

$$
(0.3)\qquad N(X, Y)=[fX, fY]-f[fX, Y]-f[X, fY]+f^2[X, Y]
$$

for arbitrary vector fields $X$ and $Y$.

Assume that a differentiable manifold $M$ with $(f, U, V, u, v, \lambda)$-structure admits a Riemannian metric $g$ such that

$$
(0.4)\qquad
\begin{aligned}
&g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y),\\
&g(U, X)=u(X), \qquad g(V, X)=v(X)
\end{aligned}
$$

for arbitrary vector fields $X$ and $Y$. We call an $(f, g, u, v, \lambda)$-structure an $(f, U, V, u, v, \lambda)$-structure with a Riemannian metric $g$ satisfying (0.4) [2].

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The tensor field of type (0, 2) defined by

$$\omega(X, Y)=g(fX, Y) \tag{0.5}$$

for arbitrary vector fields $X$ and $Y$ is a 2-form [2].

Okumura and one of the present authors [2] proved

THEOREM A. *Let $M$ be a complete manifold with normal $(f, g, u, v, \lambda)$-structure satisfying*

$$du=2\phi\omega, \qquad dv=2\omega,$$

*$\phi$ being a differentiable function on $M$. If $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function and* dim $M>2$, *then $M$ is isometric with an even-dimensional sphere.*

We put

$$T(X, Y, Z)=g(S(X, Y), Z). \tag{0.6}$$

If

$$T(X, Y, Z)-\{(d\omega)(fX, Y, Z)-(d\omega)(fY, X, Z)\}=0, \tag{0.7}$$

then we say that the $(f, g, u, v, \lambda)$-structure is *quasi-normal.*

The main purpose of the present paper is first to prove that in a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, the conditions

$$\mathcal{L}_U g=-2\alpha\lambda g \qquad \text{and} \qquad dv=2\alpha\omega$$

are equivalent, where $\mathcal{L}_U$ denotes the operator of Lie differentiation with respect to the vector field $U$ and $\alpha$ is a function, and next to prove that in a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and

$$\mathcal{L}_U g=-2c\lambda g \qquad \text{or} \qquad dv=2c\omega$$

is satisfied, $c$ being a non-zero constant, we have

$$du=-2\phi\omega, \qquad \mathcal{L}_V g=-2\phi\lambda g,$$

$\phi$ being a function.

Combining Theorem A and this result, we see that a complete manifold with normal $(f, g, u, v, \lambda)$-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, dim $M>2$ and $\mathcal{L}_U g=-2c\lambda g$ or $dv=2c\omega$ is satisfied is isometric to an even-dimensional sphere.

This result is an improvement of Theorem A.

In §1, we prove general formulas for an $(f, g, u, v, \lambda)$-structure and in §2, we specialize these formulas for a quasi-normal $(f, g, u, v, \lambda)$-structure. In §3, we prove the equivalence of $\mathcal{L}_U g=-2\alpha\lambda g$ and $dv=2\alpha\omega$ in a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure.

In the last §4, we prove that for the normal $(f, g, u, v, \lambda)$-structure, the condition $\mathcal{L}_U g=-2c\lambda g$ or $dv=2c\omega$ implies $du=-2\phi\omega$.

In the sequel, we assume that the function $\lambda(1-\lambda^2)$ is almost everywhere nonzero and we use the index notation.

## §1. General formulas.

We consider a $C^\infty$ differentiable manifold $M$ with an $(f, g, u, v, \lambda)$-structure, that is, a Riemannian manifold with metric tensor $g$ which admits a tensor field $f$ of type (1, 1), two 1-forms $u$ and $v$ (or two vector fields associated with them), and a function $\lambda$ satisfying

$$(1.1)\qquad \begin{cases} f_j{}^t f_t{}^h=-\delta_j^h+u_j u^h+v_j v^h, \\ f_j{}^t f_i{}^s g_{ts}=g_{ji}-u_j u_i-v_j v_i, \\ f_i{}^t u_t=+\lambda v_i \quad \text{or} \quad f_i{}^h u^i=-\lambda v^h, \\ f_i{}^t v_t=-\lambda u_i \quad \text{or} \quad f_i{}^h v^i=+\lambda u^h, \\ u_i u^i=v_i v^i=1-\lambda^2, \quad u_i v^i=0, \end{cases}$$

$$(1.2)\qquad f_{ji}=f_j{}^t g_{ti}$$

being skew-symmetric. Such an $M$ is even-, say, $2n$-dimensional.

