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INVARIANT SUBMANIFOLDS OF A MANIFOLD WITH (f, g, u, v, λ)-STRUCTURE

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Dedicated to Professor Y. Mutō on his sixtieth birthday

§ 0. Introduction.

Blair, Ludden and one of the present authors [1] have started the study of the structure induced on a submanifold of codimension 2 of an almost complex manifold and that induced on a hypersurface of an almost contact manifold.

In papers [4], [5], [6], we have defined the (f, g, u, v, λ)-structure on an even-dimensional differentiable manifold, and have studied normal (f, g, u, v, λ)-structures on submanifolds of codimension 2 in a Euclidean space and invariant hypersurfaces of a manifold with (f, g, u, v, λ)-structure.

In this paper, we shall study invariant submanifolds of odd and even dimension of a manifold with (f, g, u, v, λ)-structure.

In § 1, we state some of known results and formulas in the theory of submanifolds.

In § 2, we study invariant submanifolds of a manifold with (f, g, u, v, λ)-structure.

In § 3, we study invariant submanifolds of odd dimension and in § 4 we continue the study of odd dimensional invariant submanifolds of a manifold with normal (f, g, u, v, λ)-structure.

In the last § 5, we study invariant submanifolds of even dimension.

§ 1. Preliminaries.

Let M be a differentiable manifold with (f, g, u, v, λ)-structure, that is, a differentiable manifold endowed with a tensor field f of type (1, 1), a Riemannian metric g, two 1-forms u and v and a function λ satisfying

$$\begin{aligned} &f_j{}^i f_i{}^h = -\delta_j^h + u_j u^h + v_j v^h, \\ &f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i, \\ &u_i f_j{}^i = \lambda v_j, \qquad v_i f_j{}^i = -\lambda u_j, \\ &f_i{}^h u^i = -\lambda v^h, \qquad f_i{}^h v^i = \lambda u^h, \end{aligned} \tag{1.1}$$

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$$u_i u^i = 1-\lambda^2, \qquad v_i v^i = 1-\lambda^2, \qquad u_i v^i = 0,$$

$f_i{}^h$, g_{ji}, u_i, v_i and λ being respectively components of f, g, u, v and λ with respect to a local coordinate system, u^h and v^h being defined by

$$u_i = g_{ih} u^h \quad \text{and} \quad v_i = g_{ih} v^h$$

respectively, where here and throughout the paper the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2m\}$. It is known that such a manifold is even dimensional.

If we put

$$f_{ji} = f_j{}^t g_{ti}, \tag{1.2}$$

we can easily see that f_{ji} is skew-symmetric.

We put

$$S_{ji}{}^h = N_{ji}{}^h + (\nabla_j u_i - \nabla_i u_j) u^h + (\nabla_j v_i - \nabla_i v_j) v^h, \tag{1.3}$$

$N_{ji}{}^h$ denoting the Nijenhuis tensor formed with $f_i{}^h$ and ∇_i the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}. If $S_{ji}{}^h$ vanishes, we say that the (f, g, u, v, λ)-structure is *normal.*

The following two theorems are known [4]:

THEOREM 1.1. *If a normal* (f, g, u, v, λ)*-structure satisfies*

$$\nabla_j v_i - \nabla_i v_j = 2 f_{ji}, \tag{1.4}$$

then

$$f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj} = u_j(\nabla_i u_h) - u_i(\nabla_j u_h) + v_j(\nabla_i v_h) - v_i(\nabla_j v_h). \tag{1.5}$$

THEOREM 1.2. *Let M be a complete manifold with normal* (f, g, u, v, λ)*-structure satisfying* (1.4) *and*

$$\nabla_j u_i - \nabla_i u_j = 2\phi f_{ji}, \tag{1.6}$$

ϕ being a certain function. If the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere, then M is isometric with a sphere.

We consider a submanifold N of M represented by

$$x^h = x^h(y^a) \tag{1.7}$$

and put

$$B_b{}^h = \partial_b x^h, \qquad \partial_b = \partial/\partial y^b, \tag{1.8}$$

where here and throughout the paper the indices a, b, c, d, e run over the range $\{1, 2, \cdots, n\}$.

