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ON CERTAIN SUBMANIFOLDS OF CODIMENSION 2 OF A LOCALLY FUBINIAN MANIFOLD

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§ 0. Introduction.

Blair, Ludden and Yano [2] introduced a structure which is natually defined in a submanifold of codimension 2 of an almost complex manifold.

Yano and Okumura introduced what they call an (f, g, u, v, λ)-structure and gave a characterization of even-dimensional sphere [5]. They also studied submanifold of codimension 2 of an even-dimensional Euclidean space which admits a normal (f, g, u, v, λ)-structure [6]. The main theorem of [6] is the following

THEOREM. *Let a complete differentiable submanifold M of codimension* 2 *of an even-dimensional Euclidean space be such that the connection induced in the normal bundle is trivial. If the* (f, g, u, v, λ)*-structure induced on M is normal, then M is a sphere, a plane, or a product of a sphere and a plane.*

In the present paper, we study submanifolds of codimension 2 of a locally Fubinian manifold which admits an (f, g, u, v, λ)-structure.

In § 1, we consider a submanifold of codimension 2 of a Kählerian manifold and find differential equations which the induced (f, g, u, v, λ)-structure satisfies.

In § 2, we prove a series of lemmas which are valid for a certain (f, g, u, v, λ)-structure.

In § 3 we study submanifolds with normal (f, g, u, v, λ)-structure in a locally Fubinian manifold.

In the last § 4, we study a submanifold of codimehsion 2 such that the linear transformations $h_j{}^i$ and $k_j{}^i$ which are defined by the second fundamental tensors commute with $f_j{}^i$ in a locally Fubinian manifold.

§ 1. Submanifolds of codimension 2 of a Kählerian manifold ([5]).

Let $\tilde{M}$ be a $(2n+2)$-dimensional Kählerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}; y^\kappa\}$, where here and in the sequel the indices $\kappa, \lambda, \mu, \nu, \cdots$ run over the range $\{1, 2, \cdots, 2n+2\}$, and let $(F_\mu{}^\kappa, G_{\mu\lambda})$ be the Kählerian structure of $\tilde{M}$, that is,

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(1. 1) $$F_\mu{}^\kappa F_\lambda{}^\mu = -\delta_\lambda^\kappa,$$

and $G_{\mu\lambda}$ a Riemannian metric such that

(1. 2) $$G_{\beta\alpha} F_\mu{}^\beta F_\lambda{}^\alpha = G_{\mu\lambda},$$

(1. 3) $$\tilde{\nabla}_\mu F_\lambda{}^\kappa = 0,$$

where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Christoffel symbols $\widetilde{\{{}^{\kappa}_{\mu\lambda}\}}$ formed with $G_{\mu\lambda}$.

Let M be a $2n$-dimensional differentiable manifold which is covered by a system of coordinate neighborhoods $\{U; x^h\}$, where hear and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$, and, which is differentiably immersed in M as a submanifold of codimension 2 by the equations

(1. 4) $$y^\kappa = y^\kappa(x^h).$$

We put

$$B_i{}^\kappa = \partial_i y^\kappa, \qquad (\partial_i = \partial/\partial x^i)$$

then $B_i{}^\kappa$ is, for fixed i, a local vector field of $\tilde{M}$ tangent to M and the vectors $B_i{}^\kappa$ are linearly independent in each coordinate neighborhood. $B_i{}^\kappa$ is, for fixed κ, a local 1-form of M.

We choose two mutually orthogonal unit vectors C^κ and D^κ of $\tilde{M}$ normal to M in such a way that $2n+2$ vectors $B_i{}^\kappa$, C^κ, D^κ give the positive orientation of $\tilde{M}$.

The transforms $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed as linear combinations of $B_i{}^\kappa$, C^κ and D^κ, that is,

(1. 5) $$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + u_i C^\kappa + v_i D^\kappa,$$

where $f_i{}^h$ is a tensor field of type (1, 1) and u_i, v_i are 1-forms of M. Similarly, the transform $F_\lambda{}^\kappa C^\lambda$ of C^λ by $F_\lambda{}^\kappa$ and the transform $F_\lambda{}^\kappa D^\lambda$ of D^λ by $F_\lambda{}^\kappa$ can be written as

(1. 6) $$F_\lambda{}^\kappa C^\lambda = -u^i B_i{}^\kappa + \lambda D^\kappa,$$
$$F_\lambda{}^\kappa D^\lambda = -v^i B_i{}^\kappa - \lambda C^\kappa,$$

where

$$u^i = u_t g^{ti}, \qquad v^i = v_t g^{ti},$$

g_{ji} being the Riemannian metric on M induced from that of $\tilde{M}$, and λ is a function on M. We can easily verify that λ is a function globally defined on M.