We put

$$(1.3)\qquad S_{ji}{}^h=f_j{}^t\nabla_t f_i{}^h-f_i{}^t\nabla_t f_j{}^h-(\nabla_j f_i{}^t-\nabla_i f_j{}^t)f_t{}^h+u_{ji}u^h+v_{ji}v^h,$$

where

$$(1.4)\qquad u_{ji}=\nabla_j u_i-\nabla_i u_j, \qquad v_{ji}=\nabla_j v_i-\nabla_i v_j,$$

$\nabla_j$ denoting the operator of covariant differentiation with respect to the Levi-Civita connection. If the tensor $S_{ji}{}^h$ vanishes, the $(f, g, u, v, \lambda)$-structure is said to be *normal.*

Transvecting (1.3) with $u_h$, and using (1.1), we find

$$\begin{aligned} S_{ji}{}^h u_h=&f_j{}^t[\nabla_t(f_i{}^h u_h)-f_i{}^h\nabla_t u_h]-f_i{}^t[\nabla_t(f_j{}^h u_h)-f_j{}^h\nabla_t u_h] \\ &-\lambda[\nabla_j(f_i{}^t v_t)-f_i{}^t\nabla_j v_t-\nabla_i(f_j{}^t v_t)+f_j{}^t\nabla_i v_t]+(1-\lambda^2)u_{ji} \\ =&f_j{}^t[(\nabla_t\lambda)v_i+\lambda\nabla_t v_i-f_i{}^h\nabla_t u_h]-f_i{}^t[(\nabla_t\lambda)v_j+\lambda\nabla_t v_j-f_j{}^h\nabla_t u_h] \\ &-\lambda[-(\nabla_j\lambda)u_i-\lambda\nabla_j u_i-f_i{}^t\nabla_j v_t+(\nabla_i\lambda)u_j+\lambda\nabla_i u_j+f_j{}^t\nabla_i v_t]+(1-\lambda^2)u_{ji}, \end{aligned}$$

that is,

$$
\begin{aligned}
S_{ji}{}^{h}u_h = {} & u_{ji} - f_j{}^t f_i{}^s u_{ts} + \lambda(f_j{}^t v_{ti} - f_i{}^t v_{tj}) \\
& + (f_j{}^t v_i - f_i{}^t v_j)\nabla_t \lambda + \lambda[(\nabla_j \lambda)u_i - (\nabla_i \lambda)u_j].
\end{aligned} \tag{1.5}
$$

Similarly, we can prove

$$
\begin{aligned}
S_{ji}{}^{h}v_h = {} & v_{ji} - f_j{}^t f_i{}^s v_{ts} - \lambda(f_j{}^t u_{ti} - f_i{}^t u_{tj}) \\
& - (f_j{}^t u_i - f_i{}^t u_j)\nabla_t \lambda + \lambda[(\nabla_j \lambda)v_i - (\nabla_i \lambda)v_j].
\end{aligned} \tag{1.6}
$$

We now put

$$
f_{jih} = \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji} \tag{1.7}
$$

and consider the covariant components of $S$:

$$
S_{jih} = f_j{}^t \nabla_t f_{ih} - f_i{}^t \nabla_t f_{jh} + (\nabla_j f_{it} - \nabla_i f_{jt}) f_h{}^t + u_{ji}u_h + v_{ji}v_h. \tag{1.8}
$$

Then we have

$$
\begin{aligned}
S_{jih} = {} & f_j{}^t(f_{tih} - \nabla_i f_{ht} - \nabla_h f_{ti}) - f_i{}^t(f_{tjh} - \nabla_j f_{ht} - \nabla_h f_{tj}) \\
& + (\nabla_j f_{it} - \nabla_i f_{jt}) f_h{}^t + u_{ji}u_h + v_{ji}v_h \\
= {} & f_j{}^t f_{tih} - f_i{}^t f_{tjh} - \nabla_i(f_j{}^t f_{ht}) + \nabla_j(f_i{}^t f_{ht}) \\
& - f_j{}^t \nabla_h f_{ti} + f_i{}^t \nabla_h f_{tj} + u_{ji}u_h + v_{ji}v_h \\
= {} & f_j{}^t f_{tih} - f_i{}^t f_{tjh} - \nabla_i(g_{jh} - u_j u_h - v_j v_h) + \nabla_j(g_{ih} - u_i u_h - v_i v_h) \\
& - f_j{}^t \nabla_h f_{ti} + f_i{}^t \nabla_h f_{tj} + u_{ji}u_h + v_{ji}v_h,
\end{aligned}
$$

from which

$$
\begin{aligned}
& S_{jih} - (f_j{}^t f_{tih} - f_i{}^t f_{tjh}) \\
& = -(f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj}) + u_j(\nabla_i u_h) - u_i(\nabla_j u_h) + v_j(\nabla_i v_h) - v_i(\nabla_j v_h).
\end{aligned} \tag{1.9}
$$

Transvecting (1. 9) with $u^j$ and using (1. 1), we find

$$
\begin{aligned}
& u^j[S_{jih} - (f_j{}^t f_{tih} - f_i{}^t f_{tjh})] \\
& = \lambda[\nabla_h(f_{ti}v^t) - f_{ti}\nabla_h v^t] + f_i{}^t[\nabla_h(f_{tj}u^j) - f_{tj}\nabla_h u^j] \\
& \quad + (1-\lambda^2)\nabla_i u_h - u_i(u^j \nabla_j u_h) - v_i(u^j \nabla_j v_h) \\
& = \lambda[(\nabla_h \lambda)u_i + \lambda \nabla_h u_i + f_i{}^t \nabla_h v_t] \\
& \quad + f_i{}^t[(\nabla_h \lambda)v_t + \lambda \nabla_h v_t - f_t{}^j \nabla_h u_j] \\
& \quad + (1-\lambda^2)\nabla_i u_h - u_i(u^j \nabla_j u_h) - v_i(u^j \nabla_j v_h)
\end{aligned}
$$

