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DEFICIENCIES OF AN ENTIRE ALGEBROID FUNCTION, III

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1. It is known that in the theory of algebroid functions a remarkable phenomenon occurs, while any corresponding fact never occurs in the one-valued theory. The phenomenon is characterized by the fact that certain conditions only on the deficiencies imply the appearance of Picard's exceptional values. It is curious that the discovery of this phenomenon has been done quite recently although the theory of algebroid functions has its long history. A typical one of this phenomenon is the following:

Let $f(z)$ be a two-valued transcendental entire algebroid function and let a_1, a_2, a_3 be three different finite numbers satisfying $\sum_{j=1}^{3}\delta(a_j, f)>2$. Then at least one of $\{a_j\}$ is a Picard exceptional value of f.

In [3], [4] several extensions of the above fact were given and Toda [5], [6] has also given some substantial extensions. However a general conjecture still remains unsettled.

In [4] we proved the following result:

Let $f(z)$ be a four-valued transcendental entire algrbroid function defined by an irreducible equation

$$F(z, f)\equiv f^4+A_3f^3+A_2f^2+A_1f+A_0=0,$$

where A_j are entire. Let $\{a_j\}_{j=1}^{6}$ be six different finite numbers satisfying $\sum_{j=1}^{6}\delta(a_j, f)>5$. Further assume that any two of $\{F(z, a_j)\}$ are not proportional. Then at least two of $\{a_j\}$ are Picard exceptional values of f.

It is still hoped to give a condition, involving only five deficiencies, which implies the existence of at least one Picard exceptional value of f. In this tendency we shall prove the following:

THEOREM 1. *Let $f(z)$ be the same as in the above. Let a_j, $j=1, \cdots, 5$ be five different finite numbers satisfying*

$$\sum_{j=1}^{5}\delta(a_j f)>5-\delta(a_l, f)$$

for every l ($l=1, 2, 3, 4, 5$). Further assume that any three of $\{F(z, a_j)\}$ are not linearly dependent. Then there is at least one Picard exceptional value of f among $\{a_j\}$.

In our opinion Theorem 1 is still an unsatisfactory one, since the condition we want to give is $\sum_{j=1}^{5}\delta(a_j, f)>4$. Correspondingly we can give the following:

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THEOREM 2. *Let $f(z)$ be the same as in Theorem 1. Let $\{a_j\}_{j=1}^6$ be six different finite numbers satisfying $\sum_{j=1}^5 \delta(a_j, f) > 5 - \delta(a_l, f)$ for every l $(l=1, 2, 3, 4, 5)$ and $\sum_{j=1}^4 \delta(a_j, f) + \delta(a_6, f) > 4$. Further assume that any two of $\{F(z, a_j)\}$ are not proportional. Then there are at least two Picard exceptional values of f among $\{a_j\}$.*

The counter-example listed in [4] shows that non-proportionality condition cannot be omitted in Theorem 2.[1] Further a slight modification of this example shows that we cannot replace the appearance of two Picard exceptional values by that of two lacunary values.

Our main tool is Nevanlinna's method [1], [2] of proof for the impossibility of Borel's identity. Almost all parts in this paper do use it without any explicite statement. [3] and [4] indicate how to use Nevanlinna's method.

2. *Proof of Theorem* 1. We put $g_j(z) = F(z, a_j)$ and assume that all g_j are transcendental. By the assumption

$$\sum_{j=1}^{5} \delta(a_j, f) > 4$$

and

$$\sum_{j=1}^{5} \alpha_j g_j = 1, \qquad \alpha_j = 1 \Big/ \prod_{k=1, \neq j}^{5} (a_j - a_k).$$

Then we have the linear dependency of $\{g_j\}$, that is

$$\sum_{j=1}^{5} \alpha'_j g_j = 0.$$

Here $\alpha'_j \neq 0$ for at least four indices. Hence we may assume that $\alpha'_1\alpha'_3\alpha'_4\alpha'_5 \neq 0$ and $\alpha'_5 = \alpha_5$. We now divide into several cases:

1) $\alpha_1 \neq \alpha'_1, \quad \alpha_2 \neq \alpha'_2, \quad \alpha_3 \neq \alpha'_3, \quad \alpha_4 \neq \alpha'_4;$

2) $\alpha_1 \neq \alpha'_1, \quad \alpha_2 \neq \alpha'_2, \quad \alpha_3 \neq \alpha'_3, \quad \alpha_4 = \alpha'_4;$

(i) $\alpha'_2 = 0, \quad \alpha_1\alpha'_3 = \alpha'_1\alpha_3,$

(ii) $\alpha'_2 = 0, \quad \alpha_1\alpha'_3 \neq \alpha'_1\alpha_3,$

(iii) $\alpha'_2 \neq 0, \quad \alpha_1\alpha'_3 = \alpha'_1\alpha_3, \quad \alpha_2\alpha'_3 = \alpha'_2\alpha_3,$

(iv) $\alpha'_2 \neq 0, \quad \alpha_1\alpha'_3 \neq \alpha'_1\alpha_3, \quad \alpha_2\alpha'_3 = \alpha'_2\alpha_3,$

(v) $\alpha'_2 \neq 0, \quad \alpha_1\alpha'_3 \neq \alpha'_1\alpha_3, \quad \alpha_2\alpha'_3 \neq \alpha'_2\alpha_3.$

3) $\alpha_1 \neq \alpha'_1, \quad \alpha_2 \neq \alpha'_2, \quad \alpha_3 = \alpha'_3, \quad \alpha_4 = \alpha'_4;$

(i) $\alpha'_2 = 0,$

(ii) $\alpha'_2 \neq 0, \quad \alpha_1\alpha'_2 = \alpha_2\alpha'_1,$

(iii) $\alpha'_2 \neq 0, \quad \alpha_1\alpha'_2 \neq \alpha_2\alpha'_1.$

The case 1), 2) (ii), 2) (iv), 2) (v), 3) (i) and 3) (iii) lead us to an equation of the

1) In [4] there are two misprints. In page 186 A_1 should be replaced by $A_1 = (1/6)(12 - 3g_1 + 6g_2 - 2g_3 - g_4)$. In page 187 $F(z, 3) = g_3$ should be read as $F(z, 3) = g_6$.

following type

$$\text{(a)} \qquad \lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3+\lambda_4 g_4=1, \qquad \lambda_1\lambda_2\lambda_3\lambda_4 \neq 0.$$

The case 2) (i) leads us to a simultaneous equation of the following type

$$\text{(b)} \qquad \begin{cases} \lambda_1 g_1+\lambda_2 g_2+\lambda_3 g_3=1, \\ \lambda_2 g_2+\lambda_4 g_4+\lambda_5 g_5=1, \end{cases} \qquad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \neq 0.$$

The cases 2) (iii) and 3) (ii) lead us to a simultaneous equation of the following type

$$\text{(c)} \qquad \begin{cases} \lambda_1 g_1+\lambda_2 g_2=1, \\ \lambda_3 g_3+\lambda_4 g_4+\lambda_5 g_5=1, \end{cases} \qquad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \neq 0.$$

The case (a). In this case we have

$$\lambda_1' g_1+\lambda_2' g_2+\lambda_3' g_3+\lambda_4' g_4=0.$$

Here $\lambda_1'\lambda_2'\lambda_3'\lambda_4' \neq 0$ and $\lambda_4'=\lambda_4$. If $\lambda_1 \neq \lambda_1'$, $\lambda_2 \neq \lambda_2'$, $\lambda_3 \neq \lambda_3'$ or $\lambda_1 \neq \lambda_1'$, $\lambda_2 \neq \lambda_2'$, $\lambda_3=\lambda_3'$, $\lambda_1'\lambda_2 \neq \lambda_1\lambda_2'$, then we have