Applying $F_\kappa{}^\mu$ again to (1. 5) and taking account of (1. 5) itself and (1. 6), we find

(1. 7) $$f_j{}^h f_i{}^j = -\delta_i^h + u_i u^h + v_i v^h,$$

(1. 8) $$u_h f_i{}^h = \lambda v_i, \qquad v_h f_i{}^h = -\lambda u_i.$$

Applying $F_\kappa{}^\mu$ again to (1. 6) and taking account of (1. 5) and (1. 6) itself, we get

$$f_i{}^h u^i = -\lambda v^h, \qquad u_i u^i = 1-\lambda^2, \qquad u_i v^i = 0, \tag{1. 9}$$

$$f_i{}^h v^i = \lambda u^h, \qquad v_i u^i = 0, \qquad v_i v^i = 1-\lambda^2. \tag{1. 10}$$

On the other hand, we have, from (1. 2)

$$g_{kh} f_j{}^k f_i{}^h = g_{ji} - u_j u_i - v_j v_i. \tag{1. 11}$$

If we put $f_{it} = f_i{}^r g_{rt}$, then we can easily verify that f_{it} is skew-symmetric.

We call an (f, g, u, v, λ)-structure of M the set of f, g, u, v, and λ satisfying (1. 7)-(1. 11).

We denote by $\{{}_j{}^h{}_i\}$ and ∇_i the Christoffel symbols formed with g_{ji} and the operator of covariant differentiation with respect to $\{{}_j{}^h{}_i\}$, respectively.

Then the equations of Gauss of M are

$$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa + \widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}} B_j{}^\mu B_i{}^\lambda - B_h{}^\kappa \left\{ {h \atop j\ i} \right\} = h_{ji} C^\kappa + k_{ji} D^\kappa, \tag{1. 12}$$

where h_{ji} and k_{ji} are the second fundamental tensors of M with respect to the normals C^κ and D^κ respectively.

The equations of Weingarten are

$$\begin{aligned} \nabla_j C^\kappa &= \partial_j C^\kappa + \widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}} B_j{}^\mu C^\lambda = -h_j{}^i B_i{}^\kappa + l_j D^\kappa, \\ \nabla_j D &= \partial_j D^\kappa + \widetilde{\left\{ {\kappa \atop \mu\lambda} \right\}} B_j{}^\mu D^\lambda = -k_j{}^i B_i{}^\kappa - l_j C^\kappa, \end{aligned} \tag{1. 13}$$

where $h_j{}^i = h_{jt} g^{ti}$, $k_j{}^i = k_{jt} g^{ti}$ and l_j is the so-called third fundamental tensor.

Differentiating (1. 5) covariantly along M and taking account of (1. 12) and (1. 13), we get

$$\begin{aligned} &(\tilde{\nabla}_\mu F_\lambda{}^\kappa) B_j{}^\mu B_i{}^\lambda - (h_{ji} u^h + k_{ji} v^h) B_h{}^\kappa - \lambda k_{ji} C^\kappa + \lambda h_{ji} D^\kappa \\ &= (\nabla_j f_i{}^h - h_j{}^h u_i - k_j{}^h v_i) B_h{}^\kappa + (\nabla_j u_i - h_{jt} f_i{}^t - l_j v_i) C^\kappa + (\nabla_j v_i - k_{jt} f_i{}^t + l_j u_i) D^\kappa. \end{aligned}$$

Since $\tilde{M}$ is a Kählerian manifold, we have

$$\nabla_j f_i{}^h = -h_{ji} u^h + h_j{}^h u_i - k_{ji} v^h + k_j{}^h v_i, \tag{1. 14}$$

$$\nabla_j u_i = -h_{jt} f_i{}^t - \lambda k_{ji} + l_j v_i. \tag{1. 15}$$

$$\nabla_j v_i = -k_{jt} f_i{}^t + \lambda h_{ji} - l_j u_i. \tag{1. 16}$$

Similarly, differentiating (1. 6) covariantly along M, we find

$$\nabla_j \lambda = k_{ji} u^i - h_{ji} v^i. \tag{1. 17}$$

§ 2. Some lemmas on (f, g, u, v, λ)-structure.