$$=-\lambda^2 u_{ih}+\mathcal{L}_u g_{ih}+2\lambda f_i{}^t\nabla_h v_t-u_i u^t\mathcal{L}_u g_{th}-v_i u^t v_{th}$$

$$=-\lambda^2 u_{ih}+\mathcal{L}_u g_{ih}+\lambda f_i{}^t[\mathcal{L}_v g_{th}-v_{th}]-u_i u^t\mathcal{L}_u g_{th}-v_i u^t v_{th},$$

where $\mathcal{L}_u$ and $\mathcal{L}_v$ denote Lie differentiation with respect to $u^h$ and $v^h$ respectively, from which,

$$(1.10)\quad \begin{aligned} &u^j[S_{jih}-(f_j{}^t f_{tih}-f_i{}^t f_{tjh})]\\ &=\mathcal{L}_u g_{ih}-u_i u^t\mathcal{L}_u g_{th}+\lambda f_i{}^t\mathcal{L}_v g_{th}-\lambda^2 u_{ih}-(\lambda f_i{}^t+v_i u^t)v_{th},.\end{aligned}$$

Similarly, we have

$$(1.11)\quad \begin{aligned} &v^j[S_{jih}-(f_j{}^t f_{tih}-f_i{}^t f_{tjh})]\\ &=\mathcal{L}_v g_{ih}-v_i v^t\mathcal{L}_v g_{th}-\lambda f_i{}^t\mathcal{L}_u g_{th}-\lambda^2 v_{ih}+(\lambda f_i{}^t-u_i v^t)u_{th}.\end{aligned}$$

## §2. Formulas for quasi-normal $(f, g, u, v, \lambda)$-structures.

If the condition

$$(2.1)\qquad S_{jih}-(f_j{}^t f_{tih}-f_i{}^t f_{tjh})=0$$

is satisfied, then we say that the $(f, g, u, v, \lambda)$-structure is *quasi-normal.*

If the structure is quasi-normal, we have, from (1.10) and (1.11),

$$(2.2)\qquad \mathcal{L}_u g_{ih}-u_i u^t\mathcal{L}_u g_{th}+\lambda f_i{}^t\mathcal{L}_v g_{th}=\lambda^2 u_{ih}+(\lambda f_i{}^t+v_i u^t)v_{th}$$

and

$$(2.3)\qquad \mathcal{L}_v g_{ih}-v_i v^t\mathcal{L}_v g_{th}-\lambda f_i{}^t\mathcal{L}_u g_{th}=\lambda^2 v_{ih}-(\lambda f_i{}^t-u_i v^t)u_{th}$$

respectively.

From (2.2), we find

$$(2.4)\qquad \lambda^2 u_{th}=\mathcal{L}_u g_{th}-u_t u^s\mathcal{L}_u g_{sh}+\lambda f_t{}^s\mathcal{L}_v g_{sh}-(\lambda f_t{}^s+v_t u^s)v_{sh}.$$

From (2.3), we have

$$\lambda^2\mathcal{L}_v g_{ih}-\lambda^2 v_i v^t\mathcal{L}_v g_{th}-\lambda^3 f_i{}^t\mathcal{L}_u g_{th}=\lambda^4 v_{ih}-(\lambda f_i{}^t-u_i v^t)(\lambda^2 u_{th}).$$

Substituting (2.4) into this equation, we have

$$\begin{aligned} &\lambda^2\mathcal{L}_v g_{ih}-\lambda^2 v_i v^t\mathcal{L}_v g_{th}-\lambda^3 f_i{}^t\mathcal{L}_u g_{th}\\ &=\lambda^4 v_{ih}-\lambda f_i{}^t[\mathcal{L}_u g_{th}-u_t u^s\mathcal{L}_u g_{sh}+\lambda f_t{}^s\mathcal{L}_v g_{sh}-(\lambda f_t{}^s+v_t u^s)v_{sh}]\\ &\quad+u_i v^t[\mathcal{L}_u g_{th}-u_t u^s\mathcal{L}_u g_{sh}+\lambda f_t{}^s\mathcal{L}_v g_{sh}-(\lambda f_t{}^s+v_t u^s)v_{sh}],\end{aligned}$$

$$\lambda^2\mathcal{L}_v g_{ih}-\lambda^2 v_i v^t\mathcal{L}_v g_{th}-\lambda^3 f_i{}^t\mathcal{L}_u g_{th}$$

$$=\lambda^4 v_{ih}-\lambda f_i{}^t \mathcal{L}_u g_{th}+\lambda^2 v_i u^s \mathcal{L}_u g_{sh}-\lambda^2(-\delta_i^s+u_i u^s+v_i v^s)\mathcal{L}_v g_{sh}$$

$$+\lambda^2(-\delta_i^s+u_i u^s+v_i v^s)v_{sh}-\lambda^2 u_i u^s v_{sh}$$

$$+u_i v^t \mathcal{L}_u g_{th}+\lambda^2 u_i u^s \mathcal{L}_v g_{sh}-\lambda^2 u_i u^s v_{sh}-(1-\lambda^2)u_i u^s v_{sh},$$

from which,

$$\lambda^2(1-\lambda^2)v_{ih}+(u_i u^s-\lambda^2 v_i v^s)v_{sh}=\{u_i v^t+\lambda^2 v_i u^t-\lambda(1-\lambda^2)f_i{}^t\}\mathcal{L}_u g_{th}. \tag{2.5}$$

