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DISTRIBUTION AND CRITICAL CURVES IN A RIEMANNIAN MANIFOLD

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Let $\mathcal{D}$ be a C^∞ distribution in a C^∞ Riemannian manifold M. In the present paper a curve of M where every tangent vector lies in $\mathcal{D}$ is called a $\mathcal{D}$-curve. Let P and Q be two points of M such that there exist $\mathcal{D}$-curves joining P and Q. We call a $\mathcal{D}$-curve C a critical $\mathcal{D}$-curve with the fixed end points P, Q if the length l of C takes a critical value in the set of $\mathcal{D}$-curves joining P and Q. The purpose of the present paper is to find differential equations of critical $\mathcal{D}$-curves when $n-m=\dim \mathcal{D}$ satisfies $n<2(n-m)$, where $n=\dim M$, and to study properties of such critical $\mathcal{D}$-curves in some special cases.

§1. The differential equations of a critical $\mathcal{D}$-curve.

Let M be an n-dimensional Riemannian manifold and $\mathcal{D}$ (or $\mathcal{D}^{n-m}$) an $(n-m)$-dimensional distribution given locally by $n-m$ linearly independent C^∞ vector fields $\underset{\lambda}{X}\,(\lambda=m+1, \cdots, n)$.[1] Their components with respect to a local coordinate system will be denoted by $\underset{\lambda}{X}{}^h$. The distribution $\mathcal{D}$ will also be represented by m linearly independent covector fields $\overset{\alpha}{\varphi}\,(\alpha=1, \cdots, m)$ whose components $\overset{\alpha}{\varphi}_i$ satisfy

$$\overset{\alpha}{\varphi}_i \underset{\lambda}{X}{}^i=0.$$

A $\mathcal{D}$-curve C is by definition a curve $x^h=x^h(t)$ such that

$$\overset{\alpha}{\varphi}_i \frac{dx^i}{dt}=0 \tag{1.1}$$

holds throughout the curve.

We assume that $2m$ covectors

$$\overset{1}{\varphi}_i, \cdots, \overset{m}{\varphi}_i, \overset{1}{\psi}_i, \cdots, \overset{m}{\psi}_i \tag{1.2}$$

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1) We let the indices $h, i, j, \cdots$ run over the range $\{1, \cdots, n\}$, $\alpha, \beta, \gamma, \cdots$ over the range $\{1, \cdots, m\}$ and $\kappa, \lambda, \mu, \cdots$ over the range $\{m+1, \cdots, n\}$. The summation convention is used for all such indices.

are linearly independent at every point of C, where $\overset{\alpha}{\phi}_i$ are defined by

$$\overset{\alpha}{\phi}_i=(\partial_j\overset{\alpha}{\varphi}_i-\partial_i\overset{\alpha}{\varphi}_j)\frac{dx^j}{dt}. \tag{1. 3}$$

Let P and Q be the end points of C and the parameter t be such that $t=0$ and $t=1$ correspond respectively to P and Q. Then the length l of C is given by the integral

$$J(C)=\int_C ds=\int_0^1\left[g_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}\right]^{1/2}dt. \tag{1. 4}$$

Let us consider an infinitesimal deformation of the curve C with the points P and Q fixed assuming that any curve obtained is also a $\mathscr{D}$-curve. Then the vector of deformation $\xi^h(t)$ must satisfy

$$\xi^j\frac{dx^i}{dt}\partial_j\overset{\alpha}{\varphi}_i+\overset{\alpha}{\varphi}_i\frac{d\xi^i}{dt}=0. \tag{1. 5}$$

As the points P and Q are fixed, ξ^h must also satisfy

$$\xi^h(0)=\xi^h(1)=0. \tag{1. 6}$$

Then it is a consequence of an ordinary argument in the calculus of variations that C is a critical $\mathscr{D}$-curve if and only if

$$\int_0^l\left[\frac{d^2x^i}{ds^2}+\left\{{i \atop k\ j}\right\}\frac{dx^k}{ds}\frac{dx^j}{ds}\right]g_{ih}\xi^h(s)ds=0 \tag{1. 7}$$

is satisfied by every set of functions $\xi^h(t)$ satisfying (1. 5) and (1. 6). Notice that the arc length s is used in (1. 7) as the parameter and that l is the length of C.

