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MINIMAL SURFACES WITH M-INDEX 2, T_1-INDEX 2 AND T_2-INDEX 2

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For a minimal submanifold of dimension greater than 2 and with M-index 2 in a Riemannian manifold of constant curvature, Ōtsuki [4] gave a condition that its geodesic codimension is 3 and some examples of such minimal submanifolds under certain additional conditions in Euclidean, spherical and hyperbolic non-Euclidean spaces. For a minimal surface with M-index 2 in a Riemannian manifold of constant curvature, the author [2] proved that we may put formally $p=0$ and $n=2$ in the result of [4] when the ambiant spaces are spheres and solved the differential equations. Ōtsuki [5] gave a condition that the geodesic codimension becomes 4 and some examples in space forms.

In the present paper, the author will study minimal surfaces with M-index 2, T_1-index 2 and T_2-index 2 in a Riemannian manifold of constant curvature, where T_1-index and T_2-index are analogous to those of Ōtsuki [3]. Furthermore, he will give a condition that the geodesic codimension becomes 6 and a general solution of such minimal surfaces.

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§1. **Minimal surfaces with M-index 2.** Let $\bar{M}=\bar{M}^{2+\nu}$ be a $(2+\nu)$-dimensional Riemannian manifold of constant curvature $\bar{c}$ and $M=M^2$ be a 2-dimensinal submanifold in $\bar{M}$ with the Riemannian metric induced from $\bar{M}$, where both manifolds are C^∞. Let $\bar{\omega}_A$, $\bar{\omega}_{AB}=-\bar{\omega}_{BA}$ $A, B=1, 2, \cdots, 2+\nu$, be the basic and connection forms of $\bar{M}$ on the orthonormal frame bundle $F(\bar{M})$ which satify the structure equations:

$$(1.1)\qquad d\bar{\omega}_A=\sum_A \bar{\omega}_{AB}\wedge\bar{\omega}_B, \qquad d\bar{\omega}_{AB}=\sum_C \bar{\omega}_{AC}\wedge\bar{\omega}_{CB}-\bar{c}\bar{\omega}_A\wedge\bar{\omega}_B.$$

Let B be the subbundle of $F(\bar{M})$ over M such that $B\ni b=(x, e_1, e_2, e_3, \cdots, e_{2+\nu})$ $\in F(\bar{M})$ and $(x, e_1, e_2)\in F(M)$, where $F(M)$ is the orthonormal frame bundle of M. Then, deleting bars of $\bar{\omega}_A$, $\bar{\omega}_{AB}$, on B we have

$$(1.2)\qquad \omega_\alpha=0, \qquad \omega_{i\alpha}=\sum_j A_{\alpha ij}\omega_j, \qquad A_{\alpha ij}=A_{\alpha ji},$$

$$d\omega_i=\omega_{ik}\wedge\omega_k, \qquad d\omega_{ik}=\sum_\alpha \omega_{i\alpha}\wedge\omega_{\alpha k}-\bar{c}\omega_i\wedge\omega_k,$$

$$(1.3)\qquad d\omega_{i\alpha}=\omega_{ik}\wedge\omega_{k\alpha}+\sum_\beta \omega_{i\beta}\wedge\omega_{\beta\alpha},$$

$$d\omega_{\alpha\beta}=\sum_i \omega_{\alpha i}\wedge\omega_{i\beta}+\sum_\gamma \omega_{\alpha\gamma}\wedge\omega_{\gamma\beta},$$

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where $i, j, k=1, 2, i \neq k, \alpha, \beta, \gamma=3, 4, \cdots, 2+\nu$.

For any point $x \in M$, let N_x be the normal space to $M_x=T_xM$ in $\bar{M}_x=T_x\bar{M}$. For any $b \in B$, we define a linear mapping ψ_b from N_x into the set of all symmetric matrices of order 2 by

$$\psi_b(\sum_\alpha v_\alpha e_\alpha)=\sum_\alpha v_\alpha A_\alpha, \qquad A_\alpha=(A_{\alpha ij}). \tag{1.4}$$

We call the $\dim \psi_b$ ($\ker \bar{m}$) at x *M-index* of M at x in $\bar{M}$, where $\bar{m}$ is a linear mapping from N_x into R defined by $\bar{m}(\sum_\alpha v_\alpha e_\alpha)=(1/2)\,\mathrm{trace}\,(\sum_\alpha v_\alpha A_\alpha)$.

Now we suppose that M is minimal in $\bar{M}$ and of M-index 2 at each point. Then N_x is decomposed as

$$N_x=N'_x+O, \qquad N'_x \perp O_x, \tag{1.5}$$

where $O_x=\psi_b^{-1}(0)$ and $\dim N'_x=2$, which does not depend on the choice of b over x and is smooth. Let B_1 be the set of $b \in B$ such that $e_3, e_4 \in N'_x$. From the definition of M-index, on B_1 we have

$$\omega_{i\beta}=0, \qquad i=1, 2, \qquad 4<\beta. \tag{1.6}$$

LEMMA 1. *On B_1, for a fixed $\beta>4$, we have $\omega_{3\beta}\equiv\omega_{4\beta}\equiv 0$ $(\mathrm{mod}\,\omega_1, \omega_2)$ and $\omega_{3\beta}=\omega_{4\beta}=0$ or else $\omega_{3\beta}\wedge\omega_{4\beta}\neq 0$.*

LEMMA 2. *We can choose a frame $b \in B_1$ such that*

$$\omega_{13}=\lambda\omega_1, \quad \omega_{23}=-\lambda\omega_2, \quad \omega_{14}=\mu\omega_2, \quad \omega_{24}=\mu\omega_1, \quad \lambda\mu\neq 0. \tag{1.7}$$

Proof. Rotating 2-frames (x, e_1, e_2) and (x, e_3, e_4) suitably, we can choose a frame $b \in B_1$ such that

$$\omega_{13}=\lambda\omega_1, \qquad \omega_{23}=-\lambda\omega_2, \qquad \lambda\neq 0, \qquad \langle A_3, A_4\rangle=0.$$

Then, putting $\omega_{14}=a\omega_1+\mu\omega_2$ and $\omega_{24}=\mu\omega_1+b\omega_2$, we have $\langle A_3, A_4\rangle=(1/2)\lambda(a-b)=0$. It follows from $\lambda\neq 0$ that $a=b$. On the other hand, since trace $A_4=0$, we have $a+b=0$. Hence we have $a=b=0$. Since M-index is 2, μ must not be zero. Q.E.D.