We now compute

$$S_{ji}{}^{h}=N_{ji}{}^{h}+(\nabla_j u_i-\nabla_i u_j)u^h+(\nabla_j v_i-\nabla_i v_j)v^h, \tag{2. 1}$$

where $N_{ji}{}^{h}$ is the Nijenhuis tensor formed with $f_i{}^h$.

Substituting (1. 14), (1. 15) and (1. 16) into (2. 1), we get

$$\begin{aligned} S_{ji}{}^{h}=&(f_j{}^t h_t{}^h-h_j{}^t f_t{}^h)u_i-(f_i{}^t h_t{}^h-h_i{}^t f_t{}^h)u_j \\ &+(f_j{}^t k_t{}^h-k_j{}^t f_t{}^h)v_i-(f_i{}^t k_t{}^h-k_i{}^t f_t{}^h)v_j+(l_j v_i-l_i v_j)u^h-(l_j u_i-l_i u_j)v^h. \end{aligned} \tag{2. 2}$$

When the tensor $S_{ji}{}^{h}$ vanishes identicallly, the (f, g, u, v, λ)-structure is said to be normal.

If the connection induced in the normal bundle of M is flat, then we can choose C^κ, D^κ in such a way that we have $l_j=0$, and we say that the connection induced in the normal bundle is trivial.

In this case, (2. 2) can be written as

$$\begin{aligned} &(f_j{}^t h_t{}^h-h_j{}^t f_t{}^h)u_i-(f_i{}^t h_t{}^h-h_i{}^t f_t{}^h)u_j \\ &+(f_j{}^t k_t{}^h-k_j{}^t f_t{}^h)v_i-(f_i{}^t k_t{}^h-k_i{}^t f_t{}^h)v_j=0. \end{aligned} \tag{2. 3}$$

We see that left hand side of (2. 3) is independent of the choice of mutually orthogonal unit normal vectors C^κ and D^κ.

Let M be a submanifold of codimension 2 of a Kählerian manifold such that connection induced in the normal bundle is trival. Assuming that the function $\lambda(1-\lambda^2)$ does not vanish almost everywhere on M, we prove the following two lemmas.

LEMMA 2. 1. *For the normal (f, g, u, v, λ)-structure of M such that the connection induced in the normal bundle is trivial, we have*

$$\begin{aligned} h_{ji}u^i&=\alpha u_j+\beta v_j, & h_{ji}v^i&=\beta u_j+\gamma v_j, \\ k_{ji}u^i&=\bar{\alpha} u_j+\bar{\beta} v_j, & k_{ji}v^i&=\bar{\beta} u_j+\bar{\gamma} v_j, \end{aligned} \tag{2. 4}$$

α, β, γ, $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ being scalars of M.

Proof. See [6].

LEMMA 2. 2. *In Lemma 2. 1, we have*

$$2\beta=\bar{\alpha}-\bar{\gamma}, \qquad 2\bar{\beta}=\gamma-\alpha. \tag{2. 5}$$

Proof. In (2. 3), we contract with respect to h and i, then we have

$$f_j{}^t(h_t{}^i u_i)+f_j{}^t(k_t{}^i v_i)-h_j{}^t(f_t{}^i u_i)-k_j{}^t(f_t{}^i v_i)=0.$$

Substituting (1. 8) and (2. 4) into this equation, we find

$$-\lambda(2\beta+\bar{\gamma}-\bar{\alpha})u_j+\lambda(2\bar{\beta}+\alpha-\gamma)v_j=0$$

from which, we obtain (2. 5).

Next, we consider a submanifold M^{2n} of codimension 2 of a Kählerian manifold satisfying the following conditions:

(2. 6) $$f_j{}^t h_t{}^h=h_j{}^t f_t{}^h,$$

and

(2. 7) $$f_j{}^t k_t{}^h=k_j{}^t f_t{}^h.$$

We see that (2. 6) and (2. 7) are independent of the choice of mutnally orthogonal unit normal vectors C^ϵ and D^ϵ and consequently that (2. 6) and (2. 7) are globally defined over M^{2n}.