Now let $\underset{\alpha}{f}(t)$ $(\alpha=1, \cdots, m)$ be a set of arbitrary C^∞ functions. Then we find that

$$\int_0^1\left[(\underset{\alpha}{f}(t)\partial_j\overset{\alpha}{\varphi}_i)\xi^j\frac{dx^i}{dt}+\underset{\alpha}{f}(t)\overset{\alpha}{\varphi}_i\frac{d\xi^i}{dt}\right]dt=0 \tag{1. 8}$$

is equivalent to (1. 5). (1. 8) is also equivalent to

$$\int_0^1\left[\left(\frac{d}{dt}\underset{\alpha}{f}\right)\overset{\alpha}{\varphi}_i+\underset{\alpha}{f}(t)(\partial_j\overset{\alpha}{\varphi}_i-\partial_i\overset{\alpha}{\varphi}_j)\frac{dx^j}{dt}\right]\xi^i(t)dt=0$$

and again to

$$\int_0^l\left[\left(\frac{d}{ds}\underset{\alpha}{f}\right)\overset{\alpha}{\varphi}_i+\underset{\alpha}{f}(s)(\partial_j\overset{\alpha}{\varphi}_i-\partial_i\overset{\alpha}{\varphi}_j)\frac{dx^j}{ds}\right]\xi^i(s)ds=0. \tag{1. 9}$$

If we put

$$\overset{\alpha}{\psi}_i=\frac{dx^j}{ds}(\partial_j\overset{\alpha}{\varphi}_i-\partial_i\overset{\alpha}{\varphi}_j),$$

we can write (1. 9) in the form

$$\int_0^l\left[\overset{\alpha}{\varphi}_i(s)\frac{d}{ds}\underset{\alpha}{f}+\underset{\alpha}{f}(s)\overset{\alpha}{\psi}_i(s)\right]\xi^i(s)ds=0. \tag{1. 10}$$

We prove in § 2 the following lemma.

LEMMA 1. 1. *In an n-dimensional Euclidean space let there be given* $2m+1$ C^∞ *vector functions* $A_i(t)$, $\overset{\alpha}{\varphi}_i(t)$, $\overset{\alpha}{\psi}_i(t)$ $(\alpha=1, \cdots, m)$ *where* $2m$ *vectors*

$$\overset{1}{\varphi}(t), \cdots, \overset{m}{\varphi}(t), \overset{1}{\psi}(t), \cdots, \overset{m}{\psi}(t)$$

are linearly independent at each value of t, $0\leqq t\leqq a$. *If, for every functions* $\xi^i(t)$ *which satisfy*

$$\xi^i(0)=\xi^i(a)=0 \tag{1. 11}$$

and

$$\int_0^a\left\{\left(\frac{d}{dt}\underset{\alpha}{f}\right)\overset{\alpha}{\varphi}_i(t)+\underset{\alpha}{f}(t)\overset{\alpha}{\psi}_i(t)\right\}\xi^i(t)dt=0 \tag{1. 12}$$

for every choice of C^∞ *functions* $\underset{\alpha}{f}(t)$, *we have*

$$\int_0^a A_i(t)\xi^i(t)dt=0, \tag{1. 13}$$

then there exist functions $\underset{1}{\chi}(t), \cdots, \underset{m}{\chi}(t)$ *such that*

$$A_i(t)=\left(\frac{d}{dt}\underset{\alpha}{\chi}\right)\overset{\alpha}{\varphi}_i(t)+\underset{\alpha}{\chi}(t)\overset{\alpha}{\psi}_i(t). \tag{1. 14}$$

REMARK. It is easily found that (1. 13) is a consequence of (1. 12) and (1. 14).

Applying Lemma 1. 1 to the case of $\mathcal{D}$-curves, we easily obtain the following lemma.

LEMMA 1. 2. *Let* M *be an n-dimensional Riemannian manifold equipped with an* $(n-m)$*-dimensional distribution* $\mathcal{D}$ *determined locally by* m *covector fields* $\overset{\alpha}{\varphi}_i$. *Let* C *be a* $\mathcal{D}$*-curve* $x^h=x^h(s)$, $0\leqq s\leqq l$, *such that* $2m$ *covectors*

$$\overset{\alpha}{\varphi}_i,\ \frac{dx^j}{ds}(\nabla_j\overset{\alpha}{\varphi}_i-\nabla_i\overset{\alpha}{\varphi}_j) \qquad (\alpha=1, \cdots, m)$$

are linearly independent at each point of C. *A necessary and sufficient condition* *for the curve C to be a critical $\mathfrak{D}$-curve with fixed end points is that there exist functions $\underset{\alpha}{\chi}(s)$ satisfying the equations*

$$\frac{d^2x^h}{ds^2}+\left\{ {h \atop j\ i} \right\}\frac{dx^j}{ds}\frac{dx^i}{ds}=\left[\left(\frac{d}{ds}\underset{\alpha}{\chi}\right)\overset{\alpha}{\varphi}_i+\underset{\alpha}{\chi}\frac{dx^j}{ds}(\nabla_j\overset{\alpha}{\varphi}_i-\nabla_i\overset{\alpha}{\varphi}_j)\right]g^{ih}. \tag{1.15}$$