Let B_2 be the set of all $b \in B_1$ satisfying (1.7). Then since M-index is 2 everywhere on M, B_2 is a smooth submanifold of B_1. Making use of (1.3) and (1.6), we have $\omega_{i3}\wedge\omega_{3\beta}+\omega_{i4}\wedge\omega_{4\beta}=0$. Substituting (1.7) into these equations, we have

$$\begin{aligned}&\lambda\omega_1\wedge\omega_{3\beta}+\mu\omega_2\wedge\omega_{4\beta}=0,\\ &\lambda\omega_2\wedge\omega_{3\beta}-\mu\omega_1\wedge\omega_{4\beta}=0,\end{aligned} \tag{1.8}$$

which imply that we can put

$$\begin{aligned}&\lambda\omega_{3\beta}=f_\beta\omega_1+g_\beta\omega_2,\\ &\mu\omega_{4\beta}=g_\beta\omega_1-f_\beta\omega_2.\end{aligned} \tag{1.9}$$

Now, by virtue of Lemma 1, we can define two linear mappings φ_{11} and φ_{12}

corresponding to the normal vector fields e_3 and e_4 from M_x into O_x as follows: for any $X \in M_x$

(1. 10) $$\varphi_{11}(X)=\sum_\beta \omega_{3\beta}(X)e_\beta, \qquad \varphi_{12}(X)=\sum_\beta \omega_{4\beta}(X)e_\beta.$$

By means of (1. 9), the two linear mappings

(1. 11) $$\tilde\varphi_{11}=\lambda\varphi_{11} \qquad \text{and} \qquad \tilde\varphi_{12}=\mu\varphi_{12}$$

have the same images of the tangent unit sphere $S_x^1=\{X\in M_x \mid \|X\|=1\}$ and $\tilde\varphi_{11}(X)$ and $\tilde\varphi_{12}(X)$ are conjugate to each other with respect to the image. The mappings $\tilde\varphi_{11}$ and $\tilde\varphi_{12}$ may be called *the 1st torsion operators* of M in $\bar M$. We define *the second curvature* $k_2(x)$ of M at x by

$$k_2(x)=\operatorname*{Max}_{X\in S_x^1}\|\tilde\varphi_{11}(X)\|=\operatorname*{Max}_{X\in S_x^1}\|\tilde\varphi_{12}(X)\|$$

and call the dimension of the image of M_x by $\tilde\varphi_{11}$ (or $\tilde\varphi_{12}$) *the 1st torsion index* of M at x and denote it by *T_1-index $_xM$*. It is trival that T_1-index $\leqq 2$ at each point of M. If $k_2(x)=0$, then T_1-index $_xM=0$. Hence if $k_2(x)=0$ at each point $x\in M$, then the geodesic codimension of M is 2. If T_1-index$=1$ at each $x\in M$, then the geodesic codimension is 3, which is the case treated by Ōtsuki [4] for a minimal submanifold of general dimension n.

§2. Minimal surfaces with M-index 2 and T_1-index 2. In this section we will consider minimal surfaces with M-index 2 and T_1-index 2 in $\bar M$. Then we can choose a frame $b\in B_2$ and local function θ_1 on M such that

(2. 1) $$\begin{aligned}&\tilde\varphi_{11}(e_1\cos\theta_1+e_2\sin\theta_1)=k_{21}e_5,\\ &\tilde\varphi_{11}(-e_1\sin\theta_1+e_2\cos\theta_1)=k_{22}e_6,\end{aligned}$$

where $k_{21}=k_2>0$ and $k_{22}=\mathrm{Min}_{X\in S_x^1}\|\tilde\varphi_{11}(X)\|=\mathrm{Min}_{X\in S_x^1}\|\tilde\varphi_{12}(X)\|$. If $k_{21}\neq k_{22}$, then k_{21} and k_{22} are differentiable functions on M. We suppose that they are differentiable functions. Let B_3 be the set of all such frames $b\in B_2$. Then B_3 is a submanifold of B_2. On B_3 we have

(2. 2) $$\begin{aligned}&\omega_{35}=\frac{k_{21}}{\lambda}(\cos\theta_1\omega_1+\sin\theta_1\omega_2),\\ &\omega_{36}=\frac{k_{22}}{\lambda}(-\sin\theta_1\omega_1+\cos\theta_1\omega_2), \qquad \omega_{3\gamma}=0, \qquad 6<\gamma.\end{aligned}$$

Making use of (1. 7) and (2. 2), we have

(2. 3) $$\begin{aligned}&\omega_{45}=\frac{k_{21}}{\mu}(\sin\theta_1\omega_1-\cos\theta_1\omega_2),\\ &\omega_{46}=\frac{k_{22}}{\mu}(\cos\theta_1\omega_1+\sin\theta_1\omega_2), \qquad \omega_{4\gamma}=0, \qquad 6<\gamma.\end{aligned}$$

Then, by means of (1. 3), (1. 7), (2. 2) and (2.3), we can verify the following

LEMMA 3. *Under the above condition, on* B_3 *we have*

$$\{d\log\lambda-i(2\omega_{12}-\sigma\tilde{\omega}_1)\}\wedge(\omega_1+i\omega_2)=0, \tag{2. 4}$$

$$\{d\sigma+i(1-\sigma^2)\tilde{\omega}_1\}\wedge(\omega_1+i\omega_2)=0, \tag{2. 5}$$

$$d\omega_{12}=-(\bar{c}-\lambda^2-\mu^2)\omega_1\wedge\omega_2, \tag{2. 6}$$

$$d\tilde{\omega}=-\left(2\lambda\mu-\frac{k_{21}^2+k_{22}^2}{\lambda\mu}\right)\omega_1\wedge\omega_2, \tag{2. 7}$$

where $\tilde{\omega}_1=\omega_{34}$ *is the connetion form of the vector bundle* $N'=\cup_{x\in M}N'_x$ *over* M.

Furthermore, making use of (2. 2) and (2. 3), we have

LEMMA 4. *On* B_3, *for a fixed* $\gamma>6$, *we have* $\omega_{5\gamma}\equiv\omega_{6\gamma}\equiv0$ (mod ω_1, ω_2) *and* $\omega_{5\gamma}=\omega_{6\gamma}=0$ *or else* $\omega_{5\gamma}\wedge\omega_{6\gamma}\neq0$.

Now, since T_1-index is 2 everywhere on M, the image of M_x by $\tilde{\varphi}_{11}$ (or $\tilde{\varphi}_{12}$) spans 2-dimensional subspace in O_x, which we denote by N''_x. Let $N''=\cup_{x\in M}N''_x$. Then $N''$is a 2-dimensional normal vector bundle over M like N'. We can orthogonally decompose N_x as

$$N_x=N'_x+N''_x+O'_x,\qquad O_x=N''_x+O'_x,\qquad N''_x\perp O'_x. \tag{2. 8}$$

By virtue of Lemma 4, we can define two linear mappings φ_{21} and φ_{22} from M_x into O'_x corresponding to the normal vector field e_5 and e_6 respectively as follows: for any $X\in M_x$

$$\varphi_{21}(X)=\sum_{6<\gamma}\omega_{5\gamma}(X)e_\gamma,\qquad \varphi_{22}(X)=\sum_{6<\gamma}\omega_{6\gamma}(X)e_\gamma.$$

On the other hand, since $\omega_{3\gamma}=\omega_{4\gamma}=0$, we have

$$\omega_{35}\wedge\omega_{5\gamma}+\omega_{36}\wedge\omega_{6\gamma}=0,\qquad 6<\gamma,$$
$$\omega_{45}\wedge\omega_{5\gamma}+\omega_{46}\wedge\omega_{6\gamma}=0.$$