Transvecting this equation with $u^i$, we find

$$\lambda^2(1-\lambda^2)u^i v_{ih}+(1-\lambda^2)u^s v_{sh}=\{(1-\lambda^2)v^t+\lambda^2(1-\lambda^2)v^t\}\mathcal{L}_u g_{th},$$

from which,

$$u^s v_{sh}=v^s \mathcal{L}_u g_{sh}, \tag{2.6}$$

which shows that

$$(\mathcal{L}_u g_{ji})u^j v^i=0 \tag{2.7}$$

and

$$(\mathcal{L}_u g_{ji})v^j v^i=v_{ji}u^j v^i. \tag{2.8}$$

Substituting (2. 6) into (2. 5), we obtain

$$\lambda(1-\lambda^2)v_{ih}-\lambda v_i v^s v_{sh}=\{\lambda v_i u^t-(1-\lambda^2)f_i{}^t\}\mathcal{L}_u g_{th}. \tag{2.9}$$

Transvecting this equation with $v^h$, we find

$$\lambda(1-\lambda^2)v_{ih}v^h=\{\lambda v_i u^t-(1-\lambda^2)f_i{}^t\}(\mathcal{L}_u g_{th})v^h,$$

or, using (2. 7),

$$\lambda v_{hi}v^h=f_i{}^t(\mathcal{L}_u g_{th})v^h$$

because of the skew-symmetry of $v_{hi}$.

Thus (2. 9) can be written as

$$\lambda(1-\lambda^2)v_{ih}-v_i f_h{}^t(\mathcal{L}_u g_{ts})v^s=\{\lambda v_i u^t-(1-\lambda^2)f_i{}^t\}\mathcal{L}_u g_{th},$$

or

$$\lambda(1-\lambda^2)v_{ih}=v_i f_h{}^t(\mathcal{L}_u g_{ts})v^s+\{\lambda v_i u^t-(1-\lambda^2)f_i{}^t\}\mathcal{L}_u g_{th}. \tag{2.10}$$

Similarly, from (2. 2) and (2. 3), we obtain

$$\begin{aligned}&\lambda^2(1-\lambda^2)u_{ih}+(v_i v^s-\lambda^2 u_i u^s)u_{sh}\\&=\{v_i u^t+\lambda^2 u_i v^t+\lambda(1-\lambda^2)f_i{}^t\}\mathcal{L}_v g_{th}.\end{aligned} \tag{2.11}$$

Transvecting this equation with $v^i$, we find

$$\lambda^2(1-\lambda^2)v^i u_{ih}+(1-\lambda^2)v^s u_{sh}=\{(1-\lambda^2)u^t+\lambda^2(1-\lambda^2)u^t\}\mathcal{L}_v g_{th},$$

from which,

$$v^s u_{sh}=u^s\mathcal{L}_v g_{sh}, \tag{2.12}$$

which shows that

$$(\mathcal{L}_v g_{ji})u^j v^i=0 \tag{2.13}$$

and

$$(\mathcal{L}_v g_{ji})u^j u^i=-u_{ji}u^j v^i. \tag{2.14}$$

Substituting (2.12) into (2.11), we find

$$\lambda(1-\lambda^2)u_{ih}-\lambda u_i u^s u_{sh}=\{\lambda u_i v^t+(1-\lambda^2)f_i{}^t\}\mathcal{L}_v g_{th}. \tag{2.15}$$

Transvecting this equation with $u^h$, we find

$$\lambda(1-\lambda^2)u_{ih}u^h=\{\lambda u_i v^t+(1-\lambda^2)f_i{}^t\}(\mathcal{L}_v g_{th})u^h,$$

or, using (2.13),

$$\lambda u_{ih}u^h=f_i{}^t(\mathcal{L}_v g_{th})u^h.$$

Thus (2.15) can be written as

$$\lambda(1-\lambda^2)u_{ih}+u_i f_h{}^t(\mathcal{L}_v g_{ts})u^s=\{\lambda u_i v^t+(1-\lambda^2)f_i{}^t\}\mathcal{L}_v g_{th},$$

or

$$\lambda(1-\lambda^2)u_{ih}=-u_i f_h{}^t(\mathcal{L}_v g_{ts})u^s+\{\lambda u_i v^t+(1-\lambda^2)f_i{}^t\}\mathcal{L}_v g_{th}. \tag{2.16}$$

## §3. Equivalence of $\mathcal{L}_u g_{ji}=-2\alpha\lambda g_{ji}$ and $v_{ji}=2\alpha f_{ji}$ in a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure.