The induced Riemannian metric is given by

$$g_{cb} = g_{ji} B_c{}^j B_b{}^i. \tag{1.9}$$

We denote by $C_x{}^h$ $2m-n$ mutually orthogonal unit normals to N. Then equations of Gauss and those of Weingarten are respectively

$$\nabla_c B_b{}^h = \sum_x h_{cbx} C_x{}^h \tag{1.10}$$

and

$$\nabla_c C_x{}^h = -h_c{}^a{}_x B_a{}^h + \sum_y l_{cxy} C_y{}^h, \tag{1.11}$$

where

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j B_b{}^i - \left\{ {a \atop c\ b} \right\} B_a{}^h \tag{1.12}$$

is the van der Waerden-Bortolotti covariant differentiation of $B_b{}^h$, $\{{}_c{}^a{}_b\}$ being Christoffel symbols formed with g_{cb},

$$\nabla_c C_x{}^h = \partial_c C_x{}^h + \left\{ {h \atop j\ i} \right\} B_c{}^j C_x{}^i, \tag{1.13}$$

h_{cbx} components of the second fundamental tensors with respect to normals $C_x{}^h$,

$$h_c{}^a{}_x = h_{cbx} g^{ba}, \tag{1.14}$$

g^{ba} being contravariant components of the induced Riemannian metric tensor and l_{cxy} components of the third fundamental tensor with respect to normals $C_x{}^h$.

§ 2. Invariant submanifolds of a manifold with (f, g, u, v, λ)-structure.

We assume that the submanifold N of M is f-invariant, that is, the transform of a vector tangent to N by the linear transformation f is always tangent to N:

$$f_i{}^h B_b{}^i = f_b{}^a B_a{}^h, \tag{2.1}$$

$f_b{}^a$ being a tensor field of type (1, 1) of N.

This shows that

$$f_{ih} B_b{}^i C_x{}^h = 0,$$

that is, $f_i{}^h C_x{}^i$ is normal to the submanifold N. Thus, we put

$$f_i{}^h C_x{}^i = \sum_y \gamma_{xy} C_y{}^h. \tag{2.2}$$

Since

$$f_{ih} C_x{}^i C_y{}^h = \gamma_{xy},$$

we see that

$$\gamma_{xy} = -\gamma_{yx}. \tag{2.3}$$

We put

$$u^h = B_a{}^h u^a + \sum_x \alpha_x C_x{}^h, \tag{2.4}$$

and

(2. 5) $$v^h=B_a{}^h v^a+\sum_x \beta_x C_x{}^h,$$

u^a and v^a being vector fields of N and α_x and β_x being functions of N.

Now, from the first equation of (1. 1) and (2. 1), we find

$$(-\delta_i^h+u_i u^h+v_i v^h)B_b{}^i=f_b{}^c f_c{}^a B_a{}^h,$$

from which

$$f_b{}^c f_c{}^a=-\delta_b^a+u_b u^a+v_b v^a$$

and

$$u_b\alpha_x+v_b\beta_x=0.$$

From the second equation of (1. 1) and (2. 1), we find

$$f_c{}^e f_b{}^d B_e{}^t B_d{}^s g_{ts}=(g_{ji}-u_j u_i-v_j v_i)B_c{}^j B_b{}^i,$$

from which

$$f_c{}^e f_b{}^d g_{ed}=g_{cb}-u_c u_b-v_c v_b.$$

From (2. 2), we find

$$(-\delta_i^h+u_i u^h+v_i v^h)C_x{}^i=\sum_{y,z}\gamma_{xy}\gamma_{yz}C_z{}^h,$$

from which

$$\alpha_x u^a+\beta_x v^a=0$$

and

$$\sum_y \gamma_{xy}\gamma_{yz}=-\delta_{xz}+\alpha_x\alpha_z+\beta_x\beta_z.$$

From the fourth equations of (1. 1), (2. 4) and (2. 5), we find

$$-\lambda v^h=f_b{}^a B_a{}^h u^b+\sum_{x,y}\alpha_x\gamma_{xy}C_y{}^h.$$

and

$$\lambda u^h=f_b{}^a B_a{}^h v^b+\sum_{x,y}\beta_x\gamma_{xy}C_y{}^h,$$

from which

$$f_b{}^a u^b=-\lambda v^a, \qquad \sum_x \alpha_x\gamma_{xy}=-\lambda\beta_y$$

and

$$f_b{}^a v^b=\lambda u^a, \qquad \sum_x \beta_x\gamma_{xy}=\lambda\alpha_y$$

respectively.

Finally, from (2. 4) and (2. 5), we obtain respectively

$$u_a u^a=1-\lambda^2-\sum_x \alpha_x{}^2,$$

$$v_a v^a=1-\lambda^2-\sum_x \beta_x{}^2$$

and

$$u_a v^a=-\sum_x \alpha_x\beta_x.$$

Summing up these results, we have

(2.6) $$f_b{}^c f_c{}^a = -\delta_b^a + u_b u^a + v_b v^a,$$

(2.7) $$f_c{}^e f_b{}^d g_{ed} = g_{cb} - u_c u_b - v_c v_b,$$

(2.8) $$f_b{}^a u^b = -\lambda v^a, \qquad f_b{}^a v^b = \lambda u^a,$$

(2.9) $$u_a u^a = 1 - \lambda^2 - \sum_x \alpha_x{}^2, \qquad v_a v^a = 1 - \lambda^2 - \sum_x \beta_x{}^2,$$