Substituting (2. 2) and (2. 3) into these equations, we may put

$$\begin{aligned} k_{21}\omega_{5\gamma}&=a_\gamma(\cos\theta_1\omega_1+\sin\theta_1\omega_2)+b_\gamma(-\sin\theta_1\omega_1+\cos\theta_1\omega_2),\\ k_{22}\omega_{6\gamma}&=b_\gamma(\cos\theta_1\omega_1+\sin\theta_1\omega_2)-a_\gamma(-\sin\theta_1\omega_1+\cos\theta_1\omega_2),\end{aligned} \tag{2. 9}$$

which imply that the two linear mappings

$$\tilde{\varphi}_{21}=k_{21}\varphi_{21}\qquad\text{and}\qquad\tilde{\varphi}_{22}=k_{22}\varphi_{22}$$

have the same images of S_x^1 in M_x. We call $\tilde{\varphi}_{21}$ and $\tilde{\varphi}_{22}$ *the second torsion operators* of M in $\bar{M}$. We define *the third curvature* $k_3(x)$ of M at x by

(2. 10) $$k_3(x)=\underset{X\in S_x^1}{\mathrm{Max}}\,\|\tilde{\varphi}_{21}(X)\|=\underset{X\in S_x^1}{\mathrm{Max}}\,\|\tilde{\varphi}_{22}(X)\|$$

and call the dimension of the image of M_x by $\tilde{\varphi}_{21}$ (or $\tilde{\varphi}_{22}$) *the second torsion index* of M at x, which we denote by T_2-*index* ${}_xM$. It is trivial that $k_3(x)=0$ if and only if T_2-index ${}_xM=0$. If $k_3(x)=0$ identically on M, then the geodesic codimension of M is 4. If T_2-index is identically 1 on M, the geodesic codimension will be 5. In the next section we shall give a condition for the geodedsic codimension to be 6 when T_2-index is identically 2 on M.

§3. Minimal surfaces with M-index 2, T_1-index 2 and T_2-index 2. In this section we shall consider the minimal surfaces with M-index 2, T_1-index 2 and T_2-index 2 and give a condition that the geodesic codimension is 6. Under the above conditions, we can choose a frame $b\in B_3$ and a local function θ_2 on M such that

(3. 1) $$\tilde{\varphi}_{21}(e_1\cos\theta_2+e_2\sin\theta_2)=k_{31}e_7,$$
$$\tilde{\varphi}_{21}(-e_1\sin\theta_2+e_2\cos\theta_2)=k_{32}e_8,$$

where $k_{31}=k_3>0$ and $k_{32}=\mathrm{Min}_{X\in S_x^1}\|\tilde{\varphi}_{21}(X)\|=\mathrm{Min}_{X\in S_x^1}\|\tilde{\varphi}_{22}(X)\|$. If $k_{31}\neq k_{32}$, then both k_{31} and k_{32} are differentiable functions on M. From now on, we suppose that they are differentiable functions on M. Then, B_4 being the set of all such frames of B_3, B_4 is a smooth submanifold of B_3. On B_4 we have

(3. 2) $$\omega_{57}=\frac{k_{31}}{k_{21}}(\cos\theta_2\omega_1+\sin\theta_2\omega_2),$$
$$\omega_{58}=\frac{k_{31}}{k_{21}}(-\sin\theta_2\omega_1+\cos\theta_2\omega_2),\qquad \omega_{5\gamma}=0,\qquad 8<\gamma.$$

From (3. 2) and (2. 9), we get

(3. 3) $$\omega_{67}=\frac{k_{31}}{k_{22}}(\sin\theta_2\omega_1-\cos\theta_2\omega_2),$$
$$\omega_{68}=\frac{k_{32}}{k_{22}}(\cos\theta_2\omega_1+\sin\theta_2\omega_2),\qquad \omega_{6\gamma}=0,\qquad 8<\gamma.$$

Making use of (3. 2) and (3. 3), we have the following

LEMMA 5. *On B_4, for a fixed $\gamma>8$, we have $\omega_{7\gamma}\equiv\omega_{8\gamma}\equiv 0$ (mod ω_1,ω_2) and $\omega_{7\gamma}=\omega_{8\gamma}=0$ or else $\omega_{7\gamma}\wedge\omega_{8\gamma}\neq 0$.*

Proof. Since $\omega_{5\gamma}=\omega_{6\gamma}=0$, we have

$$\omega_{57}\wedge\omega_{7\gamma}+\omega_{58}\wedge\omega_{8\gamma}=\omega_{67}\wedge\omega_{7\gamma}+\omega_{68}\wedge\omega_{8\gamma}=0.$$

Substituting (3. 2) and (3. 3) into these equations, we get

$$k_{31}(\cos\theta_2\omega_1+\sin\theta_2\omega_2)\wedge\omega_{7\gamma}+k_{32}(-\sin\theta_2\omega_1+\cos\theta_2\omega_2)\wedge\omega_{8\gamma}=0,$$

$$k_{31}(-\sin\theta_2\omega_1+\cos\theta_2\omega_2)\wedge\omega_{7\gamma}-k_{32}(\cos\theta_2\omega_1+\sin\theta_2\omega_2)\wedge\omega_{8\gamma}=0,$$

which imply that we may put

$$(3.4)\qquad \begin{aligned} k_{31}\omega_{7\gamma}&=a'_\gamma(\cos\theta_2\omega_1+\sin\theta_2\omega_2)+b'_\gamma(-\sin\theta_2\omega_1+\cos\theta_2\omega_2),\\ k_{32}\omega_{8\gamma}&=b'_\gamma(\cos\theta_2\omega_1+\sin\theta_2\omega_2)-a'_\gamma(-\sin\theta_2\omega_1+\cos\theta_2\omega_2).\end{aligned}$$

Then we have $k_{31}k_{32}\omega_{7\gamma}\wedge\omega_{8\gamma}=-(a'^2_\gamma+b'^2_\gamma)\omega_1\wedge\omega_2$, which completes the proof.

Now, since T_2-index is identically 2 on M, the image of M_x by $\tilde\varphi_{21}$ (or $\tilde\varphi_{22}$) spans 2-dimensional linear subspace in O'_x, which we denote by N''_x. Then we can decompose N_x as follows:

$$(3.5)\qquad N_x=N'_x+N''_x+N'''_x+O''_x,\quad O_x=N''_x+N'''_x+O''_x,\quad O'_x=N'''_x+O''_x,\quad N'''_x\perp O''_x.$$

By virtue of Lemma 5, we can define two linear mappings φ_{31} and φ_{32} from M_x into O''_x corresponding to the normal vector fields e_7 and e_8 respectively as follows: for any $X\in M_x$

$$\varphi_{31}(X)=\sum_{8<\gamma}\omega_{7\gamma}(X)e_\gamma,\qquad \varphi_{32}(X)=\sum_{8<\gamma}\omega_{8\gamma}(X)e_\gamma.$$