In this section, we assume that the $(f, g, u, v, \lambda)$-structure is quasi-normal, that is,

$$S_{jih}-(f_j{}^t f_{tih}-f_i{}^t f_{tjh})=0. \tag{3.1}$$

We moreover assume that

$$\mathcal{L}_u g_{ji}=-2\alpha\lambda g_{ji}, \tag{3.2}$$

where $\alpha$ is a function, that is, the vector field $u^h$ defines an infinitesimal conformal transformation with dilatation factor $-\alpha\lambda$.

Then we have, from (2.10),

$$\lambda(1-\lambda^2)v_{ih}=-2\alpha\lambda v_i f_h{}^t g_{ts}v^s-2\alpha\lambda\{\lambda v_i u^t-(1-\lambda^2)f_i{}^t\}g_{th},$$

or

$$v_{ih}=2\alpha f_{ih}. \tag{3.3}$$

Conversely, suppose that (3. 3) is satisfied, $\alpha$ being a function. Then from (1. 10) we obtain

$$0=\mathcal{L}_u g_{ih}-u_i u^t\mathcal{L}_u g_{th}+\lambda f_i{}^t\mathcal{L}_v g_{th}-\lambda^2 u_{ih}-2\alpha(\lambda f_i{}^t+v_i u^t)f_{th},$$

$$0=\mathcal{L}_u g_{ih}-u_i u^t\mathcal{L}_u g_{th}+\lambda f_i{}^t\mathcal{L}_v g_{th}-\lambda^2 u_{ih}-2\alpha\lambda(-g_{ih}+u_i u_h+v_i v_h)+2\alpha\lambda v_i v_h,$$

that is,

$$\lambda f_i{}^t\mathcal{L}_v g_{th}=-\mathcal{L}_u g_{ih}-2\alpha\lambda g_{ih}+\lambda^2 u_{ih}+u_i(u^t\mathcal{L}_u g_{th}+2\alpha\lambda u_h). \tag{3.4}$$

We have also, from (1. 11),

$$0=\mathcal{L}_v g_{ih}-v_i v^t\mathcal{L}_v g_{th}-\lambda f_i{}^t\mathcal{L}_u g_{th}-2\alpha\lambda^2 f_{ih}+(\lambda f_i{}^t-u_i v^t)u_{th},$$

that is,

$$2\lambda f_i{}^t\nabla_h u_t=\mathcal{L}_v g_{ih}-2\alpha\lambda^2 f_{ih}-u_i v^t u_{th}-v_i v^t\mathcal{L}_v g_{th}. \tag{3.5}$$

Writing (3. 5) as

$$2\lambda f_t{}^s\nabla_h u_s=\mathcal{L}_v g_{th}-2\alpha\lambda^2 f_{th}-u_t v^s v_{sh}-v_t v^s\mathcal{L}_v g_{sh}$$

and transvecting this with $\lambda f_i{}^t$, we find

$$2\lambda^2(-\delta_i^s+u_i u^s+v_i v^s)\nabla_h u_s$$
$$=\lambda f_i{}^t\mathcal{L}_v g_{th}-2\alpha\lambda^3(-g_{ih}+u_i u_h+v_i v_h)-\lambda^2 v_i v^s u_{sh}+\lambda^2 u_i v^s\mathcal{L}_v g_{sh}.$$

Substituting (3. 4) into this equation, we find

$$2\lambda^2[-\nabla_h u_i+u_i u^s(\nabla_h u_s)+v_i v^s(\nabla_h u_s)]$$
$$=-\mathcal{L}_u g_{ih}-2\alpha\lambda g_{ih}+\lambda^2(\nabla_i u_h-\nabla_h u_i)+u_i(u^t\mathcal{L}_u g_{th}+2\alpha\lambda u_h)$$
$$+2\alpha\lambda^3(g_{ih}-u_i u_h-v_i v_h)-\lambda^2 v_i v^s(\nabla_s u_h-\nabla_h u_s)+\lambda^2 u_i v^s\mathcal{L}_v g_{sh},$$

$$(1-\lambda^2)\mathcal{L}_u g_{ih}$$
$$=-2\alpha\lambda(1-\lambda^2)g_{ih}$$
$$+u_i[-2\lambda^2 u^s(\nabla_h u_s)+u^t\mathcal{L}_u g_{th}+2\alpha\lambda(1-\lambda^2)u_h+\lambda^2 v^s\mathcal{L}_v g_{sh}]$$
$$+v_i[-2\lambda^2 v^s(\nabla_h u_s)-2\alpha\lambda^3 v_h-\lambda^2 v^s(\nabla_s u_h-\nabla_h u_s)],$$

or

$$(1-\lambda^2)\mathcal{L}_u g_{ih}$$
$$=-2\alpha\lambda(1-\lambda^2)g_{ih}$$
$$+u_i[-2\lambda^2 u^s(\nabla_h u_s)+u^t\mathcal{L}_u g_{th}+2\alpha\lambda(1-\lambda^2)u_h+\lambda^2 v^s\mathcal{L}_v g_{sh}]$$
$$-\lambda^2 v_i[(\mathcal{L}_u g_{th})v^t+2\alpha\lambda v_h],$$

or, using (2. 6) and (3. 3),

$$(1-\lambda^2)\mathcal{L}_u g_{ih}$$
$$=-2\alpha\lambda(1-\lambda^2)g_{ih} \tag{3.6}$$
$$+u_i[-2\lambda^2 u^s(\nabla_h u_s)+u^t\mathcal{L}_u g_{th}+2\alpha\lambda(1-\lambda^2)u_h+\lambda^2 v^s\mathcal{L}_v g_{sh}].$$