(2.10) $$u_a v^a = -\sum_x \alpha_x \beta_x,$$

(2.11) $$\alpha_x u_b + \beta_x v_b = 0,$$

(2.12) $$\sum_y \gamma_{xy} \gamma_{yz} = -\delta_{xz} + \alpha_x \alpha_z + \beta_x \beta_z,$$

(2.13) $$\sum_x \gamma_{xy} \alpha_x = -\lambda \beta_y, \qquad \sum_x \gamma_{xy} \beta_x = \lambda \alpha_y.$$

We also have, from (2.1),

$$f_{ji} B_c{}^j B_b{}^i = f_c{}^e g_{eb}.$$

Thus putting

$$f_c{}^e g_{eb} = f_{cb},$$

we have

(2.14) $$f_{ji} B_c{}^j B_b{}^i = f_{cb},$$

which shows that f_{cb} is skew-symmetric.

Equations (2.6)~(2.11) show that a necessary and sufficient condition for $f_b{}^a$, g_{cb}, u_b, v_b and λ to define an (f, g, u, v, λ)-structure is that

$$\sum_x \alpha_x{}^2 = 0, \qquad \sum_x \beta_x{}^2 = 0,$$

that is,

$$\alpha_x = 0, \qquad \beta_x = 0,$$

or, what amounts to the same, the vectors u^h and v^h are always tangent to the submanifold.

We now compute $S_{ji}{}^h B_c{}^j B_b{}^i$. Since

$$(\nabla_j u_i - \nabla_i u_j) B_c{}^j B_b{}^i = \nabla_c (u_i B_b{}^i) - u_i \nabla_c B_b{}^i - \nabla_b (u_j B_c{}^j) + u_j \nabla_b B_c{}^j,$$

that is,

(2.15) $$(\nabla_j u_i - \nabla_i u_j) B_c{}^j B_b{}^i = \nabla_c u_b - \nabla_b u_c,$$

and similarly

(2.16) $$(\nabla_j v_i - \nabla_i v_j) B_c{}^j B_b{}^i = \nabla_c v_b - \nabla_b v_c,$$

we have

(2.17) $$\begin{aligned} S_{ji}{}^h B_c{}^j B_b{}^i = & \{N_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a\} B_a{}^h \\ & + \{\sum_x (\nabla_c u_b - \nabla_b u_c) \alpha_x + (\nabla_c v_b - \nabla_b v_c) \beta_x\} C_x{}^h, \end{aligned}$$

$N_{ji}{}^{h}B_c{}^{j}B_b{}^{i}$ being equal to $N_{cb}{}^{a}B_a{}^{h}$ by virtue of (2. 1), where $N_{cb}{}^{a}$ is the Nijenhuis tensor of $f_b{}^{a}$.

Thus, if the (f, g, u, v, λ)-structure of the ambient manifold is normal and the inuced structure on the invariant submanifold is again an (f, g, u, v, λ)-structure, then the induced structure is also normal.

§ 3. Invariant submanifolds of odd dimension.

First of all we prove the

LEMMA 3. 1. *Let N be an invariant submanifold of a manifold with (f, g, u, v, λ)-structure. If there exists a point* P *of N such that λ does not vanish at* P, *then the submanifold N is even-dimensional.*

Proof. Suppose that there exists a point P of N such that $\lambda(\mathrm{P})\neq 0$. Then from (2. 8) and the fact that f_{cb} is skew-symmetric, we have

$$(u_a v^a)(\mathrm{P})=0, \tag{3. 1}$$

from which, taking account of (2. 10), we have

$$\sum_x \alpha_x \beta_x(\mathrm{P})=0. \tag{3. 2}$$

On the other hand, from (2. 13) and the skew-symmetry of γ_{xy}, we find

$$\lambda \sum_x (\alpha_x{}^2-\beta_x{}^2)=0,$$

from which

$$\sum_x \alpha_x{}^2(\mathrm{P})=\sum_x \beta_x{}^2(\mathrm{P}). \tag{3. 3}$$

Multiplying (2. 11) by α_x and summing up over x, we get

$$(\sum_x \alpha_x{}^2(\mathrm{P}))u^a(\mathrm{P})=0, \tag{3. 4}$$

because of (3. 2).

Thus we have

$$\alpha_x(\mathrm{P})=0 \qquad \text{or} \qquad u^a(\mathrm{P})=0.$$

Suppose first that $\alpha_x(\mathrm{P})=0$. Then, because of (3. 3), we have $\beta_x(\mathrm{P})=0$. So, (2. 12) shows that

$$\sum_y \gamma_{xy}\gamma_{yz}=-\delta_{xz}$$

at P. This means that the normal space of N at P admits an almost complex structure and consequently that N is even-dimensional.