By means of (3.4) we have two linear mappings

$$(3.6)\qquad \tilde\varphi_{31}=k_{31}\varphi_{31}\quad\text{and}\quad \tilde\varphi_{32}=k_{32}\varphi_{32}$$

which have the same images of S^1_x. We call $\tilde\varphi_{31}$ and $\tilde\varphi_{32}$ *the third torsion operators* of M in $\bar M$. We define *the forth curvature* $k_4(x)$ of M at x by

$$(3.7)\qquad k_4(x)=\operatorname*{Max}_{X\in S^1_x}\|\tilde\varphi_{31}(X)\|=\operatorname*{Max}_{X\in S^1_x}\|\tilde\varphi_{32}(X)\|$$

and call the dimension of the image of M_x by $\tilde\varphi_{31}$ (or $\tilde\varphi_{32}$) *the third torsion index* of M at x and denote it by *T_3-index $_xM$.* Then we get a condition that the geodesic codimension is 6 as follows:

THEOREM 1. *Let $M=M^2$ be a minimal surface with M-index 2, T_1-index 2 and T_2-index 2 in $\bar M$. The geodesic codimension of M is 6 if and only if T_3-index $_xM$ $=0$ at each point $x\in M$.*

Proof. The necessity is trival. Let us suppose that T_3-index $_xM=0$ at each point x of M. Then we have $\omega_{7\gamma}=\omega_{8\gamma}=0$, $8<\gamma$. It follows from (3.2), (3.3), (2.2), (2.3) and (1.6) that the geodesic codimension is 6. Q.E.D.

§4. Minimal surfaces with *M*-index 2, T_1-index 2, T_2-index 2 and T_3-index 0.

We shall consider minimal surfaces with M-index 2, T_1-index 2, T_2-index 2 and T_3-index 0. Then, by virtue of Theorem 1, we may put $\nu=6$, i.e., $\bar M=\bar M^8$. Making use of (2.2), (2.3), (3.2) and (3.3), we have the following

LEMMA 6. *Under the above conditions, on B_5 we have the following equations*

$$d\tilde{\omega}_2=d\omega_{56}=\left(\frac{k_{31}^2+k_{32}^2}{k_{21}k_{22}}-k_{21}k_{22}\left(\frac{1}{\lambda^2}+\frac{1}{\mu^2}\right)\right)\omega_1\wedge\omega_2, \tag{4. 1}$$

$$d\tilde{\omega}_3=d\omega_{78}=-k_{31}k_{32}\left(\frac{1}{k_{21}^2}+\frac{1}{k_{22}^2}\right)\omega_1\wedge\omega_2. \tag{4. 2}$$

THEOREM 2. *Let M be a minimal surface with M-index 2, T_1-index 2, T_2-index 2 and T_3-index 0 in a Riemannian manifald of constant curvature $\bar{c}$. If we have*

(α) $\quad\tilde{\omega}_1\neq 0,\quad \sigma=\mu/\lambda=$ *constant on M,*

(β) $\quad$ *M is of constant curvature c,*

(γ) $\quad k_2=$ *constant and* $k_3=$ *constant on M,*

then we have

$$\sigma=1 \quad \textit{or} \quad -1, \tag{4. 3}$$

$$c=\bar{c}-2\lambda^2, \tag{4. 4}$$

$$\tilde{\omega}_1=2\omega_{12},\qquad \tilde{\omega}_2=d\theta_1+3\omega_{12},\qquad \tilde{\omega}_3=d\theta_1+d\theta_2+4\omega_{12}, \tag{4. 5}$$

$$k_2=k_{21}=k_{22} \quad \textit{and} \quad k_3=k_{31}=k_{32}, \tag{4. 6}$$

$$\lambda^2=\frac{9}{2}c,\qquad \frac{k_2^2}{\lambda^2}=\frac{7}{2}c,\qquad \frac{k_3^2}{k_2^2}=2c \quad \textit{and} \quad \bar{c}=10c. \tag{4. 7}$$

Furthermore the Frenet formula of M is

$$\begin{aligned} &dx=R((e_1^*+ie_2^*)(\omega_1^*-i\omega_2^*)),\\ &\bar{D}(e_1^*+ie_2^*)=-i(e_1^*+ie_2^*)\omega_{12}^*+\lambda(e_3^*+ie_4^*)(\omega_1^*-i\omega_2^*),\\ &\bar{D}(e_3^*+ie_4^*)=-2i(e_3^*+ie_4^*)\omega_{12}^*-\lambda(e_1^*+ie_2^*)(\omega_1^*+i\omega_2^*)+\frac{k_2}{\lambda}(e_5^*+ie_6^*)(\omega_1^*-i\omega_2^*),\\ &\bar{D}(e_5^*+ie_6^*)=-3i(e_5^*+ie_6^*)\omega_{12}^*-\frac{k_2}{\lambda}(e_3^*+ie_4^*)(\omega_1^*+i\omega_2^*)+\frac{k_3}{k_2}(e_7^*+ie_8^*)(\omega_1^*-i\omega_2^*),\\ &\bar{D}(e_7^*+ie_8^*)=-4i(e_7^*+ie_8^*)\omega_{12}^*-\frac{k_3}{k_2}(e_5^*+ie_6^*)(\omega_1^*+i\omega_2^*), \end{aligned} \tag{4. 8}$$

where $e_j^*=e_j$ ($j=1, \cdots, 4$), $e_5^*+ie_6^*=e^{i\theta_1}(e_5+ie_6)$, $e_7^*+ie_8^*=e^{i(\theta_1+\theta_2)}(e_7+ie_8)$.

Proof. From (α) and (2. 5) we have $\sigma^2=1$. Hence we have $\lambda^2=\mu^2$, which together with (2. 6) implies that $c=\bar{c}-2\lambda^2$. We may suppose $\sigma=1$. Since we have $\lambda=$const. from (4. 4), (2. 4) implies $\tilde{\omega}_1=2\omega_{12}$. From (2. 7) and (4. 5), we get

$$2c=2\lambda^2-\frac{k_{21}^2+k_{22}^2}{\lambda^2}. \tag{4. 9}$$

Since $k_2=k_{21}$ is constant, so is k_{22}. Thus, since $\lambda=\mu$ is constant, k_{21}=constant and k_{22}=constant, making use of (2. 2) and (2.3), we have

$$(4.10)\qquad \begin{aligned} k_{21}(d\theta_1+\omega_{12}+\tilde{\omega}_1)-k_{22}\tilde{\omega}_2=0,\\ k_{22}(d\theta_1+\omega_{12}+\tilde{\omega}_1)-k_{21}\tilde{\omega}_2=0. \end{aligned}$$

Hence, making use of $\tilde{\omega}_1=2\omega_{12}$ and (4. 1), we have

$$(4.\ 11)\qquad 3k_{21}c=k_{22}\left\{\frac{2k_{21}k_{22}}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{21}k_{22}}\right\},$$

$$(4.\ 12)\qquad 3k_{22}c=k_{21}\left\{\frac{2k_{21}k_{22}}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{21}k_{22}}\right\},$$

which together with $k_3=k_{31}$=constant imply that k_{32} is constant and hence we have

$$(4.\ 13)\qquad c=\frac{1}{3}\left\{\frac{2k_{21}^2}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{22}^2}\right\}=\frac{1}{3}\left\{\frac{2k_{22}^2}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{21}^2}\right\}.$$

From the second equality of (4. 13), we have

$$(4.\ 14)\qquad (k_{21}^2-k_{22}^2)\left\{\frac{2k_{21}k_{22}}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{21}k_{22}}\right\}=0.$$