LEMMA 2. 3. *For (f, g, u, v, λ)-structure of M^{2n} with* (2. 6) *and* (2. 7), *we have*

(2. 8) $$h_{ji}u^i=\alpha u_j, \qquad h_{ji}v^i=\alpha v_j,$$

(2. 9) $$k_{ji}u^i=\bar{\alpha}u_j, \qquad k_{ji}v^i=\bar{\alpha}v_j,$$

where α and $\bar{\alpha}$ are scalars of M^{2n} and λ does not vanish almost everywhere on M^{2n}.

Proof. From (2. 6), we see that $h_j{}^t f_{ti}$ is skew-symmetric in j and i. Thus

$$h_j{}^t f_{ti}u^j u^i=\lambda h_{ji}u^j v^i=0$$

by virtue of (1. 9) and consequently

(2. 10) $$h_{ji}u^j v^i=0.$$

Transvecting (2. 6) with $f_h{}^i$ and taking account of (1. 7), we get

$$h_j{}^t(-\delta_t^i+u^i u_t+v^i v_t)=h_{ts}f_j{}^t f^{si},$$

or

$$-h_{st}f_j{}^s f_i{}^t=-h_{ji}+(h_{jt}u^t)u_i+(h_{jt}v^t)v_i.$$

Since $h_{st}f_j{}^s f_i{}^t$ is symmetric in j and i, we have

(2. 11) $$(h_{jt}u^t)u_i-(h_{it}u^t)u_j+(h_{jt}v^t)v_i-(h_{jt}v^t)v_j=0.$$

Transvecting (2. 11) with u^i and using (2. 10), we get

$$h_{jt}u^t(1-\lambda^2)-(h_{st}u^s u^t)u_j=0,$$

and consequently

$$h_{ji}u^i=\alpha u_j,$$

where we have put

$$h_{st}u^s u^t=(1-\lambda^2)\alpha. \tag{2. 12}$$

On the other hand, transvecting (2. 6) with u^h and taking account of (1. 9) and (2. 12), we have

$$\lambda h_j{}^t v_t=\alpha u_t f_j{}^t=\lambda\alpha v_j,$$

frow which

$$h_{ji}v^i=\alpha v_j.$$

Similarly we can prove (2. 9). This completes the proof of Lemma 2. 3.

LEMMA 2. 4. *Under the same assumptions as those in Lemma* 2. 3, *we have*

$$\bar{\alpha}h_{ji}=\alpha k_{ji}, \tag{2. 13}$$

where $\lambda(1-\lambda^2)$ *does not vanish almost everywhere on* M^{2n}.

Proof. From (2. 8) and (2. 9), (1. 17) can be written as

$$\nabla_i\lambda=\bar{\alpha}u_j-\alpha v_j. \tag{2. 14}$$

Differentiating (2. 14) covariantly, we have

$$\nabla_k\nabla_j\lambda=(\nabla_k\bar{\alpha})u_j-(\nabla_k\alpha)v_j+\bar{\alpha}\nabla_k u_j-\alpha\nabla_k v_j.$$

If we subtract this from the equation obtained by interchanging the indices j and k in this and making use of (1. 15) and (1. 16), we find

$$\begin{aligned}&(\nabla_k\bar{\alpha}+\alpha l_k)u_j-(\nabla_j\bar{\alpha}+\alpha l_j)u_k-2\bar{\alpha}h_{kt}f_j{}^t\\&=(\nabla_k\alpha-\bar{\alpha}l_k)v_j-(\nabla_j\alpha-\bar{\alpha}l_j)v_k-2\alpha k_{kt}f_j{}^t.\end{aligned} \tag{2. 15}$$

Transvecting (2. 15) with u^j and v^j respectively, we have

$$\nabla_k\bar{\alpha}+\alpha l_k=Au_k+Bv_k,$$

$$\nabla_k\alpha-\bar{\alpha}l_k=Cu_k+Dv_k.$$

Substituting these into (2.15), we get

$$(B+C)(v_k u_j-v_j u_k)-2\bar{\alpha}h_{kt}f_j{}^t=-2\alpha k_{kt}f_j{}^t.$$

Applying u^j to these and making use of (2. 8) and (2. 9), we find $B+C=0$. It follows that

$$\bar{\alpha}h_{kt}f_j{}^t=\alpha k_{kt}f_j{}^t. \tag{2.16}$$

Transvecting (2. 16) with $f_i{}^j$ and taking account of (1. 7) and (1. 8), we have (2. 13).