Suppose next that $u^a(\mathrm{P})=0$. Then using (2. 11), we have

$$\beta_x(\mathrm{P})v^a(\mathrm{P})=0.$$

If $v^a(\mathrm{P})=0$, then the tangent space of N at P admits an almost complex structure and so N is of even dimension. If $\beta_x(\mathrm{P})=0$, then at P, $\alpha_x=0$ because of (3.3). Hence, as in the first case, N is even-dimensional. This completes the proof.

By virtue of this lemma we have only to consider, in this section, the case in which λ vanishes identically on the submanifold N.

In this case, we have, from (2.6)$\sim$(2.10),

$$f_b{}^c f_c{}^a = -\delta_b^a + u_b u^a + v_b v^a, \tag{3.5}$$

$$f_c{}^e f_b{}^d g_{ed} = g_{cb} - u_c u_b - v_c v_b, \tag{3.6}$$

$$f_b{}^a u^b = 0, \qquad f_b{}^a v^b = 0, \tag{3.7}$$

$$u_a u^a = 1 - \sum_x \alpha_x{}^2, \qquad v_a v^a = 1 - \sum_x \beta_x{}^2, \tag{3.8}$$

$$u_a v^a = -\sum_x \alpha_x \beta_x. \tag{3.9}$$

From (2.11), we find

$$(\sum_x \alpha_x{}^2) u_b + (\sum_x \alpha_x \beta_x) v_b = 0 \tag{3.10}$$

and

$$(\sum_x \alpha_x \beta_x) u_b + (\sum_x \beta_x{}^2) v_b = 0, \tag{3.11}$$

from which

$$(\sum_x \alpha_x{}^2) u_b u^b + (\sum_x \alpha_x \beta_x) u_b v^b = 0.$$

Thus substituting (3.8) and (3.9) into this equation, we have

$$(\sum_x \alpha_x{}^2)^2 + (\sum_x \alpha_x \beta_x)^2 = \sum_x \alpha_x{}^2. \tag{3.12}$$

Similarly, we have

$$(\sum_x \beta_x{}^2)^2 + (\sum_x \alpha_x \beta_x)^2 = \sum_x \beta_x{}^2. \tag{3.13}$$

Now we recall the fact that α_x and β_x depend on the choice of the mutually orthogonal unit normal vectors $C_x{}^h$. However, we prove the

LEMMA 3.2. *$\sum_x \alpha_x{}^2$ and $\sum_x \beta_x{}^2$ are both independent of the choice of the mutually orthogonal unit normal vectors to N and consequently both of them are globally defined fuctions on N.*

Proof. Let $\overline{C}_x{}^h$ be another choice of the mutually orthogonal unit normal vectors to N. Then we can write

$$u^h = B_a{}^h u^a + \sum_x \bar{\alpha}_x \overline{C}_x{}^h \tag{3.14}$$

and

$$v^h = B_a{}^h v^a + \sum_x \bar{\beta}_x \overline{C}_x{}^h. \tag{3.15}$$

Hence we have

(3.16)
$$\sum_x \alpha_x C_x{}^h = \sum_x \bar{\alpha}_x \bar{C}_x{}^h.$$

Since $\bar{C}_x{}^h$ are mutually orthogonal unit normals to N, using an orthogonal transformation, we have

(3.17)
$$\bar{C}_x{}^h = \sum_y A_{xy} C_y{}^h.$$

Substituting (3.17) into (3.16), we get

(3.18)
$$\alpha_y = \sum_x \bar{\alpha}_x A_{xy}.$$

Thus we have

$$\sum_y \alpha_y{}^2 = \sum_{x,y,z} \bar{\alpha}_z \bar{\alpha}_x A_{zy} A_{xy} = \sum_x \bar{\alpha}_x{}^2,$$

because (A_{xy}) is an orthogonal matrix. This shows that $\sum_x \alpha_x{}^2$ is independent of the choice of unit normals.

Similarly $\sum_x \beta_x{}^2$ is independent of the choice of unit normals.

We put

$$N_\alpha = \{\mathrm{P} \in N \mid \sum_x \alpha_x{}^2 \neq 0\} \quad \text{and} \quad N_\beta = \{\mathrm{P} \in N \mid \sum_x \beta_x{}^2 \neq 0\}.$$

Then N_α, N_β are open in N and satisfy $N_\alpha \cup N_\beta = N$, because of the fact that N is odd-dimensional.

In N_α, we find, from (3.10),

(3.19)
$$u_b = -\frac{\sum_x \alpha_x \beta_x}{\sum_x \alpha_x{}^2} v_b.$$

Substituting (3.19) into

$$u_b u^a + v_b v^a,$$

we find

$$u_b u^a + v_b v^a = \frac{(\sum_x \alpha_x{}^2)^2 + (\sum_x \alpha_x \beta_x)^2}{(\sum_x \alpha_x{}^2)^2} v_b v^a,$$

or using (3.12)

(3.20)
$$u_b u^a + v_b v^a = \frac{1}{\sum_x \alpha_x{}^2} v_b v^a.$$

In N_β, we find, from (3.11),

(3.21)
$$v_b = -\frac{\sum_x \alpha_x \beta_x}{\sum_x \beta_x{}^2} u_b,$$

from which

$$u_b u^a + v_b v^a = \frac{1}{\sum_x \beta_x{}^2} u_b u^a,$$

because of (3.13).