Now we assume that

$$\frac{2k_{21}k_{22}}{\lambda^2}-\frac{k_{21}^2+k_{32}^2}{k_{21}k_{22}}=0.$$

From (4.11) and (4.12), we have $c=0$. On the other hand, making use of (3. 2) (3. 3), we have

$$(4.\ 15)\qquad \begin{aligned} d\omega_{57}&=\frac{k_{31}}{k_{21}}d\theta_2\wedge(-\sin\theta_2\omega_1+\cos\theta_2\omega_2)+\frac{k_{31}}{k_{21}}\omega_{12}\wedge(\cos\theta_2\omega_2-\sin\theta_2\omega_1)\\ &=\frac{k_{31}}{k_{22}}\tilde{\omega}_2\wedge(\sin\theta_2\omega_1-\cos\theta_2\omega_2)+\frac{k_{32}}{k_{21}}\tilde{\omega}_3\wedge(-\sin\theta_2\omega_1+\cos\theta_2\omega_2),\\ d\omega_{58}&=\frac{k_{32}}{k_{21}}d\theta_2\wedge(-\cos\theta_2\omega_1-\sin\theta_2\omega_2)-\frac{k_{32}}{k_{21}}\omega_{12}\wedge(\sin\theta_2\omega_2+\cos\theta_2\omega_1)\\ &=\frac{k_{32}}{k_{22}}\tilde{\omega}_2\wedge(\cos\theta_2\omega_1+\sin\theta_2\omega_2)-\frac{k_{31}}{k_{21}}\tilde{\omega}_3\wedge(\cos\theta_2\omega_1+\sin\theta_2\omega_2),\\ d\omega_{67}&=\frac{k_{31}}{k_{22}}d\theta_2\wedge(\cos\theta_2\omega_1+\sin\theta_2\omega_2)+\frac{k_{31}}{k_{22}}\omega_{12}\wedge(\cos\theta_2\omega_2+\sin\theta_2\omega_1)\\ &=-\frac{k_{31}}{k_{21}}\tilde{\omega}_2\wedge(\cos\theta_2\omega_1+\sin\theta_2\omega_2)+\frac{k_{32}}{k_{22}}\tilde{\omega}_3\wedge(\cos\theta_2\omega_1+\sin\theta_2\omega_2), \end{aligned}$$

$$d\omega_{68}=\frac{k_{32}}{k_{22}}d\theta_2\wedge(-\sin\theta_2\omega_1+\cos\theta_2\omega_2)+\frac{k_{32}}{k_{22}}\omega_{12}\wedge(\cos\theta_2\omega_2-\sin\theta_2\omega_1)$$
$$=-\frac{k_{32}}{k_{21}}\tilde{\omega}_2\wedge(\cos\theta_2\omega_2-\sin\theta_2\omega_1)-\frac{k_{31}}{k_{22}}\tilde{\omega}_3\wedge(\sin\theta_2\omega_1-\cos\theta_2\omega_2).$$

Hence, if we suppose that $k_{21}=k_{22}$, we have

$$k_{31}(d\theta_2+\omega_{12}+\tilde{\omega}_2)-k_{32}\tilde{\omega}_3=0, \tag{4. 16}$$
$$k_{32}(d\theta_2+\omega_{12}+\tilde{\omega}_2)-k_{31}\tilde{\omega}_3=0.$$

From (4. 10), we have

$$\tilde{\omega}_2=d\theta_1+\omega_{12}+\tilde{\omega}_1=d\theta_1+3\omega_{12}.$$

This together with (4. 2), (4. 16) implies that

$$4k_{31}c=\frac{2k_{31}k_{32}^2}{k_2^2} \quad \text{and} \quad 4k_{32}c=\frac{2k_{31}^2k_{32}}{k_2^2} \tag{4. 17}$$

which contradict $c=0$. Thus it must be $k_{21}\neq k_{22}$ when

$$\frac{2k_{21}k_{22}}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{21}k_{22}}=0.$$

Then, from (4. 10) we get $\tilde{\omega}_2=0$ and $d\theta_1+\omega_{12}+\tilde{\omega}_1=0$. Since $\tilde{\omega}_2=0$, (4. 15) implies

$$k_{31}(d\theta_2+\omega_{12})-k_{32}\tilde{\omega}_3=k_{32}(d\theta_2+\omega_{12})-k_{31}\tilde{\omega}_3=0.$$

Making use of (4. 2), we get

$$c=k_{32}^2\left(\frac{1}{k_{21}^2}+\frac{1}{k_{22}^2}\right)=k_{31}^2\left(\frac{1}{k_{21}^2}+\frac{1}{k_{22}^2}\right),$$

which contradicts $c=0$. Thus it must be

$$\frac{2k_{21}k_{22}}{\lambda^2}-\frac{k_{31}^2+k_{32}^2}{k_{21}k_{22}}\neq 0,$$

and hence $k_2=k_{21}=k_{22}$. Then from (4. 17) we have

$$c=\frac{k_{32}^2}{2k_2^2}=\frac{k_{31}^2}{2k_2^2},$$

which implies $k_{32}=k_{31}=k_3$ and $k_3^2/k_2^2=2c$. From $\tilde{\omega}_1=2\omega_{12}$, (4. 10) and (4. 16), we have $\tilde{\omega}_2=d\theta_1+3\omega_{12}$ and $\tilde{\omega}_3=d\theta_1+d\theta_2+4\omega_{12}$. From (4. 1) and (4. 5), we have $c=(2/3)(k_2^2/\lambda^2-k_3^2/k_2^2)$, which together with $k_3^2/k_2^2=2c$ implies that $k^2/\lambda^2=(7/2)c$. Substituting this equality into (4. 9), we have $2c=2\lambda^2-7c$ and hence $\lambda^2=(9/2)c$. Furthermore, from (4. 4) and $\lambda^2=(9/2)c$ we have $\bar{c}=c+2\lambda^2=10c$.

Now we choose a new frame $b^*=(x, e_1^*, e_2^*, e_3^* \cdots, e_8^*)$ such that $e_j^*=e_j$, $j=1, 2,$

$\cdots, 4,\ e_5^*+ie_6^*=e^{i\theta_1}(e_5+ie_6),\ e_7^*+ie_8^*={}^{i(\theta_1+\theta_2)}(e_7+ie_8)$. Then, with respect to this new frame we have

$$\omega_{35}^*=\frac{k_2}{\lambda}\omega_1^*,\qquad \omega_{36}^*=\frac{k_2}{\lambda}\omega_2^*,\qquad \omega_2^*=3\omega_{12}^*,$$

$$\omega_{45}^*=-\frac{k_2}{\lambda}\omega_2^*,\qquad \omega_{46}^*=\frac{k_2}{\lambda}\omega_1^*,$$

and

$$\omega_{57}^*=\frac{k_3}{k_2}\omega_1^*,\qquad \omega_{58}^*=\frac{k_3}{k_2}\omega_2^*,\qquad \omega_3^*=4\omega_{12}^*,$$

$$\omega_{67}^*=-\frac{k_3}{k_2}\omega_2^*,\qquad \omega_{68}^*=\frac{k_3}{k_2}\omega_1^*.$$

Therefore the Frenet formula of M is (4. 8). Q.E.D.