Differentiating the equations

$$\overset{\alpha}{\varphi}_i\frac{dx^i}{ds}=0 \tag{1.16}$$

covariantly along the curve C, we get

$$(\nabla_j\overset{\alpha}{\varphi}_i)\frac{dx^j}{ds}\frac{dx^i}{ds}+\overset{\alpha}{\varphi}_h\left(\frac{d^2x^h}{ds^2}+\left\{ {h \atop j\ i} \right\}\frac{dx^j}{ds}\frac{dx^i}{ds}\right)=0.$$

Then applying (1.15) we obtain

$$\overset{\alpha}{\varphi}_i\overset{\beta}{\varphi}{}^i\frac{d}{ds}\underset{\beta}{\chi}+\frac{dx^j}{ds}\overset{\alpha}{\varphi}{}^i(\nabla_j\overset{\beta}{\varphi}_i-\nabla_i\overset{\beta}{\varphi}_j)\underset{\beta}{\chi}+\frac{dx^j}{ds}\frac{dx^i}{ds}\nabla_j\overset{\alpha}{\varphi}_i=0. \tag{1.17}$$

Let us consider a system of differential equations composed of (1. 15) and (1. 17) in the unknown functions $x^h(s)$ and $\underset{\alpha}{\chi}(s)$. As far as only these equations are considered, s may not be the arc length and the curve $x^h=x^h(s)$ may not be a $\mathfrak{D}$-curve. But, if the initial condition is chosen in such a way that

$$g_{ih}\frac{dx^i}{ds}\frac{dx^h}{ds}=1,\qquad \overset{\alpha}{\varphi}_i\frac{dx^i}{ds}=0$$

hold at $s=0$, then we can easily see that s is the arc length of the curve $x^h=x^h(s)$ and (1. 6) is satisfied by the curve.

Thus we obtain the

THEOREM 1. 3. *Let M and $\mathfrak{D}$ be the same as those assumed in Lemma* 1. 2. *A necessary and sufficient condition for a $\mathfrak{D}$-curve C, for which the same is also assumed as in Lemma* 1. 2 *and parametrized by the arc length s, to be a critical $\mathfrak{D}$-curve with the fixed end points is that the functions $x^h(s)$ satisfy with some functions $\underset{\alpha}{\chi}(s)$ the differential equations* (1. 15), (1. 16) *and* (1. 17). *If a solution $x^h=x^h(s)$, $\underset{\alpha}{\chi}=\underset{\alpha}{\chi}(s)$ of the system of differential equations composed of* (1. 15) *and* (1. 17) *satisfies the initial condition*

$$\left(g_{ih}\frac{dx^i}{ds}\frac{dx^h}{ds}\right)_0=1,\qquad \left(\overset{\alpha}{\varphi}_i\frac{dx^i}{ds}\right)_0=0$$

and the $2m$ *covectors*

$$\overset{\alpha}{\varphi}_i,\qquad \frac{dx^j}{ds}(\nabla_j\overset{\alpha}{\varphi}_i-\nabla_i\overset{\alpha}{\varphi}_j)$$

are linearly independent at each point $x^h(s)$ $(0\leqq s\leqq l)$, then the curve $x^h=x^h(s)$ is a critical $\mathcal{D}$-curve with the fixed end points $x^h(0)$, $x^h(l)$ and s is the arc length.

§2. Proof of Lemma 1. 1.

Let τ be any number such that $0<\tau<a$ and put

$$\xi^h(t)=a^h\delta(t-\tau) \tag{2. 1}$$

where a^h is a constant vector and δ is the Dirac function. Then (1. 12) becomes

$$\frac{d}{dt}\underset{\alpha}{f}(\tau)\overset{\alpha}{\varphi}_i(\tau)a^i+\underset{\alpha}{f}(\tau)\overset{\alpha}{\psi}_i(\tau)a^i=0. \tag{2. 2}$$

As we can take arbitrary C^∞ functions as $\underset{\alpha}{f}(t)$, we get

$$\overset{\alpha}{\varphi}_i(\tau)a^i=0, \qquad \overset{\alpha}{\psi}_i(\tau)a^i=0 \tag{2. 3}$$

from (2. 2).

On the other hand we have

$$A_i(\tau)a^i=0 \tag{2. 4}$$

from (1. 13). Since any vector a^h satisfying (2. 3) must satisfy (2. 4) by assumption, there exist $2m$ numbers $\underset{\alpha}{\rho}(\tau)$, $\underset{\alpha}{\sigma}(\tau)$ such that

$$A_i(\tau)=\underset{\alpha}{\rho}(\tau)\overset{\alpha}{\varphi}_i(\tau)+\underset{\alpha}{\sigma}(\tau)\overset{\alpha}{\psi}_i(\tau).$$

Thus we obtain

$$A_i(t)=\underset{\alpha}{\rho}(t)\overset{\alpha}{\varphi}_i(t)+\underset{\alpha}{\sigma}(t)\overset{\alpha}{\psi}_i(t) \tag{2. 5}$$

where $\underset{\alpha}{\rho}(t)$ and $\underset{\alpha}{\sigma}(t)$ are C^∞ functions, for $\overset{\alpha}{\varphi}_i(t)$ and $\overset{\alpha}{\psi}_i(t)$ are linearly independent.