Now we define a 1-form η_a on N in the following way: in N_α we put

$$\eta_b{}^{(\alpha)}=\frac{1}{\sqrt{\sum_x \alpha_x{}^2}}v_b \tag{3.22}$$

and in N_β

$$\eta_b{}^{(\beta)}=\frac{-1}{\sqrt{\sum_x \beta_x{}^2}}u_b. \tag{3.23}$$

Since in $N_\alpha \cap N_\beta$ we have

$$u_b=-\frac{\sum_x \alpha_x\beta_x}{\sum_x \alpha_x{}^2}v_b, \qquad v_b=-\frac{\sum_x \alpha_x\beta_x}{\sum_x \beta_x{}^2}u_b,$$

it follows that

$$u_b=\frac{(\sum_x \alpha_x\beta_x)^2}{(\sum_x \alpha_x{}^2)(\sum_x \beta_x{}^2)}u_b,$$

from which

$$(\sum_x \alpha_x\beta_x)^2=(\sum_x \alpha_x{}^2)(\sum_x \beta_x{}^2). \tag{3.24}$$

If $\sum_x \alpha_x\beta_x=0$ in $N_\alpha \cap N_\beta$, from (3.19) and (3.21), we have $u^a=0$, $v^a=0$. This shows that N is even-dimensional. So, in $N_\alpha \cap N_\beta$, $\sum_x \alpha_x\beta_x$ has no zero point. Without loss of generality we may suppose that

$$\sum_x \alpha_x\beta_x>0. \tag{3.25}$$

Thus, in $N_\alpha \cap N_\beta$, we have

$$\begin{aligned}\eta_b{}^{(\alpha)}&=\frac{1}{\sqrt{\sum_x \alpha_x{}^2}}v_b=-\frac{\sqrt{(\sum_x \alpha_x\beta_x)^2}}{\sqrt{\sum_x \alpha_x{}^2(\sum_x \beta_x{}^2)}}u_b\\ &=-\frac{1}{\sqrt{\sum_x \beta_x{}^2}}u_b=\eta_b{}^{(\beta)},\end{aligned}$$

because of (3.21), (3.24) and (3.25). Hence, η_b is a well defined 1-form on N.

Computing $u_bu^a+v_bv^a$, we find

$$u_bu^a+v_bv_v=\eta_b\eta^a, \tag{3.26}$$

and consequently, (3.5) and (3.7) give

$$f_b{}^cf_c{}^a=-\delta_b^a+\eta_b\eta^a \tag{3.27}$$

and

$$f_b{}^a\eta^b=0 \tag{3.28}$$

respectively.

Thus, from (3.27), we have, using (3.28),

(3. 29) $$-\eta^a+(\eta_b\eta^b)\eta^a=0,$$

from which

(3. 30) $$\eta_b\eta^b=1.$$

Thus the structure defined by $(f_b{}^a,\ g_{cb},\ \eta_b)$ is an almost contact metric structure.

§ 4. Odd dimensional invariant submanifolds of a manifold with normal (f, g, u, v, λ)-structure.

In § 2 we have calculated $S_{ji}{}^h B_c{}^j B_b{}^i$ and got

$$S_{ji}{}^h B_c{}^j B_b{}^i=\{N_{cb}{}^a+(\nabla_c u_b-\nabla_b u_c)u^a+(\nabla_c v_b-\nabla_b v_c)v^a\}B_a{}^h$$
$$+\sum_x\{(\nabla_c u_b-\nabla_b u_c)\alpha_x+(\nabla_c v_b-\nabla_b v_c)\beta_x\}C_x{}^h.$$

Consequently, if the (f, g, u, v, λ)-structure of the ambient manifold is normal we have

(4. 1) $$N_{cb}{}^a+(\nabla_c u_b-\nabla_b u_c)u^a+(\nabla_c v_b-\nabla_b v_c)v^a=0$$

and

(4. 2) $$(\nabla_c u_b-\nabla_b u_c)\alpha_x+(\nabla_c v_b-\nabla_b v_c)\beta_x=0.$$

Equations (3. 22), (3. 23) and (3. 30) say that

(4. 3) $$v_b v^b=\sum_x \alpha_x{}^2$$

in N_α and that

(4. 4) $$u_b u^b=\sum_x \beta_x{}^2$$

in N_β.