§ 3. Submanifold of codimension 2 with normal (f, g, u, v, λ)-structure in a locally Fubinian manifold.

A Kählerian manifold $\tilde{M}^{2n+2}$ is called a locally Fubinian manifold if the holomorphic sectional curvature at every point is indepent of the holomorphic section at the point. Its curvature tensor is given by

$$\tilde{R}_{\nu\mu\lambda\kappa}=k(G_{\nu\kappa}G_{\mu\lambda}-G_{\mu\kappa}G_{\nu\lambda}+F_{\nu\kappa}F_{\mu\lambda}-F_{\mu\kappa}F_{\nu\lambda}-2F_{\nu\mu}F_{\lambda\kappa}), \tag{3. 1}$$

k being a constant.

In this section we consider a submanifold M^{2n} in a locally Fubinian manifold. Substituting (3. 1) into the Gauss, Codazzi, Ricci-equations

$$\tilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda B_h{}^\kappa=R_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki}-k_{kh}k_{ji}+k_{jh}k_{ki},$$

$$\begin{cases}\tilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda C^\kappa=\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki},\\ \tilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu B_i{}^\lambda D^\kappa=\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki},\end{cases}$$

$$\tilde{R}_{\nu\mu\lambda\kappa}B_k{}^\nu B_j{}^\mu C^\lambda D^\kappa=\nabla_k l_j-\nabla_j l_k+h_{kt}k_j{}^t-h_{jt}k_k{}^t,$$

we have respectively

$$\begin{aligned}&k(g_{kh}g_{ji}-g_{jh}g_{ki}+f_{kh}f_{ji}-f_{jh}f_{ki}-2f_{kj}f_{ih})\\ &=R_{kjih}-h_{kh}h_{ji}+h_{jh}h_{ki}-k_{kh}k_{ji}+k_{jh}k_{ki},\end{aligned} \tag{3. 2}$$

and

$$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}=k(u_k f_{ji}-u_j f_{ki}-2u_i f_{kj}),$$

$$\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}=k(v_k f_{ji}-v_j f_{ki}-2v_i f_{kj})$$

and

$$\nabla_k l_i-\nabla_j l_k+h_{kt}k_j{}^t-h_{jt}k_k{}^t=k(v_k u_j-v_j u_k-2\lambda f_{kj}). \tag{3. 4}$$

We now prove the following

THEOREM 3. 1. *Let a submanifold M^{2n} of codimension 2 of a locally Fubinian manifold $\tilde{M}^{2n+2}$ be such that the connection induced in the normal bundle of M^{2n} is trivial. If the (f, g, u, v, λ)-structure is normal and λ is a constant different from 0 and 1, then there is no such a M^{2n} unless $\tilde{M}^{2n+2}$ is locally Fuclidean.*

Proof. Since λ is a constant on M^{2n}, we have, from (1. 17)

$$h_{ji}v^i-k_{ji}u^i=0. \tag{3. 5}$$

Making use of (2. 5) and (3. 5), we can write (2. 4) as

$$\begin{aligned}&h_{ji}u^i=\alpha u_j+\beta v_j, \qquad h_{ji}v^i=\beta u_j-\alpha v_j,\\ &k_{ji}u^i=\beta u_j-\alpha v_j, \qquad k_{ji}v^i=-\alpha u_j-\beta v_j,\end{aligned} \tag{3. 6}$$

from which,

$$h_{ji}u^i + k_{ji}v^i = 0. \tag{3. 7}$$

Differentiating (3. 7) covariantly, we obtain

$$(\nabla_k h_{ji})u^i + h_{ji}\nabla_k u^i + (\nabla_k k_{ji})v^i + k_{ji}\nabla_k v^i = 0.$$

If we subtract this from the equation obtained by interchanging the indices j and k in this and take account of (1. 15), (1. 16) and (3. 3), we have

$$\begin{aligned} &k(u_k f_{ji} - u_j f_{ki} - 2u_i f_{kj})u^i + h_{ji}(-h_{kt}f^{it} - \lambda k_k{}^i) \\ &- h_{ki}(-h_{jt}f^{it} - \lambda k_j{}^i) + k(v_k f_{ji} - v_j f_{ki} - 2v_i f_{kj})v^i \\ &+ k_{ji}(-k_{kt}f^{it} + \lambda h_k{}^i) - k_{ki}(-k_{jt}f^{it} + \lambda h_j{}^i) = 0, \end{aligned}$$