We now proceed to find a relation between $\underset{\alpha}{\rho}(t)$ and $\underset{\alpha}{\sigma}(t)$.

From (1. 13) and (2. 5) we get

$$\int_0^a[\underset{\alpha}{\rho}(t)\overset{\alpha}{\varphi}_i(t)\xi^i(t)+\underset{\alpha}{\sigma}(t)\overset{\alpha}{\psi}_i(t)\xi^i(t)]dt=0. \tag{2. 6}$$

Let λ be an arbitrary number, $0<\lambda<a$, and $\varepsilon>0$ a sufficiently small number such that $[\lambda-\varepsilon, \lambda+\varepsilon]\subset(0, a)$ and such that a determinant of order $2m$ composed of some components of the $2m$ covectors $\overset{\alpha}{\varphi}$, $\overset{\alpha}{\psi}$ does not vanish at any point of $[\lambda-\varepsilon, \lambda+\varepsilon]$. Then we can consider for example

$$\begin{vmatrix} \overset{1}{\varphi}_1(t) & \cdots & \overset{m}{\varphi}_1(t) & \overset{1}{\psi}_1(t) & \cdots & \overset{m}{\psi}_1(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overset{1}{\varphi}_{2m}(t) & \cdots & \overset{m}{\varphi}_{2m}(t) & \overset{1}{\psi}_{2m}(t) & \cdots & \overset{m}{\psi}_{2m}(t) \end{vmatrix} \neq 0.$$

In this case, if we take C^∞ functions $\overset{\alpha}{h}(t)$ such that

$$\overset{2}{h}(t) = \cdots = \overset{m}{h}(t) = 0,$$

$$\overset{1}{h}(t) = 0 \qquad 0 \leqq t \leqq \lambda - \varepsilon, \qquad \lambda + \varepsilon \leqq t \leqq a,$$

$$\overset{1}{h}(t) > 0 \qquad \lambda - \varepsilon < t < \lambda + \varepsilon$$

and determine $\xi^h(t)$ by

$$\xi^{2m+1}(t) = \cdots = \xi^n(t) = 0,$$

$$\xi^1(t) = \cdots = \xi^{2m}(t) = 0 \qquad 0 \leqq t \leqq \lambda - \varepsilon, \qquad \lambda + \varepsilon \leqq t \leqq a,$$

$$\left.\begin{aligned} &\overset{\alpha}{\varphi}_i(t)\xi^i(t) = \overset{\alpha}{h}(t), \\ &\overset{\alpha}{\psi}_i(t)\xi^i(t) = \frac{d}{dt}\overset{\alpha}{h}(t) \end{aligned}\right\} \lambda - \varepsilon < t < \lambda + \varepsilon,$$

then $\xi^h(t)$ satisfy $\xi^h(0) = \xi^h(a) = 0$ and (1. 12). On the other hand we get from (2. 6)

$$\int_0^a \left[\underset{\alpha}{\rho}(t)\overset{\alpha}{h}(t) + \underset{\alpha}{\sigma}(t)\frac{d}{dt}\overset{\alpha}{h}(t) \right] dt = 0,$$

and consequently,

$$\int_0^a \left\{ \underset{1}{\rho}(t) - \frac{d}{dt}\underset{1}{\sigma}(t) \right\} \overset{1}{h}(t)dt = 0.$$

As we can take the positive valued function $\overset{1}{h}(t)$ arbitrarily, and, as we can take the number λ $(0 < \lambda < a)$ arbitrarily, we have

$$\underset{1}{\rho}(t) = \frac{d}{dt}\underset{1}{\sigma}(t).$$

Similarly we have

$$\underset{\alpha}{\rho}(t) = \frac{d}{dt}\underset{\alpha}{\sigma}(t).$$

Hence we get (1. 14) and the lemma is proved.

§ 3. Some examples.

In § 3 some examples are given. Another example which is concerned with the normal contact metric structure of S^{2n-1} is studied in § 4.

1° A distribution which is orthogonal to a Killing vector field of constant magnitude.