Now we define α and β by

(4. 5) $$\alpha^2=\sum_x \alpha_x{}^2,\qquad \beta^2=\sum_x \beta_x{}^2,$$

then, by virtue of Lemma 3. 2, they are globally defined functions on N and we can put

(4. 6) $$u^a=-\beta\eta^a,\qquad v^a=\alpha\eta^a,$$

because, when α or β vanishes, v^a or u^a vanishes.

Then

$$(\nabla_c u_b-\nabla_b u_c)u^a+(\nabla_c v_b-\nabla_b v_c)v^a$$
$$=\beta^2(\nabla_c\eta_b-\nabla_b\eta_c)\eta^a+\alpha^2(\nabla_c\eta_b-\nabla_b\eta_c)\eta^a$$
$$+\{(\nabla_c\beta)\eta_b-(\nabla_b\beta)\eta_c\}\beta\eta^a+\{(\nabla_c\alpha)\eta_b-(\nabla_b\alpha)\eta_c\}\alpha\eta^a,$$

or

(4. 7) $$(\nabla_c u_b - \nabla_b u_c)u^a + (\nabla_c v_b - \nabla_b v_c)v^a = (\nabla_c \eta_b - \nabla_b \eta_c)\eta^a,$$

by virtue of

(4. 8) $$\alpha^2 + \beta^2 = 1,$$

which is obtained from (3. 5), (4. 3) and (4. 5).

Thus (4. 1) becomes

(4. 9) $$N_{cb}{}^a + (\nabla_c \eta_b - \nabla_b \eta_c)\eta^a = 0.$$

Thus we have

THEOREM 4. 1. *Let N be an odd-dimensional invariant submanifold of a manifold with normal (f, g, u, v, λ)-structure. Then the submanifold N admits a normal almost contact metric structure.*

We now assume that the (f, g, u, v, λ)-structure of the ambient manifold is normal and satisfies

(4. 10) $$\nabla_j v_i - \nabla_i v_j = 2f_{ji}.$$

Then we have, by Theorem 1. 1,

(4. 11) $$f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj} = u_j(\nabla_i u_h) - u_i(\nabla_j u_h) + v_j(\nabla_i v_h) - v_i(\nabla_j v_h).$$

From (4. 10), we have, by transvection with $B_c{}^j B_b{}^i$,

(4. 12) $$\nabla_c v_b - \nabla_b v_c = 2f_{cb}.$$

Also we have, from (4. 11),

$$f_c{}^d \nabla_a f_{db} - f_b{}^d \nabla_a f_{dc} = u_c(\nabla_b u_a) - u_b(\nabla_c u_a) + v_c(\nabla_b v_a) - v_b(\nabla_c v_a),$$

from which

$$\nabla_a(f_c{}^d f_{db}) - (\nabla_a f_c{}^d) f_{db} - f_b{}^d \nabla_a f_{dc} = u_c(\nabla_b u_a) - u_b(\nabla_c u_a) + v_c(\nabla_b v_a) - v_b(\nabla_c v_a),$$

$$\nabla_a(-g_{cb} + u_c u_b + v_c v_b) + 2f_b{}^d(\nabla_a f_{cd}) = u_c(\nabla_b u_a) - u_b(\nabla_c u_a) + v_c(\nabla_b v_a) - v_b(\nabla_c v_a),$$

or

(4.13) $$\begin{aligned} 2(\nabla_a f_{cd}) f_b{}^d = {} & u_c(\nabla_b u_a - \nabla_a u_b) - u_b(\nabla_c u_a + \nabla_a u_c) \\ & + v_c(\nabla_b v_a - \nabla_a v_b) - v_b(\nabla_c v_a + \nabla_a v_c). \end{aligned}$$

On the other hand, using (4. 6) and (4. 8), we have

$$\begin{aligned} & u_c(\nabla_b u_a - \nabla_a u_b) - u_b(\nabla_c u_a + \nabla_a u_c) + v_c(\nabla_b v_a - \nabla_a v_b) - v_b(\nabla_c v_a + \nabla_a v_c) \\ & = \eta_c(\nabla_b \eta_a - \nabla_a \eta_b) - \eta_b(\nabla_c \eta_a + \nabla_a \eta_c). \end{aligned}$$

Substituting this into (4. 12), we get

(4. 14) $$2f_b{}^d(\nabla_a f_{cd}) = \eta_c(\nabla_b \eta_a - \nabla_a \eta_b) - \eta_b(\nabla_c \eta_a + \nabla_a \eta_c).$$