or, using (1. 9), (1. 10) and (3. 4),

$$(h_{ji}h_{kt} + k_{ji}k_{kt})f^{it} = -2kf_{kj}.$$

Transvecting this equation with u^j and using (1. 9), (1. 10) and (3. 6), we get

$$\lambda(-\alpha v_t + \beta u_t)h_k{}^t + \lambda(\beta v_t - \alpha u_t)k_k{}^t = -2\lambda k v_k,$$

and consequently

$$\alpha^2 + \beta^2 = -k. \tag{3. 8}$$

On the other hand, transvecting (3. 4) with u^k and (1. 9) and (3. 6), we find

$$2(\alpha^2 + \beta^2) = k(1 - 3\lambda^2).$$

From this and (3. 8), it follows that $k=0$. This means that $\tilde{M}^{2n+2}$ is locally Euclidean.

§ 4. Submanifold of codimension 2 with certain $(f, g, u, v\ \lambda)$-structure in a locally Fubinian manifold.

It this section, we consider a submanifold M^{2n} of codimension 2 of a locally Fubinian manifold satisfying the conditions (2. 6) and (2. 7). We assume that $\lambda(1-\lambda^2)$ does not vanish almost everywhere on M^{2n}.

Differentiating the second equation of (2. 8) covariantly, we have

$$(\nabla_k h_{ji})v^i + h_{ji}\nabla_k v^i = (\nabla_k \alpha)v_j + \alpha \nabla_k v_j.$$

If we subtract this from the equation obtained by interchanging the indices j and k in this and take account of (1. 16), (2. 9) and (3. 3), we get

$$\begin{aligned} &\bar{\alpha}(l_k v_j - l_j v_k) + (-h_{ji}k_{kt} + h_{ki}k_{jt})f^{it} + \alpha(l_j u_k - l_k u_j) \\ &= (\nabla_k \alpha)v_j - (\nabla_j \alpha)v_k - \alpha k_{kt}f_j{}^t + \alpha k_{jt}f_k{}^t + \alpha(-l_k u_j + l_j u_k), \end{aligned}$$

or, using (2. 6),

$$(h_{ki}k_{jt}-h_{ji}k_{kt})f^{it}=(\nabla_k\alpha-\bar{\alpha}l_k)v_j-(\nabla_j\alpha-\bar{\alpha}l_j)v_k-2\alpha k_{kt}f_j{}^t. \tag{4.1}$$

Transvecting (4. 1) with v^k and using (2. 8) and (2. 9), we get

$$(1-\lambda^2)(\nabla_k\alpha-\bar{\alpha}l_k)=v^t(\nabla_t\alpha-\bar{\alpha}l_t)v_k, \tag{4.2}$$

from which,

$$u^t(\nabla_t\alpha-\bar{\alpha}l_t)=0. \tag{4.3}$$

Substituting (4. 2) into (4. 1), we find

$$(h_{ki}k_{jt}-h_{ji}k_{kt})f^{it}=-2\alpha k_{kt}f_j{}^t, \tag{4.4}$$

or, using (2. 6) and (2. 7),

$$(h_{ji}k_t{}^i+k_{ji}h_t{}^i)f_k{}^t=2\alpha k_{jt}f_k{}^t. \tag{4.5}$$

Transvecting (4. 5) with $f_h{}^k$ and using (2. 8) and (2. 9), we get

$$h_{jt}k_i{}^t+k_{jt}h_i{}^t=2\alpha k_{ji}. \tag{4.6}$$

From (2. 13) and (4. 6), we find

$$h_{jt}k_i{}^t+k_{jt}h_i{}^t=2\bar{\alpha}h_{ji}. \tag{4.7}$$

Differentiating the first equation of (2. 8) covariantly, we have

$$(\nabla_k h_{ji})u^i+h_{ji}\nabla_k u^i=(\nabla_k\alpha)u_j+\alpha(\nabla_k u_j),$$

from which,

$$(\nabla_k h_{ji}-\nabla_j h_{ki})u^i+h_{ji}\nabla_k u^i-h_{ki}\nabla_j u^i=(\nabla_k\alpha)u_j-(\nabla_j\alpha)u_k+\alpha(\nabla_k u_j-\nabla_j u_k).$$

Substituting (1. 15) and (3. 3) into this, we get

$$\begin{aligned}&k(\lambda u_k v_j-\lambda u_j v_k-2(1-\lambda^2)f_{kj})+\bar{\alpha}(l_k u_j-l_j u_k)+2h_{ji}h_k{}^t f_t{}^i+\lambda(h_{kt}k_j{}^t-h_{jt}k_k{}^t)\\&=(\nabla_k\alpha)u_j-(\nabla_j\alpha)u_k+2\alpha h_{jt}f_k{}^t,\end{aligned} \tag{4.8}$$

because of (2. 6) and (2. 8).