Transvecting (3. 6) with $u^h$ and using (2. 13), we find

$$(1-\lambda^2)(\mathcal{L}_u g_{ih})u^h$$
$$=-2\alpha\lambda(1-\lambda^2)u_i+u_i[(1-\lambda^2)(\mathcal{L}_u g_{ts})u^t u^s+2\alpha\lambda(1-\lambda^2)^2], \tag{3.7}$$

from which,

$$(1-\lambda^2)(\mathcal{L}_u g_{ih})u^i u^h$$
$$=-2\alpha\lambda(1-\lambda^2)^2+(1-\lambda^2)^2(\mathcal{L}_u g_{ts})u^t u^s+2\alpha\lambda(1-\lambda^2)^3,$$

that is,

$$(\mathcal{L}_u g_{ih})u^i u^h=-2\alpha\lambda(1-\lambda^2). \tag{3.8}$$

Thus, from (3. 7), we find

$$(1-\lambda^2)(\mathcal{L}_u g_{ih})u^h$$
$$=-2\alpha\lambda(1-\lambda^2)u_i+u_i[-2\alpha\lambda(1-\lambda^2)^2+2\alpha\lambda(1-\lambda^2)^2],$$

that is,

$$(\mathcal{L}_u g_{ih})u^h=-2\alpha\lambda u_i. \tag{3.9}$$

Thus (3. 6) becomes

$$(1-\lambda^2)\mathcal{L}_u g_{ih}$$
$$=-2\alpha\lambda(1-\lambda^2)g_{ih}$$
$$+u_i[-2\lambda^2 u^s(\nabla_h u_s)-2\alpha\lambda u_h+2\alpha\lambda(1-\lambda^2)u_h+\lambda^2 v^s(\mathcal{L}_v g_{sh})],$$

that is,

$$
\begin{aligned}
&(1-\lambda^2)\mathcal{L}_u g_{ih} \\
(3.10)\qquad &= -2\alpha\lambda(1-\lambda^2)g_{ih} \\
&\quad -\lambda^2 u_i[2u^s(\nabla_h u_s)+2\alpha\lambda u_h - v^s(\mathcal{L}_v g_{sh})],
\end{aligned}
$$

from which, taking the skew-symmetric part,

$$
\begin{aligned}
&u_i[2u^s(\nabla_h u_s)+2\alpha\lambda u_h - v^s(\mathcal{L}_v g_{sh})] \\
&-u_h[2u^s(\nabla_i u_s)+2\alpha\lambda u_i - v^s(\mathcal{L}_v g_{si})]=0.
\end{aligned}
$$

Transvecting this equation with $u^i$, we find

$$
\begin{aligned}
&(1-\lambda^2)[2u^s(\nabla_h u_s)+2\alpha\lambda u_h - v^s(\mathcal{L}_v g_{sh})] \\
&-u_h[(\mathcal{L}_u g_{is})u^i u^s + 2\alpha\lambda(1-\lambda^2)-(\mathcal{L}_v g_{si})u^s v^i]=0,
\end{aligned}
$$

from which, using (2.13) and (3.10),

$$2u^s(\nabla_h u_s)+2\alpha\lambda u_h - v^s(\mathcal{L}_v g_{sh})=0.$$

Thus (3.10) becomes

$$\mathcal{L}_u g_{ih} = -2\alpha\lambda g_{ih}. \tag{3.11}$$

Thus we have proved

THEOREM 3.1. *In a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, the conditions*

$$\mathcal{L}_u g_{ji} = -2\alpha\lambda g_{ji} \qquad and \qquad v_{ji}=2\alpha f_{ji}$$

*are equivalent, $\alpha$ being a function.*

Now we assume that $\alpha$ is a non-zero constant.
Since $v_{ji}=2\alpha f_{ji}$ implies

$$f_{jih}=0,$$

we have, as a corollary to this theorem,

COROLLARY 3.2. *A quasi-normal $(f, g, u, v, \lambda)$-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and*

$$\mathcal{L}_u g_{ji} = -2\alpha\lambda g_{ji},$$

*$\alpha$ being a non-zero constant, is normal.*

## §4. Normal $(f, g, u, v, \lambda)$-structures satisfying $\mathcal{L}_u g_{ji} = -2c\lambda g_{ji}$ or $v_{ji}=2cf_{ji}$.