Let X be a Killing vector field in an odd dimensional Riemannian manifold such that

$$g_{ji}X^jX^i=1$$

and such that the rank of the matrix (∇_jX_i) is $n-1$. X_i satisfies

$$(\nabla_jX_i-\nabla_iX_j)X^i=2X^i\nabla_jX_i=0$$

and, since the rank of (∇_jX_i) is $n-1$, $Y^j\nabla_jX_i$ does not vanish if $Y^iX_i=0$ and $Y\neq 0$. Hence the covectors X_i and $Y^j(\nabla_jX_i-\nabla_iX_j)$ are linearly independent. Consider the $(n-1)$-dimensional distribution $\mathcal{D}$ determined by the covector field X_i. Then from the above argument, for any $\mathcal{D}$-curve C: $x^h=x^h(s)$, the covectors

$$X_i,\ \frac{dx^j}{ds}(\nabla_jX_i-\nabla_iX_j)$$

are linearly independent on C.

The differential equations of the a critical $\mathcal{D}$-curve are

$$\frac{d^2}{ds^2}x^h+\left\{\begin{matrix} h \\ j\ i\end{matrix}\right\}\frac{dx^j}{ds}\frac{dx^i}{ds}=\left(\frac{d}{ds}\chi\right)X^h+2\chi\frac{dx^j}{ds}\nabla_jX^h,$$

but it is easily seen from (1. 17) that χ is a constant. Hence we have

$$\frac{d^2x^h}{ds^2}+\left\{\begin{matrix} h \\ j\ i\end{matrix}\right\}\frac{dx^j}{ds}\frac{dx^i}{ds}=c\frac{dx^j}{ds}\nabla_jX^h.$$

2° A distribution in the Euclidean 3-space.

Let $\mathcal{D}$ be a distribution orthogonal to a Killing vector field defined by

$$\varphi_1=-y, \qquad \varphi_2=x, \qquad \varphi_3=1.$$

Then we have

$$\frac{d^2x}{ds^2}=\frac{d\chi}{ds}(-y)-2\chi\frac{dy}{ds},$$

$$\frac{d^2y}{ds^2}=\frac{d\chi}{ds}x+2\chi\frac{dx}{ds},$$

$$\frac{d^2z}{ds^2}=\frac{d\chi}{ds}$$

for (1.15),

$$-y\frac{dx}{ds}+x\frac{dy}{ds}+\frac{dz}{ds}=0$$

for (1.16) and

$$(x^2+y^2+1)\frac{d\chi}{ds}+2\left(x\frac{dx}{ds}+y\frac{dy}{ds}\right)\chi=0$$

for (1.17). Then we get

$$\chi=\frac{c}{x^2+y^2+1}$$

and χ is not a constant in general, although there exist some critical $\mathscr{D}$-curves where χ is constant.

Suppose

$$a\varphi_i+b\frac{dx^j}{ds}(\partial_j\varphi_i-\partial_i\varphi_j)=0$$

for some a and b. Then we have

$$-ay-2b\frac{dy}{ds}=0,\quad ax+2b\frac{dx}{ds}=0,\quad a=0,$$

and consequently

$$b=0\qquad\text{or}\qquad\frac{dx}{ds}=\frac{dy}{ds}=0.$$

But the latter contradicts

$$\frac{dz}{ds}=y\frac{dx}{ds}-x\frac{dy}{ds},\quad\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2=1.$$

Thus we see that

$$\varphi_i,\ \frac{dx^j}{ds}(\partial_j\varphi_i-\partial_i\varphi_j)$$

are linearly independent for all $\mathscr{D}$-curves.

3° A distribution in a contact metric manifold.

A contact metric manifold M is a Riemannian manifold of odd dimension endowed with a vector field φ^h satisfying the following conditions,

(i) $$\varphi^i\varphi_i=1 \text{ where } \varphi_i=g_{ih}\varphi^h,$$

(ii) $$(\nabla_j\varphi_i-\nabla_i\varphi_j)\varphi^i=0,$$

(iii) $$\frac{1}{4}(\nabla_j\varphi^i-\nabla^i\varphi_j)(\nabla_i\varphi^h-\nabla^h\varphi_i)=-\delta_j^h+\varphi_j\varphi^h.$$

Let $\mathcal{D}$ be a distribution which is orthogonal to the vector field φ^h. Let $x^h=x^h(s)$ be a $\mathcal{D}$-curve.

Suppose

$$a\varphi^h+b\frac{dx^j}{ds}(\nabla_j\varphi^h-\nabla^h\varphi_j)=0.$$

Transvecting φ_h we get

$$a=0.$$

Transvecting with $\nabla_h\varphi^i-\nabla^i\varphi_h$ we get

$$b\left(-\frac{dx^i}{ds}+\frac{dx^j}{ds}\varphi_j\varphi^i\right)=0.$$

But, as we have

$$\varphi_j\frac{dx^j}{ds}=0$$

for a $\mathcal{D}$-curve, we get $b=0$. Hence

$$\varphi_i,\ \frac{dx^j}{ds}(\nabla_j\varphi_i-\nabla_i\varphi_j)$$

are linearly independent for all $\mathcal{D}$-curves.