Now, in order to solve (4. 8), we wish to write (4. 8) in terms of an isothermal coorinate of M. Since we may put $\sigma=1$ from (4. 7), M may be considered locally the unit sphere S^2.

On the other hand, for the unit sphere S^2, considering it the Riemann sphere, as is well known, we have the following formulas:

$$ds^2=\frac{4dzd\bar{z}}{(1+z\bar{z})^2}=\omega_1^2+\omega_2^2, \tag{4. 19}$$

$$\omega_1+i\omega_2=\frac{2dz}{1+z\bar{z}},\qquad \omega_{12}=i\frac{\bar{z}dz-zd\bar{z}}{1+z\bar{z}}, \tag{4. 20}$$

where ω_{12} is the connection form of S^2.

Hence we may put

$$\omega_1^*+i\omega_2^*=e^{-i\varphi}(\omega_1+i\omega_2). \tag{4. 21}$$

Substituting this into

$$d(\omega_1^*+i\omega_2^*)=-i\omega_{12}^*\wedge(\omega_1^*+i\omega_2^*),$$

we have

$$\omega_{12}^*=\omega_{12}+d\varphi. \tag{4. 22}$$

Putting $\xi_1=e^{i\varphi}(e_1^*+ie_2^*)$, $\xi_2=e^{2i\varphi}(e_3^*+ie_4^*)$, $\xi_3=e^{3i\varphi}(e_5^*+ie_6^*)$ and $\xi_4=e^{4i\varphi}(e_7^*+ie_8^*)$, (4. 8) can be written as follows:

$$\begin{cases} dx=\dfrac{1}{h}(\bar{\xi}_1dz+\xi_1d\bar{z}), & h=1+z\bar{z},\\[2mm] \bar{D}\xi_1=\dfrac{1}{h}\xi_1(\bar{z}dz+zd\bar{z})+\dfrac{2\lambda}{h}\xi_2d\bar{z}, & \\[2mm] \bar{D}\xi_2=-\dfrac{2\lambda}{h}\xi_1dz+\dfrac{2}{h}\xi_2(\bar{z}dz+zd\bar{z})+\dfrac{2k_2}{\lambda h}\xi_3d\bar{z}, & \end{cases} \tag{4. 23}$$

$$\left\{\begin{aligned}\bar{D}\xi_3&=-\frac{2k_2}{\lambda h}\xi_2dz+\frac{3}{h}\xi_3(\bar{z}dz+zd\bar{z})+\frac{2k_3}{k_2h}\xi_4d\bar{z},\\ \bar{D}\xi_4&=-\frac{2k_3}{k_2h}\xi_3dz+\frac{4}{h}\xi_4(\bar{z}dz+zd\bar{z}).\end{aligned}\right.$$

§5. **Solution of (4. 23).** In this section, we shall give a solution of (4. 23). As stated in §4, since we put $c=1$, from (4. 7) we have

$$\lambda=\frac{3}{\sqrt{2}},\quad \frac{k_2}{\lambda}=\frac{\sqrt{14}}{2},\quad \frac{k_3}{k_2}=\sqrt{2}\quad\text{and}\quad \bar{c}=10.$$

Hence we may regard $\bar{M}=\bar{M}^8$ as $S^8(1/\sqrt{10})\subset E^9$. Putting

$$e_9=\sqrt{10}x, \tag{5. 1}$$

we have the Frenet formula of M as follows:

$$de_9=\frac{\sqrt{10}}{h}(\bar{\xi}_1dz+\xi_1d\bar{z}), \tag{5. 2}$$

$$\left\{\begin{aligned}d\xi_1&=-\frac{2\sqrt{10}}{h}e_9dz+\frac{1}{h}\xi_1(\bar{z}dz-zd\bar{z})+\frac{3\sqrt{2}}{h}\xi_2d\bar{z},\\ d\bar{\xi}_1&=-\frac{2\sqrt{10}}{h}e_9d\bar{z}-\frac{1}{h}\bar{\xi}_1(\bar{z}dz-zd\bar{z})+\frac{3\sqrt{2}}{h}\bar{\xi}_2dz,\end{aligned}\right. \tag{5. 3}$$

$$\left\{\begin{aligned}d\xi_2&=-\frac{3\sqrt{2}}{h}\xi_1dz+\frac{2}{h}\xi_2(\bar{z}dz-zd\bar{z})+\frac{\sqrt{14}}{h}\xi_3d\bar{z},\\ d\bar{\xi}_2&=-\frac{3\sqrt{2}}{h}\bar{\xi}_1d\bar{z}-\frac{2}{h}\bar{\xi}_2(\bar{z}dz-zd\bar{z})+\frac{\sqrt{14}}{h}\bar{\xi}_3dz,\end{aligned}\right. \tag{5. 4}$$

$$\left\{\begin{aligned}d\xi_3&=-\frac{\sqrt{14}}{h}\xi_2dz+\frac{3}{h}\xi_3(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{2}}{h}\xi_4d\bar{z},\\ d\bar{\xi}_3&=-\frac{\sqrt{14}}{h}\bar{\xi}_2d\bar{z}-\frac{3}{h}\bar{\xi}_3(\bar{z}dz-zd\bar{z})+\frac{2\sqrt{2}}{h}\bar{\xi}_4dz,\end{aligned}\right. \tag{5. 5}$$

$$\left\{\begin{aligned}d\xi_4&=-\frac{2\sqrt{2}}{h}\xi_3dz+\frac{4}{h}\xi_4(\bar{z}dz-zd\bar{z}),\\ d\bar{\xi}_4&=-\frac{2\sqrt{2}}{h}\bar{\xi}_3d\bar{z}-\frac{4}{h}\bar{\xi}_4(\bar{z}dz-zd\bar{z}).\end{aligned}\right. \tag{5. 6}$$

From the first equation of (5. 6), we have

$$\xi_4=\frac{1}{h^4}F(z), \tag{5. 7}$$

where $F(z)$ is a complex analytic vector field. Substituting (5. 7) into (5. 6), we have

$$\xi_3=\frac{2\sqrt{2}\,\bar{z}}{h^4}F(z)-\frac{1}{2\sqrt{2}\,h^3}F'(z). \tag{5.8}$$

Making use of (5. 7) and (5. 8), we can verify

$$\frac{\partial\xi_3}{\partial\bar{z}}=\frac{2\sqrt{2}}{h}\xi_4-\frac{3z}{h}\xi_3.$$

From the 1st equation of (5. 5), we get

$$\xi_2=\frac{2\sqrt{7}\,\bar{z}^2}{h^4}F(z)-\frac{\sqrt{7}\,\bar{z}}{2h^3}F'(z)+\frac{1}{4\sqrt{7}\,h^2}F''(z). \tag{5.9}$$

Making use of (5. 8) and (5. 9), we can verify that

$$\frac{\partial\xi_2}{\partial\bar{z}}=\frac{\sqrt{14}}{h}\xi_3-\frac{2z}{h}\xi_2.$$

From the 1st equation of (5. 4), we get

$$\xi_1=\frac{2\sqrt{14}\bar{z}^3}{h^4}F(z)-\frac{3\sqrt{14}\bar{z}^2}{4h^3}F'(z)+\frac{3\bar{z}}{2\sqrt{14}h^2}F''(z)-\frac{1}{12\sqrt{14}h}F'''(z) \tag{5.10}$$