$$\text{(a}'\text{)} \qquad \mu_1 g_1+\mu_2 g_2+\mu_3 g_3=1, \qquad \mu_1\mu_2\mu_3 \neq 0,$$

for example. If $\lambda_1 \neq \lambda_1'$, $\lambda_2 \neq \lambda_2'$, $\lambda_3=\lambda_3'$, $\lambda_1'\lambda_2=\lambda_1\lambda_2'$, then we have

$$\text{(b}'\text{)} \qquad \begin{cases} \mu_1 g_1+\mu_2 g_2=1, \\ \mu_3 g_3+\mu_4 g_4=1, \end{cases} \qquad \mu_1\mu_2\mu_3\mu_4 \neq 0.$$

If (a$'$) occurs, then

$$\mu_1' g_1+\mu_2' g_2+\mu_3' g_3=0,$$

which is a contradiction. If (b$'$) occurs and g_1, g_2 and g_3, g_4 are linearly independent, then we have

$$\delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)+\delta(a_4, f) \leqq 3,$$

which is a contradiction.

Case (c). If g_1, g_2 and g_3, g_4, g_5 are linearly independent, then we have

$$\sum_{j=1}^{5} \delta(a_j, f) \leqq 4,$$

which is a contradiction.

Case (b). Assume that g_1, g_2, g_3 and g_2, g_4, g_5 are linearly independent. Let

$$m_{1,2}(r)=\frac{1}{2\pi}\int_0^{2\pi} \max(\log^+ |g_1|, \log^+ |g_2|)\, d\theta,$$

$$m_{2,4}(r)=\frac{1}{2\pi}\int_0^{2\pi} \max(\log^+ |g_2|, \log^+ |g_4|)\, d\theta$$

and

$$m(r)=\frac{1}{2\pi}\int_0^{2\pi}\max(\log^+|g_1|, \log^+|g_2|, \log^+|g_4|)\,d\theta.$$

Then

$$m(r)\leqq m_{1,2}(r)+m_{2,4}(r).$$

By the linear independency of g_1, g_2, g_3,

$$g_1=\frac{\varDelta_1}{\lambda_1\varDelta},\qquad g_2=\frac{\varDelta_2}{\lambda_2\varDelta},\qquad \varDelta\not\equiv 0,$$

where $\varDelta$ is the Wronskian divided by g_1, g_2, g_3 and

$$\varDelta_1=\frac{1}{g_2g_3}(g_2'g_3''-g_2''g_3'),\qquad \varDelta_2=\frac{1}{g_1g_3}(g_1''g_3'-g_1'g_3'')$$

Now we have

$$\begin{aligned}m_{1,2}(r)&\leqq m(r, \varDelta_1)+m(r, \varDelta_2)+m\left(r, \frac{1}{\varDelta}\right)+O(1)\\&\leqq m(r, \varDelta_1)+m(r, \varDelta_2)+m(r, \varDelta)+N(r;\infty, \varDelta)+O(1)\\&\leqq N(r;0, g_1)+N(r;0, g_2)+N(r;0, g_3)+o\left(\sum_{l=1}^{3}m(r, g_l)\right)\end{aligned}$$

with a negligible exceptional set. Similarly we have

$$\begin{aligned}m_{2,4}(r)&\leqq N(r;0, g_2)+N(r;0, g_4)+N(r;0, g_5)\\&\quad+o(m(r, g_2)+m(r, g_4)+m(r, g_5)).\end{aligned}$$

Hence

$$m(r)\leqq\sum_{j=1}^{5}N(r;0, g_j)+N(r;0, g_2)+o\left(\sum_{j=1}^{5}m(r, g_j)\right).$$

Let $T(r)$ be the characteristic function of f. Further let $4\mu(r, A)$ be

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\max(|A_3|, |A_2|, |A_1|, |A_0|)\,d\theta.$$

Then it is known that $|T(r)-\mu(r, A)|=O(1)$. It is very easy to prove that

$$|m(r)-4\mu(r, A)|=O(1).$$

Further

$$\sum_{j=1}^{5}m(r, g_j)=O(m(r)).$$

Hence we have

$$\sum_{j=1}^{5}\delta(a_j, f)+\delta(a_2, f)\leqq 5,$$

which is a contradiction. Therefore g_1, g_2, g_3 or g_2, g_4, g_5 are linearly dependent, which contradicts our assumption.