§4. A $(2n-2)$-dimensional distribution on S^{2n-1} and the critical $\mathcal{D}$-curves of this distribution.

In their study of normal contact metric structure Sasaki and Hatakeyama [1] showed that S^{2n-1} is an example of normal contact metric manifolds. A normal contact metric structure of S^{2n-1} induces a $(2n-2)$-dimensional distribution $\mathcal{D}$ and it is the purpose of §4 to study critical $\mathcal{D}$-curves of this distribution. On the other hand Yano and Ishihara [3] showed that S^{2n-1} is a fibred space with invariant Riemannian metric with a base space M^* which is a $(2n-2)$-dimensional Kähler manifold of constant holomorphic sectional curvature.[2] A $\mathcal{D}$-curve is a horizontal curve with respect to this fibre structure and a critical $\mathcal{D}$-curve C has a projection curve C^* on M^*. We shall study some properties of C^*.

1° When we regard S^{2n-1} as a hypersphere

2) See also Steenrod [2] where it is shown on page 108 that S^{2n-1} is a 1-sphere bundle over the projective space of n homogeneous complex variables.

$$(x^1)^2+(x^2)^2+\cdots+(x^{2n-1})^2+(x^{2n})^2=1$$

in a $2n$-dimensional Euclidean space E^{2n} where a rectangular coordinate system $(x^1, \cdots, x^{2n})$ is fixed, $x^1, \cdots, x^{2n-1}$ can be considered as local coordinates of S^{2n-1} in domains $x^{2n}>0$ and $x^{2n}<0$.

There exists on E^{2n} a complex structure induced canonically from the given rectangular coordinate system, and this complex structure and the metric of E^{2n} induce on S^{2n-1} a normal contact metric structure. The contravariant vector field φ of this structure has components

$$\varphi^1=-x^2,\quad \varphi^2=x^1,\quad \varphi^3=-x^4,\quad \varphi^4=x^3, \quad \cdots,\quad \varphi^{2n-1}=-x^{2n} \tag{4.1}$$

in the local coordinates (x^κ).[3] We consider again the distribution $\mathscr{D}$ which is orthogonal to the vecor field φ.

As the metric tensor of S^{2n-1} has components

$$g_{\mu\lambda}=\delta_{\mu\lambda}+\frac{x^\mu x^\lambda}{(x^{2n})^2} \tag{4.2}$$

in the local coordinates (x^κ), the components φ_μ of the covector field of the distribution $\mathscr{D}$ are

$$\varphi_\mu=\varphi^\mu+\frac{x^\lambda\varphi^\lambda}{(x^{2n})^2}x^\mu, \tag{4.3}$$

hence we have

$$\varphi_\lambda\varphi^\lambda=1. \tag{4.4}$$

Let $\{_\mu{}^\kappa{}_\lambda\}_g$ be the Christoffel constructed from $g_{\mu\lambda}$ and let ∇_μ be the operator of covariant differentiation with respect to the Riemannian metric of S^{2n-1}. If indices a, b, c are used in the range $\{1, \cdots, 2n-2\}$,[4] the components

$$\varphi_{\mu\lambda}=\nabla_\mu\varphi_\lambda-\nabla_\lambda\varphi_\mu=\partial_\mu\varphi_\lambda-\partial_\lambda\varphi_\mu$$

have the following values,

$$\varphi_{cb}=0 \text{ except } \quad \varphi_{12}=\varphi_{34}=\cdots=\varphi_{2n-3,2n-2}$$
$$=-\varphi_{21}=-\varphi_{43}=\cdots=-\varphi_{2n-2,2n-3}=2,$$

3) In §4 indices $\kappa, \lambda, \mu, \cdots$ run over the range $\{1, \cdots, 2n-1\}$. Summation convention is used in the usual way and also in the following way, $A^\lambda B^\lambda=A^1B^1+\cdots+A^{2n-1}B^{2n-1}$.

4) The summation convention of the following form is also used,

$$A^aB^a=A^1B^1+\cdots+A^{2n-2}B^{2n-2}.$$

$$\varphi_{c,2n-1}=-\varphi_{2n-1,c}=\frac{2x^c}{x^{2n}}.$$

The rank of $(\varphi_{\mu\lambda})$ is $2n-2$.