Making use of (5. 9) and (5. 10), we can verify

$$\frac{\partial\xi_1}{\partial\bar{z}}=\frac{3\sqrt{2}}{h}\xi_2-\frac{z}{h}\xi_1.$$

From the 1st equation of (5. 3), we have

$$e_9=\frac{\sqrt{35}\bar{z}^4}{h^4}F(z)-\frac{\sqrt{35}\bar{z}^3}{2h^3}F'(z)+\frac{3\sqrt{35}\bar{z}^2}{28h^2}F''(z)-\frac{\sqrt{35}\bar{z}}{84h}F'''(z)+\frac{\sqrt{35}}{1680}F''''(z). \tag{5.11}$$

From (5. 10) and (5. 11), we can prove that

$$\frac{\partial e_9}{\partial\bar{z}}=\frac{\sqrt{10}}{h}\xi_1.$$

Thus, if we choose $F(z)$ so that e_9 is real, then e_9, ξ_1, ξ_2, ξ_3, ξ_4 given by (5. 11), $\cdots$, (5. 7) respectively satisfy the equations (5. 2), $\cdots$, (5. 6).

From now on, we will search for $F(z)$ such that e_9 is real. Since $h=1+z\bar{z}$ is real, e_9 is real if and only if

$$\begin{aligned}\frac{1680}{\sqrt{35}}h^4e_9=&1680\bar{z}^4F(z)-840\bar{z}^3(1+z\bar{z})F'(z)+180\bar{z}^2(1+z\bar{z})^2F''(z)\\&-20\bar{z}(1+z\bar{z})^3F'''(z)+(1+z\bar{z})^4F''''(z)=:G(z,\bar{z})\end{aligned} \tag{5.12}$$

is real. Then $G(z, \bar{z})$ is a polynomial of degree at most 4 in z as well as in $\bar{z}$ since $G(z, \bar{z})=\overline{G(z, \bar{z})}$. We have easily from (5. 12)

$$G(z, \bar{z})=\{1680F(z)-840zF'(z)+180z^2F''(z)-20z^3F''''(z)+z^4F''''(z)\}\bar{z}^4$$
$$-\{840F'(z)-360zF''(z)+60z^2F'''(z)-4z^3F''''(z)\}\bar{z}^3$$
$$+\{180F''(z)-60zF'''(z)+6z^2F''''(z)\}\bar{z}^2$$
$$-\{20F'''(z)-4zF''''(z)\}\bar{z}+F''''(z),$$

which implies that $F''''(z)$ is a polynomial in z, because $G(z, \bar{z})$ is a vector valued polynomial in z and $\bar{z}$. Hence we may put

(5. 13) $$F(z)=A_0+A_1z+\cdots+A_mz^m,$$

where $A_0, A_1, \cdots, A_m$ are constant vectors in C^5. Then we have

$$G(z, \bar{z})=\{1680A_0+840A_1z+\cdots+(m-5)(m-6)(m-7)(m-8)A_mz^m\}\bar{z}^4$$
$$-\{840A_1+960A_2z+\cdots+4m(6-m)(m-7)(m-8)A_mz^{m-1}\}\bar{z}^3$$
$$+\{360A_2+720A_3z+\cdots+6m(m-1)(m-7)(m-8)A_mz^{m-2}\}\bar{z}_2$$
$$-\{120A_3+384A_4z+\cdots+4m(1-m)(m-2)(m-8)A_mz^{m-3}\}\bar{z}$$
$$+\ 24A_4+120A_5z+\cdots+\ m(m-1)(m-2)(m-3)A_mz^{m-4}.$$

Since $G(z, \bar{z})$ is a polynomial in $\bar{z}$ and z of degree at most 4, the polynomial in the first { } lacks the terms of degree 5, 6, 7, 8 in z. Hence we may suppose $m=8$. Then we have

(5. 14)
$$\begin{aligned}G(z, \bar{z})=&(1680A_0+840A_1z+360A_2z^2+120A_3z^3+24A_4z^4)\bar{z}^4\\&-(840A_1+960A_2z+720A_3z^2+384A_4z^3+120A_5z^4)\bar{z}^3\\&+(360A_2+720A_3z+864A_4z^2+720A_5z^3+360A_6z^4)\bar{z}^2\\&-(120A_3+384A_4z+720A_5z^2+960A_6z^3+840A_7z^4)\bar{z}\\&+\ 24A_4+120A_5z+360A_6z^2+840A_7z^3+1680A_8z^4,\end{aligned}$$

which implies that $G(z, \bar{z})=\overline{G(z, \bar{z})}$ is satisfied if and only if

(5. 15) $$A_4=\bar{A}_4,\quad A_5=-\bar{A}_3,\quad A_6=\bar{A}_2,\quad A_7=-\bar{A}_1,\quad A_8=\bar{A}_0.$$

Making use of (5. 12), (5. 14) and (5. 15), we have

(5. 16)
$$\begin{aligned}e_9=\frac{\sqrt{35}}{70h^4}\{&A_4(1-16z\bar{z}+36z^2\bar{z}^2-16z^3\bar{z}^3+z^4\bar{z}^4)\\&-5(\bar{A}_3z+A_3\bar{z})(1-6z\bar{z}+6z^2\bar{z}^2-z^3\bar{z}^3)\\&+5(\bar{A}_2z^2+A_2\bar{z}^2)(3-8z\bar{z}+3z^2\bar{z}^2)\\&-35(\bar{A}_1z^3+A_1\bar{z}^3)(1-z\bar{z})+70(\bar{A}_0z^4+A_0\bar{z}^4)\}.\end{aligned}$$

From (5. 7), (5. 8), (5. 9) and (5. 10), we have

(5.17) $$\xi_4=\frac{1}{h^4}\{A_4z^4+(A_3z^3-\bar{A}_3z^5)+(A_2z^2+\bar{A}_2z^6)+(A_1z-\bar{A}_1z^7)+A_0+\bar{A}_0z^8\}$$

(5.18) $$\xi_3=\frac{1}{2\sqrt{2}h^4}\{4\bar{z}F(z)-(1+z\bar{z})F'(z)\}$$

(5.19) $$\xi_2=\frac{1}{4\sqrt{7}h^4}\{56\bar{z}^2F(z)-14\bar{z}(1+z\bar{z})F'(z)+(1+z\bar{z})^2F''(z)\}$$

(5.20) $$\xi_1=\frac{1}{12\sqrt{14}h^4}\{336\bar{z}^3F(z)-126\bar{z}^2(1+z\bar{z})F'(z)+18\bar{z}(1+z\bar{z})^2F''(z)-(1+z\bar{z})^3F'''(z)\}.$$