Transvecting (4. 8) with u^k and using (2. 8), (2. 9) and (4. 3), we get

$$\nabla_j\alpha-\bar{\alpha}l_j=-3\lambda k v_j. \tag{4.9}$$

Substituting (4. 9) into (4. 8), we have

$$2k\{-\lambda u_k v_j+\lambda u_j v_k-(1-\lambda^2)f_{kj}\}+\lambda(h_{kt}k_j{}^t-h_j{}^t k_k{}^t)+2h_{ji}h_k{}^t f_t{}^i-2\alpha h_{jt}f_k{}^t=0. \tag{4.10}$$

In the first place, if we put $M_0=\{p\in M^{2n}|(h_{kt}k_j{}^t-h_{jt}k_k{}^t)(p)\neq 0\}$, then (2. 13) shows that at a point $p\in M_0$, we have $\alpha(p)=\bar{\alpha}(p)=0$.

From this and (4. 9), if there exists a point $p \in M_0$, then $k=0$ on M^{2n} because of k is a constant.

From the above discussion we know that we have to consider only the case that $k \neq 0$ and

$$h_{kt}k_j{}^t - h_{jt}k_k{}^t = 0 \tag{4. 11}$$

at every point of M^{2n}. In this case, however, we can also prove that the enveloping manifold is locally flat. In fact, (4. 10) reduced to

$$h_{ji}h_k{}^t f_t{}^i + \{\lambda u_j v_k - \lambda u_k v_j - (1-\lambda^2) f_{kj}\} = \alpha h_{jt} f_k{}^t,$$

or, using (2. 6),

$$(h_{ji}h_t{}^i - \alpha h_{jt}) f_k{}^t = k\{\lambda u_k v_j - \lambda u_j v_k + (1-\lambda^2) f_{kj}\}. \tag{4. 12}$$

Transvecting (4. 12) with $f_h{}^k$ and using (1. 7) and (1. 8), we obtain

$$(h_{ji}h_t{}^i - \alpha h_{jt})(-\delta_h{}^t + u_h u^t + v_h v^t)$$
$$= k\{\lambda^2 v_j v_h + \lambda^2 u_j u_h + (1-\lambda^2)(-g_{hj} + u_h u_j + v_j v_h)\},$$

or using (2. 8) and (2. 9),

$$h_{ji}h_h{}^i - \alpha h_{jh} = -k\{(\lambda^2-1) g_{jh} + u_j u_h + v_j v_h\}. \tag{4. 13}$$

Similarly, from the first equation of (2. 9) and (4. 11), we obtain

$$k_{ji}k_h{}^i - \bar{\alpha} k_{jh} = -k\{(\lambda^2-1) g_{jh} + u_j u_h + v_j v_h\}. \tag{4. 14}$$

From (2. 13), (4. 13) and (4. 14), we can easily find that

$$\alpha^2 = \bar{\alpha}^2. \tag{4. 15}$$

If α or $\bar{\alpha}$ is zero, then $\alpha = \bar{\alpha} = 0$ and consequently $k=0$. Therefore, we may consider that α and $\bar{\alpha}$ are not zero. Then, from (2. 13) and (4. 6), we find

$$h_{jt}h_h{}^t = \alpha h_{jh}.$$

From this and (4. 13), we have

$$k\{(\lambda^2-1) g_{jh} + u_j u_h + v_j v_h\} = 0. \tag{4. 16}$$

Transvecting (4. 16) with g^{jh}, we find

$$(n-1)(1-\lambda^2)k = 0.$$

Therefore $k=0$ for $n>1$. Hence, we have the following

THEOREM 4. 1. *If a locally Fubinian manifold $\tilde{M}^{2n+2}$ $(n>1)$ admits a submanifold of codimension 2 such that the linear transformations $h_j{}^i$ and $k_j{}^i$ which are defined by the second fundamental tensors commute with $f_j{}^i$, then it is locally flat.*

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