As we have

$$\left\{ {\kappa \atop \mu\ \lambda} \right\}_g=\delta_{\mu\lambda}x^\kappa+\frac{x^\mu x^\lambda x^\kappa}{(x^{2n})^2}=g_{\mu\lambda}x^\kappa, \tag{4.5}$$

the differential equation of a critical $\mathcal{D}$-curve is

$$\frac{d^2x^\kappa}{ds^2}+x^\kappa=C\varphi_\mu{}^\kappa\frac{dx^\mu}{ds}. \tag{4.6}$$

The study of critical $\mathcal{D}$-curves is facilitated by the use of local coordinates $y^1, \cdots, y^{2n-1}$ such that

$$\begin{aligned} x^1&=y^1\cos z+y^2\sin z,\\ x^2&=-y^1\sin z+y^2\cos z,\\ &\cdots\cdots\cdots\cdots\cdots\cdots,\\ x^{2n-3}&=y^{2n-3}\cos z+y^{2n-2}\sin z,\\ x^{2n-2}&=-y^{2n-3}\sin z+y^{2n-2}\cos z,\\ x^{2n-1}&=r\sin z,\qquad x^{2n}=r\cos z, \end{aligned} \tag{4.7}$$

where $z=y^{2n-1}$ and

$$\begin{aligned} r^2&=1-(x^1)^2-\cdots-(x^{2n-2})^2\\ &=1-(y^1)^2-\cdots-(y^{2n-2})^2. \end{aligned} \tag{4.8}$$

Notice that these coordinates are used only in the range

$$r>0,\qquad -\frac{\pi}{2}<z<\frac{\pi}{2}.$$

Let us define f_{cb} by

(4.9) $f_{cb}=0$ except $f_{12}=f_{34}=\cdots=f_{2n-3,2n-2}=-f_{21}=-f_{43}=\cdots=-f_{2n-2,2n-3}=1.$

Then the components $h_{\mu\lambda}$ of the metric tensor of S^{2n-1} in local coordinates (y^κ) are

$$\begin{aligned} h_{cb}&=\delta_{cb}+\frac{y^c y^b}{r^2},\\ h_{c,2n-1}&=f_{ct}y^t,\qquad h_{2n-1,2n-1}=1. \end{aligned} \tag{4.10}$$

If we define $h^{\mu\lambda}$ by

$$h_{\mu\lambda}h^{\lambda\kappa}=\delta^{\kappa}_{\mu},$$

we have

$$h^{ba}=\delta_{ba}-y^b y^a+\frac{1}{r^2}f_{bt}y^t f_{as}y^s,$$

(4. 11)

$$h^{b,2n-1}=\frac{-1}{r^2}f_{bt}y^t, \qquad h^{2n-1,2n-1}=\frac{1}{r^2}.$$

When we use the coordinate system (y^κ), the corresponding contravariant components of the vector φ will be denoted by ϕ^κ, hence

$$\phi^\kappa=\frac{\partial y^\kappa}{\partial x^\lambda}\varphi^\lambda.$$

Then we have

(4. 12) $$\phi^a=0, \qquad \phi^{2n-1}=-1.$$

We have for the corresponding covariant components

(4. 13) $$\phi_b=-f_{bt}y^t, \qquad \phi_{2n-1}=-1$$

2° Remember that ϕ^κ are the components of a Killing vector of unit length to which the distribution $\mathscr{D}$ is orthogonal. (4. 12) shows that the y^{2n-1}-curves (curves on which y^a are constant) are fibres of the fibred space S^{2n-1}. This fibred space which has been studied by Yano and Ishihara [3], has a base space M^* of dimension $2n-2$ and, if we use the local coordinates (y^κ), namely (y^a, y^{2n-1}), in S^{2n-1}, the projection π: $S^{2n-1}\to M^*$ is given by π: $(y^a, y^{2n-1})\to(y^a)$.

Let us introduce a metric into M^* by the standard of Yano and Ishihara. If the metric tensor of M^* is written h^*_{cb} in the coordinate system (y^a), h^*_{cb} are obtained from

$$h_{\mu\lambda}dy^\mu dy^\lambda=h^*_{cb}dy^c dy^b$$

by putting $\phi_\kappa dy^\kappa=0$. The explicit formula is

(4. 14) $$h^*_{cb}=\delta_{cb}+\frac{y^c y^b}{r^2}-f_{ct}y^t f_{bs}y^s.$$

The inverse (h^{ba}) of the matrix (h^*_{cb}) has the elements

(4. 15) $$h^{ba}=\delta_{ba}-y^b y^a+\frac{1}{r^2}f_{bt}y^t f_{as}y^s.$$

The Christoffel $\{{}_c{}^a{}_b\}^*$ is

$$
\begin{aligned}
\left\{ {a \atop c\ b} \right\}^* &= \left(\delta_{cb} + \frac{y^c y^b}{r^2}\right) y^a \\
&\quad + f_{ct} y^t f_{ba} + f_{bt} y^t f_{ca} - 2 f_{ct} y^t f_{bs} y^s y^a \\
&\quad - (f_{ct} y^t y^b + f_{bt} y^t y^c) \frac{1}{r^2} f_{as} y^s .
\end{aligned} \tag{4.16}
$$