In this section, we put the assumption that the $(f, g, u, v, \lambda)$-structure we consider is quasi-normal and satisfies $\mathcal{L}_u g_{ji}=-2c\lambda g_{ji}$ or $v_{ji}=2cf_{ji}$, that is,

A. The $(f, g, u, v, \lambda)$-structure under consideration is normal and satisfies

$$\mathcal{L}_u g_{ji}=-2c\lambda g_{ji} \qquad \text{or} \qquad v_{ji}=2cf_{ji},$$

$c$ being a non-zero constant.

Under the assumption A, (1. 5) and (1. 6) become

$$u_{ji}-f_j{}^t f_i{}^s u_{ts}+(f_j{}^t v_i-f_i{}^t v_j)\nabla_t\lambda+\lambda[(\nabla_j\lambda)u_i-(\nabla_i\lambda)u_j]=0 \tag{4. 1}$$

and

$$\begin{aligned}&2c\lambda(u_jv_i-u_iv_j)+\lambda(f_j{}^t u_{ti}-f_i{}^t u_{tj})\\&+(f_j{}^t u_i-f_i{}^t u_j)\nabla_t\lambda-\lambda[(\nabla_j\lambda)v_i-(\nabla_i\lambda)v_j]=0\end{aligned} \tag{4. 2}$$

respectively. (4. 2) can also be written as

$$\begin{aligned}&2c\lambda(u_jv_i-u_iv_j)+2\lambda(f_j{}^t\nabla_t u_i-f_i{}^t\nabla_t u_j+2c\lambda f_{ji})\\&+(f_j{}^t u_i-f_i{}^t u_j)\nabla_t\lambda-\lambda[(\nabla_j\lambda)v_i-(\nabla_i\lambda)v_j]=0,\end{aligned} \tag{4. 3}$$

since

$$u_{ti}=2\nabla_t u_i-\mathcal{L}_u g_{ti}=2\nabla_t u_i+2c\lambda g_{ti}. \tag{4. 4}$$

Also, under the assumption A, (1. 10) becomes

$$f_i{}^t\mathcal{L}_v g_{th}=\lambda u_{ih}. \tag{4. 5}$$

Now we transvect (4. 2) with $u^jv^i$ and find

$$2c\lambda(1-\lambda^2)^2-\lambda(1-\lambda^2)u^t\nabla_t\lambda-\lambda(1-\lambda^2)u^t\nabla_t\lambda=0,$$

that is,

$$u^t\nabla_t\lambda=c(1-\lambda^2). \tag{4. 6}$$

Equation (4. 5) can be written as

$$f_i{}^t(\nabla_t v_h+\nabla_h v_t)=\lambda(\nabla_i u_h-\nabla_h u_i),$$

$$f_i{}^t(2\nabla_t v_h+v_{ht})=\lambda(\mathcal{L}_u g_{ih}-2\nabla_h u_i),$$

$$f_i{}^t(\nabla_t v_h+cf_{ht})=-\lambda(c\lambda g_{ih}+\nabla_h u_i),$$

that is,

$$\lambda\nabla_h u_i+f_i{}^t\nabla_t v_h=-c(1+\lambda^2)g_{ih}+c(u_iu_h+v_iv_h). \tag{4. 7}$$

Transvecting (4. 3) with $v^j$, we find

$$-2c\lambda(1-\lambda^2)u_i+2\lambda[\lambda u^t\nabla_t u_i+f_i{}^t(\nabla_t v_j)u^j+2c\lambda^2 u_i]$$
$$+\lambda u_i u^t\nabla_t\lambda-\lambda[(v^j\nabla_j\lambda)v_i-(1-\lambda^2)\nabla_i\lambda]=0,$$

or, using (4. 6),

$$-c\lambda(1-5\lambda^2)u_i+2\lambda u^t[\lambda\nabla_t u_i+f_i{}^s(\nabla_s v_t)]$$
$$-\lambda[(v^j\nabla_j\lambda)v_i-(1-\lambda^2)\nabla_i\lambda]=0,$$

or, using (4. 7),

$$-c\lambda(1-5\lambda^2)u_i+2\lambda u^t[-c(1+\lambda^2)g_{ti}+c(u_t u_i+v_t v_i)]$$
$$-\lambda[(v^j\nabla_j\lambda)v_i-(1-\lambda^2)\nabla_i\lambda]=0,$$

that is,

$$\nabla_i\lambda=cu_i+\phi v_i, \tag{4. 8}$$

where we have put

$$v^j\nabla_j\lambda=(1-\lambda^2)\phi. \tag{4. 9}$$

We have

$$u^t(\nabla_t u_i)=u^t(\mathcal{L}_u g_{ti}-\nabla_i u_t)$$
$$=u^t(-2c\lambda g_{ti})-\frac{1}{2}\nabla_i(u_t u^t)$$
$$=-2c\lambda u_i+\lambda\nabla_i\lambda,$$

from which, using (4. 8),

$$u^t(\nabla_t u_i)=-c\lambda u_i+\lambda\phi v_i. \tag{4. 10}$$

We have

$$v^t(\nabla_t v_i)=v^t(2cf_{ti}+\nabla_i v_t)$$
$$=2c\lambda u_i+\frac{1}{2}\nabla_i(v_t v^t)$$
$$=2c\lambda u_i-\lambda\nabla_i\lambda,$$