Now we prove the

LEMMA 4.2. *Let N be an odd-dimensional invariant submanifold of a manifold with normal (f, g, u, v, λ)-structure. If the ambient manifold satisfies* (4.10), *we have*

$$\alpha(\nabla_b\eta_a-\nabla_a\eta_b)=2f_{ba}. \tag{4.15}$$

Proof. Since an almost contact metric structure (f, g, η) always satisfies

$$f_b{}^a f_a{}^b=1-n,$$

it follows that

$$N_{ca}{}^a=0. \tag{4.16}$$

If the ambient manifold admits a normal (f, g, u, v, λ)-structure, from Theorem 4.1, we have

$$(\nabla_a\eta_b-\nabla_b\eta_a)\eta^a=S_{ca}{}^a-N_{ca}{}^a=0. \tag{4.17}$$

On the other hand, (4.6) and (4.12) imply that

$$\alpha(\nabla_a\eta_b-\nabla_b\eta_a)+(\nabla_a\alpha)\eta_b-(\nabla_b\alpha)\eta_a=2f_{ab}, \tag{4.18}$$

from which

$$\nabla_b\alpha=(\eta^a\nabla_a\alpha)\eta_b, \tag{4.19}$$

because of (4.17).

Substituting (4.19) into (4.18), we have (4.15).

LEMMA 4.3. *Under the same assumptions as those in Lemma* 4.2, α *is a non-zero constant.*

Proof. Suppose that there exists a point P at which

$$\alpha(\mathrm{P})=0, \qquad \text{then, for all } x,\ \alpha_x(\mathrm{P})=0.$$

Consequently we have at P

$$(\nabla_c v_b-\nabla_b v_c)\beta_x=2f_{cb}\beta_x=0 \tag{4.20}$$

because of (4.2).

Thus $\beta_x(\mathrm{P})=0$ and this, together with (2.12), shows that N is even-dimensional.

To prove that α is a constant, we differentiate (4.19) covariantly and find

$$\nabla_a\nabla_b\alpha=\gamma\nabla_a\eta_b+(\nabla_a\gamma)\eta_b,$$

from which

$$\gamma(\nabla_a\eta_b-\nabla_b\eta_a)+(\nabla_a\gamma)\eta_b-(\nabla_b\gamma)\eta_a=0, \tag{4.21}$$

where we have put $\gamma=\eta^a\nabla_a\alpha$.

Transvecting (4.21) with f^{ba}, we have

$$(n-1)\gamma=0,$$

which, together with (4. 19), implies $\nabla_b\alpha=0$.

Thus we have proved Lemma 4. 3.

THEOREM 4. 4. *An odd dimensional invariant submanifold of a manifold with normal* (f, g, u, v, λ)*-structure satisfying*

$$\nabla_j v_i-\nabla_i v_j=2f_{ji}$$

admits a Sasakian structure.

Proof. Transvecting (4. 14) with η^b and making use of (4. 17), we have

$$\nabla_c\eta_a+\nabla_a\eta_c=0, \tag{4.22}$$

which, together with (4. 15), implies that

$$\alpha\nabla_c\eta_a=f_{ca}. \tag{4.23}$$

Substituting (4. 23) into (4. 14), we have

$$\alpha f_b{}^d(\nabla_a f_{cd})=\eta_c f_{ba}.$$

Transvecting this equation with $f_e{}^b$, we find

$$-\alpha\nabla_a f_{cb}+\alpha\eta_b\eta^d\nabla_a f_{cd}=-\eta_c g_{ab}+\eta_c\eta_b\eta_a$$

or

$$-\alpha\nabla_a f_{cb}-\alpha\eta_b f_{cd}\nabla_a\eta^d=-\eta_c g_{ab}+\eta_c\eta_b\eta_a.$$

Substituting (4. 23) into the above equation and making use of (3. 6), we have

$$\alpha\nabla_a f_{bc}=\eta_b g_{ca}-\eta_c g_{ab}.$$

Thus the submanifold admits a Sasakian structure.

§ 5. Invariant submanifolds of even dimension.

We now consider an even-dimensional invariant submanifold of a manifold with (f, g, u, v, λ)-structure.

First we assume that the function λ does not vanish almost everywhere along the submanifold. In this case, from (2. 8) and the fact that f_{cb} is skew-symmetric, we have

$$u_a v^a=0, \tag{5.1}$$

from which, taking account of (2. 10), we have

$$\sum_x \alpha_x\beta_x=0. \tag{5.2}$$

On the other hand, from (2. 13) and the skew-symmetry of γ_{xy}, we find

$$\lambda(\sum_x \alpha_x^2 - \sum_x \beta_x^2) = 0,$$

from which

(5. 3) $$\sum_x \alpha_x^2 = \sum_x \beta_x^2.$$

We assume furthermore that

(5. 4) $$\sum_x \alpha_x^2 = \sum_x \beta_x^2 \neq 0$$

almost everywhere along the submanifold.

From (2. 11) and (3. 2), we find

$$\sum_x \alpha_x^2 u_b = 0, \qquad \sum_x \beta_x^2 v_b = 0,$$

from which

(5. 5) $$u_b = 0, \qquad v_b = 0,$$

that is, the vectors u^h and v^h are normal to the submanifold.