Therefore we can conclude that one of $\{g_j\}$ is a polynomial, and hence correspondingly one of $\{a_j\}$ is a Picard exceptional value.

3. We shall give here two supplementary results.

Suppose that any four of $\{F(z, a_j)\}$ are not linearly dependent. Then

$$\sum_{j=1}^{5} \delta(a_j, f) \leqq 4.$$

Indeed suppose that

$$\sum_{j=1}^{5} \delta(a_j, f) > 4.$$

Then we have $\alpha_1' g_1 + \alpha_2' g_2 + \alpha_3' g_3 + \alpha_4' g + \alpha_5' g_5 = 0$, $\alpha_1'\alpha_2'\alpha_3'\alpha_4'\alpha_5' \neq 0$. We may put $\alpha_5' = \alpha_5$. We have only two possiblities (a) and (c). Both cases imply a contradiction similarly. Hence one of $\{g_j\}$, say g_5, must be a polynomial. Then we have

$$\alpha_1 g_1 + a_2 g_2 + \alpha_3 g_3 + \alpha_4 g_4 = 1 - \alpha_5 g_5.$$

This implies that

$$\beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3 + \beta_4 g_4 = 0.$$

Here at least two coefficients are not zero. This contradicts the assumption.

In our Theorem 1 a little more precise discussion leads to the following result: Under the same assumption there is at least one lacunary value among $\{a_j\}$. We omit its proof.

4. *Proof of Theorem* 2. As in our result in [4] we have the following five possibilities, starting from $\sum_{j=1}^{5} \alpha_j g_j = 1$, by simply $\sum_{j=1}^{5} \delta(a_j, f) > 4$ and by the transcendency assumption of all the $g_j(z)$, $j = 1, \cdots, 5$:

(A) $$\lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4 = 1, \qquad \lambda_1\lambda_2\lambda_3\lambda_4 \neq 0,$$

(B) $$\begin{cases} \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 = 1, \\ \lambda_4 g_4 + \lambda_5 g_5 = 1, \end{cases} \qquad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \neq 0,$$

(C) $$\begin{cases} \alpha_1 g_1 + \alpha_2 g_2 = 1, \\ \alpha_1 g_1 + \alpha_4 g_4 + \alpha_5 g_5 = 0, \end{cases}$$

(D) $$\begin{cases} \alpha_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 = 1, \\ \alpha_1 g_1 + \lambda_4 g_4 + \lambda_5 g_5 = 1, \end{cases} \qquad \lambda_2\lambda_3\lambda_4\lambda_5 \neq 0,$$

(E) $$\begin{cases} \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 = 1, \\ \alpha_4 g_4 + \alpha_5 g_5 = 0. \end{cases}$$

Case (A). In this case we have three possibilities:

1) $$\mu_1 g_1+\mu_2 g_2+\mu_3 g_3=1, \qquad \mu_1\mu_2\mu_3 \neq 0,$$

2) $$\begin{cases} \mu_1 g_1+\mu_2 g_2=1, \\ \mu_3 g_3+\mu_4 g_4=1, \end{cases} \qquad \mu_1\mu_2\mu_3\mu_4 \neq 0,$$

3) $$\begin{cases} \mu_1 g_1+\mu_2 g_2=1, \\ \mu_3 g_3+\mu_4 g_4=0, \end{cases} \qquad \mu_1\mu_2\mu_3\mu_4 \neq 0.$$

1) implies $\nu_1 g_1+\nu_2 g_2=1$ and hence $\nu_1' g_1+\nu_2' g_2=0$, which is absurd. 2) implies $\mu_1' g_1+\mu_2' g_2=0$ or $\mu_3' g_3+\mu_4' g_4=0$, which is again absurd. 3) is a contradiction.