from which, using (4. 8),

$$v^t(\nabla_t v_i)=c\lambda u_i-\lambda\phi v_i. \tag{4. 11}$$

We have also

$$v^t\nabla_t u_i=v^t(\mathcal{L}_u g_{ti}-\nabla_i u_t)$$

$$= -2c\lambda v_i + u^t \nabla_i v_t$$

$$= -2c\lambda v_i + u^t(2cf_{it} + \nabla_t v_i),$$

from which,

$$v^t \nabla_t u_i = u^t \nabla_t v_i. \tag{4.12}$$

On the other hand, transvecting (4. 7) with $u^h$, we find

$$\lambda u^t \nabla_t u_i - f_i{}^s(\nabla_s u_t)v^t = -c(1+\lambda^2)u_i + c(1-\lambda^2)u_i,$$

$$\lambda u^t \nabla_t u_i - f_i{}^s(\nabla_s u_t)v^t = -2c\lambda^2 u_i,$$

from which, substituting (4. 10),

$$f_i{}^s(\nabla_s u_t)v^t = \lambda^2(cu_i + \phi v_i).$$

Transvecting this equation with $f_j{}^i$, we find

$$(-\delta_j^s + u_j u^s + v_j v^s)(\nabla_s u_t)v^t = \lambda^3(cv_j - \phi u_j),$$

or

$$-(\nabla_j u_t)v^t + u_j(u^s \nabla_s u_t)v^t - v_j(v^s \nabla_s v_t)u^t = \lambda^3(cv_j - \phi u_j),$$

or, using (4. 10) and (4. 11),

$$(\nabla_i u_t)v^t = \lambda(\phi u_i - cv_i). \tag{4.13}$$

From (4. 13), we find

$$(\mathcal{L}_u g_{it} - \nabla_t u_i)v^t = \lambda(\phi u_i - cv_i),$$

that is,

$$v^t \nabla_t u_i = -\lambda(\phi u_i + cv_i). \tag{4.14}$$

From (4. 13) and (4. 14), we have

$$v^t u_{ti} = -2\lambda\phi u_i. \tag{4.15}$$

Now differentiating (4. 8) covariantly, we find

$$\nabla_j \nabla_i \lambda = c\nabla_j u_i + (\nabla_j \phi)v_i + \phi \nabla_j v_i,$$

from which,

$$cu_{ji} + (\nabla_j \phi)v_i - (\nabla_i \phi)v_j + 2c\phi f_{ji} = 0. \tag{4.16}$$

Transvecting this equation with $v^j$ and using (4. 15), we find

$$0 = -2c\lambda\phi u_i + (v^j \nabla_j \phi)v_i - (1-\lambda^2)(\nabla_i \phi) + 2c\lambda\phi u_i,$$

or

(4. 17) $$(1-\lambda^2)(\nabla_i\phi)=(v^j\nabla_j\phi)v_i,$$

which shows that $\nabla_i\phi$ is proportional to $v_i$, and consequently, (4. 16) becomes

(4. 18) $$u_{ji}=-2\phi f_{ji}.$$

From (4. 18) and

$$\nabla_j u_i+\nabla_i u_j=-2c\lambda g_{ji},$$

we have

(4. 19) $$\nabla_j u_i=-c\lambda g_{ji}-\phi f_{ji}.$$

Substituting (4. 19) into (4. 7), we find

$$f_i{}^t\nabla_t v_h=\lambda\phi f_{hi}+c(-g_{ih}+u_i u_h+v_i v_h).$$

Transvecting this with $f_j{}^i$ and using (4. 11) and (4. 14), we obtain

(4. 20) $$\nabla_j v_i=-\lambda\phi g_{ji}+c f_{ji},$$

and consequently

$$\mathcal{L}_v g_{ji}=-2\lambda\phi g_{ji}.$$

Thus we have proved

THEOREM 4. 1. *In a manifold with quasi-normal $(f, g, u, v, \lambda)$-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and*

(4. 23) $$\mathcal{L}_u g_{ji}=-2c\lambda g_{ji} \qquad or \qquad v_{ji}=2cf_{ji}$$

*is satisfied, $c$ being a non-zero constant, we have*

$$u_{ji}=-2\phi f_{ji},$$

*and*

$$\mathcal{L}_v g_{ji}=-2\lambda\phi g_{ji},$$

*$\phi$ being a function.*

If the condition of Theorem 4. 1 is satisfied, the structure is normal, so applying Theorem 7. 1 of [2], we have

Theorem 4. 2. *Let $M$ be a complete manifold with normal $(f, g, u, v, \lambda)$-structure satisfying*

$$\mathcal{L}_u g_{ji}=-2c\lambda g_{ji} \qquad or \qquad v_{ji}=2cf_{ji},$$

*$c$ being a non-zero constant. If $\lambda(1-\lambda^2)$ is almost everywhere non-zero function and $n>1$, then $M$ is isometric with an even-dimensional sphere.*

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