From (2. 6) and (5. 5), we have

(5. 6) $$f_c{}^b f_b{}^a = -\delta_c^a,$$

that is, $f_b{}^a$ defines an almost complex structure on the submanifold. If the (f, g, u, v, λ)-structure of the ambient manifold is normal, we have

$$0 = S_{ji}{}^h B_c{}^j B_b{}^i = N_{cb}{}^a B_a{}^h$$

and consequently the almost complex structure is integrable.

If the ambient manifold satisfies

$$\nabla_j v_i - \nabla_i v_j = 2f_{ji},$$

then we have

$$(\nabla_j v_i - \nabla_i v_j) B_c{}^j B_b{}^i = 2f_{ji} B_c{}^j B_b{}^i,$$

or

$$0 = 2f_{cb},$$

which contradicts (3. 6). Thus, we have

THEOREM 5. 1. *Let M be a differentiable manifold with (f, g, u, v, λ)-structure satisfying $\nabla_j v_i - \nabla_i v_j = 2f_{ji}$ and N be an invariant submanifold along which $\lambda \neq 0$ almost everywhere. Then*

$$\sum_x \alpha_x^2 = \sum_x \beta_x^2$$

cannot be different from zero almost everywhere.

We next assume that

(5. 7) $$\sum_x \alpha_x^2 = \sum_x \beta_x^2 = 0$$

everywhere along N, that is,

$$\alpha_x=0, \qquad \beta_x=0, \tag{5.8}$$

and consequently the vectors u^h and v^h are tangent to the submanifold.

Then equations (2.6)~(2.10) show that the submanifold admits an (f, g, u, v, λ)-structure.

Equation (2.12) shows that the normal bundle of the submanifold admits an almost complex structure.

In this case, we have

$$S_{ji}{}^h B_c{}^j B_b{}^i = \{N_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c)u^a + (\nabla_c v_b - \nabla_b v_c)v^a\} B_a{}^h,$$

and consequently

THEOREM 5.2. *Let M be a differentiable manifold with normal (f, g, u, v, λ)-structnre and N an invariant submanifold such that $\lambda \neq 0$ almost everywhere along N and u^h and v^h are always tangent to N. Then, the submanifold N admits also a normal (f, g, u, v, λ)-structure.*

Suppose that the (f, g, u, v, λ)-structure of M satisfies

$$\nabla_j u_i - \nabla_i u_j = 2\phi f_{ji}, \qquad \nabla_j v_i - \nabla_i v_j = 2 f_{ji},$$

then that of the submanifold N satisfies

$$\nabla_c u_b - \nabla_b u_a = 2\phi f_{cb}, \qquad \nabla_c v_b - \nabla_b v_c = 2 f_{cb}$$

and consequently we have

THEOREM 5.3. *Let S be an even-dimensional sphere with (f, g, u, v, λ)-structure naturally induced in it. An invariant complete submanifold N such that $\lambda \neq 0$ almost everywhere along N and vectors u^h and v^h are tangent to N is an even-dimentional sphere.*

We next assume that λ vanishes identically along the invariant submaniold N.

If there exists a point P of N at which one of $\sum_x \alpha_x^2$ and $\sum_x \beta_x^2$, say $\sum_x \alpha_x^2$, does not vanish, then the tangent space of N at P admits an almost contact structure such that

$$\left(f_b{}^a(\mathrm{P}), \qquad \frac{1}{\sqrt{\sum_x \alpha_x^2}} v_b(\mathrm{P}) \right)$$

is the structure tensors of it. Consequently, the submanifold is odd-dimensional.

Thus we have only to consider, in this section, the case in which both $\sum_x \alpha_x^2$ and $\sum_x \beta_x^2$ vanish.

Then, equations (2.6)~(2.10) become

$$f_b{}^c f_c{}^a = -\delta_b^a + u_b u^a + v_b v^a,$$

$$f_c{}^e f_b{}^d g_{ed} = g_{cb} - u_c u_b - v_c v_b,$$

$$f_b{}^a u^b = 0, \qquad f_b{}^a v^b = 0,$$

$$u_a u^a = 1, \qquad v_a v^a = 1,$$

$$u_a v^a = 0$$

and consequently the invariant submanifold admits the so-called framed f-structure of rank $n-2$.

If the (f, g, u, v, λ)-structure of the ambient manifold is normal, we have

$$S_{cb}{}^a = N_{cb}{}^a + (\nabla_c u_b - \nabla_b u_c) u^a + (\nabla_c v_b - \nabla_b v_c) v^a = 0.$$

Thus we have the

THEOREM 5.4. *Let N be an even-dimensional invariant submanifold of a manifold with (f, g, u, v, λ)-structure. If the function λ vanishes identically on the submanifold N, then N admits a framed f-structure of rank $n-2$. If, moreover, the (f, g, u, v, λ)-structure is normal, the f-structure of N is also normal.*

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