Case (B). In this case we have

$$\mu_1 g_1+\mu_2 g_2=1, \qquad \lambda_4 g_4+\lambda_5 g_5=1.$$

Contradiction.

Case (D). This case implies similarly as in Theorem 1 that

$$\mu_1 g_1+\mu_2 g_2=1,\ \ \alpha_1 g_1+\lambda_4 g_4+\lambda_5 g_5=1 \ \text{ or } \ \alpha_1 g_1+\lambda_2 g_2+\lambda_3 g_3=1,\ \ \mu_1 g_1+\mu_4 g_4=1.$$

Of course we must use

$$\sum_{j=1}^{5} \delta(a_j, f)>5-\delta(a_l, f), \qquad l=1, 2, 3, 4, 5.$$

Similarly we have a contradiction.

Case (E). Contradiction trivially.

Case (C). We have $\sum_{j=1}^{4} \beta_j g_j+\beta_6 g_6=1$. In this case we have to discuss two cases:

$$1)\ \begin{cases} \alpha_1 g_1+\alpha_2 g_2=1, \\ \alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=0, \end{cases} \qquad 2)\ \begin{cases} \alpha_1 g_1+\alpha_5 g_5=1, \\ \alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4=0. \end{cases}$$

Here assume that g_6 is also transcendental. Case 1) has discussed already in [4]. Case 2) implies

$$\beta_1\alpha_4 g_1+(\alpha_4\beta_2-\alpha_2\beta_4)g_2+(\alpha_4\beta_3-\alpha_3\beta_4)g_3+\alpha_4\beta_6 g_6=\alpha_4.$$

Here $\alpha_4\beta_2 \neq \alpha_2\beta_4$ and $\alpha_4\beta_3 \neq \alpha_3\beta_4$. Hence this reduces to (A) with $\sum_{j=1}^{4} \delta(a_j, f)+\delta(a_6, f)>4$. Thus we have a contradiction.

All the cases imply a contradiction. Hence one of $\{g_j\}_{j=1}^{6}$ is a polynomial. Now we may consider three cases:

a) g_1 is a polynomial, b) g_5 is a polynomial, c) g_6 is a polynomial.

a) Assume that the others are transcendental. Then

$$\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=1-\alpha_1 g_1,$$

$$\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_6 g_6=1-\beta_1 g_1.$$

Since one of $1-\alpha_1 g_1$, $1-\beta_1 g_1$, does not vanish identically, we have an equation of type (A). This leads to a contradiction. Hence one of $\{g_j\}_{j=2}^{6}$ is a polynomial. This is the desired result.

b) Assume that the others are transcendental. Then

$$\begin{cases} \alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4=1-\alpha_5 g_5, \\ \beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_6 g_6=1. \end{cases}$$

If $\alpha_5 g_5 \neq 1$, then the first equation implies a contradiction as in (A). If $\alpha_5 g_5=1$, then

$$(\beta_1\alpha_2-\alpha_1\beta_2)g_2+(\beta_1\alpha_3-\alpha_1\beta_3)g_3+(\beta_1\alpha_4-\alpha_1\beta_4)g_4-\alpha_1\beta_6 g_6=-\alpha_1.$$

Here all the coefficients are not zero. Thus this of type (A), which leads to a contradiction. Thus one of $\{g_j\}_{j\neq 5}$ is a polynomial.

c) This case is quite similar to the case b). Then one of $\{g_j\}_{j=1}^{5}$ is a polynomal.

This completes the proof of Theorem 2.

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