Now we must find the conditions that ξ_1, ξ_2, ξ_3, ξ_4 and e_9 form an orthonormal frame. In the following calculation, " $\equiv$ " denotes the equality modulus the quantities:

$$e_9\cdot\xi_j,\quad e_9\cdot\bar{\xi}_j,\quad \xi_j\cdot\xi_j,\quad \bar{\xi}_j\cdot\bar{\xi}_j,\quad \xi_j\cdot\xi_k,\quad \xi_j\cdot\bar{\xi}_k,\quad \bar{\xi}_j\cdot\bar{\xi}_k,$$

where j, $k=1, 2, \cdots, 4$ and $j\neq k$. Then we have easily the relations:

$$d(e_9\cdot e_9)\equiv d(e_9\cdot\xi_l)\equiv d(\xi_j\cdot\xi_k)\equiv d(\xi_j\cdot\bar{\xi}_j)\equiv d(\xi_1\cdot\bar{\xi}_3)\equiv d(\xi_1\cdot\bar{\xi}_4)\equiv d(\xi_2\cdot\bar{\xi}_4)\equiv 0,$$
$$l=2, 3, 4,\quad j, k=1, \cdots, 4,$$

$$d(e_9\cdot\xi_1)\equiv\frac{\sqrt{10}}{h}(\xi_1\cdot\bar{\xi}_1-2e_9\cdot e_9)dz$$

$$d(\xi_1\cdot\bar{\xi}_2)\equiv\frac{3\sqrt{2}}{h}(\xi_2\cdot\bar{\xi}_2-\xi_1\cdot\bar{\xi}_1)d\bar{z}\equiv\overline{d(\bar{\xi}_1\cdot\xi_2)}$$

$$d(\xi_2\cdot\bar{\xi}_3)\equiv\frac{\sqrt{14}}{h}(\xi_3\cdot\bar{\xi}_3-\xi_2\cdot\bar{\xi}_2)d\bar{z}\equiv\overline{d(\bar{\xi}_2\cdot\xi_3)}$$

$$d(\xi_3\cdot\bar{\xi}_4)\equiv\frac{2\sqrt{2}}{h}(\xi_4\cdot\bar{\xi}_4-\xi_3\cdot\bar{\xi}_3)d\bar{z}\equiv\overline{d(\bar{\xi}_3\cdot\xi_4)},$$

from which we see that $e_9\cdot e_9$, $\xi_j\cdot\bar{\xi}_j$ $(j=1, \cdots, 4)$ are constants. Hence, if we can choose A_0, A_1, A_2, A_3, A_4 such that $e_9\cdot e_9=1$ and $\xi_j\cdot\bar{\xi}_j=2$ $(j=1, \cdots, 4)$ at $z=0$, then the above quantities are all zero. It is sufficient to give conditions that e_9, ξ_1, ξ_2, ξ_3 and ξ_4 form an orthonormal frame at $z=0$. From (5.16), (5.17), (5.18), (5.19) and (5.20), at $z=0$, we have

$$e_9=\frac{\sqrt{35}}{70}A_4,\quad \xi_1=-\frac{A_3}{2\sqrt{14}},\quad \xi_2=\frac{A_2}{2\sqrt{7}},\quad \xi_3=-\frac{A_1}{2\sqrt{2}},\quad \xi_4=A_0.$$

Thus we have the conditions for A_0, A_1, $\cdots$, A_4:

(5.21) $$\begin{cases} A_4=\bar{A}_4,\quad A_j\cdot A_j=0 \quad (j=0, 1, 2, 3), \\ A_4\cdot A_4=140,\quad A_3\cdot\bar{A}_3=112,\quad A_2\cdot\bar{A}_2=56\quad A_1\cdot\bar{A}_1=16,\quad A_0\cdot\bar{A}_0=2, \\ A_4\cdot A_j=0,\quad A_3\cdot\bar{A}_l=0,\quad A_3\cdot\bar{A}_l=0\quad (j=0, 1, \cdots, 3)\quad (l=0, 1, 2), \end{cases}$$

$$A_2\cdot A_1=A_2\cdot\bar{A}_1=A_2\cdot A_0=A_2\cdot\bar{A}_0=A_1\cdot A_0=A_1\cdot\bar{A}_0=0.$$

Now we give the equation of M using the above result. First of all, we choose five constant vectors A_0, A_1, A_2, A_3, A_4 in C^5 which satisfy the conditions (5. 21) and determine e_9 given by (5. 16) which is a real unit vector field in $E^{10}\cong C^5$. Since we may consider $x=(1/\sqrt{10})e_9$, we have a general solution of (4. 23) as follows:

$$\begin{aligned}x=\frac{1}{10\sqrt{14}(1+z\bar{z})^4}\{&(1-16z\bar{z}+36z^2\bar{z}^2-16z^3\bar{z}^3+z^4\bar{z}^4)A_4\\ &-5(1-6z\bar{z}+6z^2\bar{z}^2-z^3\bar{z}^3)(\bar{A}_3z+A_3\bar{z})\\ &+5(3-8z\bar{z}+3z^2\bar{z}^2)(\bar{A}_2z+A_2\bar{z})\\ &-35(1-z\bar{z})(\bar{A}_1z^3+A_1\bar{z}^3)+70(\bar{A}_0z^4+A_0\bar{z}^4)\}.\end{aligned}\tag{5. 22}$$

If we put

$$A_4=2\sqrt{35}\frac{\partial}{\partial x_9},\quad A_3=-2\sqrt{14}\left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right),\quad A_2=2\sqrt{7}\left(\frac{\partial}{\partial x_3}+i\frac{\partial}{\partial x_4}\right)$$

$$A_1=-2\sqrt{2}\left(\frac{\partial}{\partial x_5}+i\frac{\partial}{\partial x_6}\right)\quad\text{and}\quad A_0=\frac{\partial}{\partial x_7}+i\frac{\partial}{\partial x_8},$$

we can write (5. 22) in the cannonical coordinates $x_1, x_2, \cdots, x_9$ as follows:

$$\begin{aligned}x_1&=\frac{1-6z\bar{z}+6\bar{z}^2z^2-z^3\bar{z}^3}{(1+z\bar{z})^4}(z+\bar{z}),\\ x_2&=-i\frac{1-6z\bar{z}+6z^2\bar{z}^2-z^3\bar{z}^3}{(1+z\bar{z})^4}(z-\bar{z}),\\ x_3&=\frac{3-8z\bar{z}+3z^2\bar{z}^2}{\sqrt{2}(1+z\bar{z})^4}(z^2+\bar{z}^2),\\ x_4&=-i\frac{3-8z\bar{z}+3z^2\bar{z}^2}{\sqrt{2}(1+z\bar{z})^4}(z^2-\bar{z}^2),\\ x_5&=\frac{\sqrt{7}(1-z\bar{z})}{(1+z\bar{z})^4}(z^3+\bar{z}^3),\\ x_6&=-i\frac{\sqrt{7}(1-z\bar{z})}{(1+z\bar{z})^4}(z^3-\bar{z}^3),\\ x_7&=\frac{\sqrt{7}}{2(1+z\bar{z})^4}(z^4+\bar{z}^4),\\ x_8&=-\frac{\sqrt{7}i}{2(1+z\bar{z})^4}(z^4-\bar{z}^4),\\ x_9&=\frac{1-16zz+36z^2\bar{z}^2-16z^3\bar{z}^3+z^4\bar{z}^4}{\sqrt{10}(1+z\bar{z})^4}\end{aligned}\tag{5. 23}$$

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