On the other hand, if we define $\phi_\mu{}^\kappa$ by

$$
\phi_\mu{}^\kappa = \left(\frac{\partial \phi_\lambda}{\partial y^\mu} - \frac{\partial \phi_\mu}{\partial y^\lambda} \right) h^{\lambda\kappa},
$$

we can write the differential equations of a critical $\mathscr{D}$-curve in the form

$$
\frac{d^2 y^\kappa}{ds^2} + \left\{ {\kappa \atop \mu\ \lambda} \right\} \frac{dy^\mu}{ds} \frac{dy^\lambda}{ds} = C \phi_\mu{}^\kappa \frac{dy^\mu}{ds} . \tag{4.17}
$$

Calculating the Christoffel $\{{}_\mu{}^\kappa{}_\lambda\}$ of $h_{\mu\lambda}$, we get from (4. 17)

$$
\begin{aligned}
y''^a &= \left\{ -y'^c y'^c - \frac{(y^c y'^c)^2}{r^2} + 2\rho^2 - 2C\rho \right\} y^a \\
&\quad + \frac{2 y^c y'^c}{r^2} (\rho - C) f_{at} y^t + 2(\rho - C) f_{at} y'^t
\end{aligned} \tag{4.18}
$$

where

$$
y'^a = \frac{dy^a}{ds}
$$

and ρ is defined by

$$
\rho = f_{ts} y'^t y^s . \tag{4.19}
$$

We can regard (4. 18) as a curve C^* in M^*, the projection of a critical $\mathscr{D}$-curve C. In order to find some properties of C^* we use (4. 16) and write (4. 18) in the form

$$
\begin{aligned}
y''^a &+ \left\{ {a \atop c\ b} \right\}^* y'^c y'^b \\
&= -2C \left(\rho y^a + \frac{1}{r^2} y^c y'^c f_{at} y^t + f_{at} y'^t \right).
\end{aligned} \tag{4.20}
$$

Differentiating (4. 20) covariantly along the curve C^* we get after some straightforward calculation

$$\frac{d}{ds}\left(y''^a+\begin{Bmatrix} a \\ c\ b \end{Bmatrix}^* y'^c y'^b\right)$$

$$(4.21)\qquad +\begin{Bmatrix} a \\ c\ b \end{Bmatrix}^*\left(y''^c+\begin{Bmatrix} c \\ t\ s \end{Bmatrix}^* y'^t y'^s\right)y'^b$$

$$=-4C^2 y'^a.$$

This shows that C^* is a Riemannian circle of curvature $2|C|$.

A Riemannian circle is by definition a curve in a Riemannian space whose development in a tangent space is a circle. Its global properties are quite various according to the enveloping manifold. Thus, for example, we cannot even guess the period of C^*.

But, as for the function $r(s)$ only, we can find its period.

As r is given by $y^a y^a=1-r^2$, we have

$$y^c y'^c=-rr',$$

$$(4.22)$$

$$y'^c y'^c+y^c y''^c=-r'r'-rr''.$$

We also get from $h^*_{cb}y'^c y'^b=1$ and (4. 14)

$$(4.23)\qquad y'^c y'^c+r'r'=1+\rho^2.$$

On the other hand, if we substitute (4. 18) into $y^c y''^c$, the second equation of (4. 22) gives

$$rr''=-r^2(1-\rho^2+2C\rho)=-r^2\{1+C^2-(\rho-C)^2\}.$$

As we assume $r>0$, we get

$$(4.24)\qquad r''=-r\{1+C^2-(\rho-C)^2\}.$$

We also obtain from (4. 18), (4. 19) and (4. 22)

$$\rho'=-\frac{2(\rho-C)r'}{r}.$$

Hence we have

$$(4.25)\qquad \rho-C=\frac{k}{r^2}$$

where k is a constant. Substituting this into (4. 24) we get

$$r''=-(1+C^2)r+\frac{k^2}{r^3}.$$

The general solution of this differential equation is

$$r^2 = C_1 + C_2 \sin(\pm 2\sqrt{1+C^2}(s-s_0))$$

where

$$k^2 = (1+C^2)(C_1{}^2 - C_2{}^2).$$

Thus we find that $r(s)$ has period $\pi/\sqrt{1+C^2}$ or $r(s)$ is reduced to a constant. The only exceptional cases will occur if $k=0$. Then we have $\rho=C$. Such cases will be studied in the appendix.

3° It was shown by Yano and Ishihara [3] that the base space M^* is a Kähler manifold of constant holomorphic sectional curvature.

Let us turn to the Euclidean space E^{2n} equipped with a fixed rectangular coordinate system $(x^1, \cdots, x^{2n})$ and introduce a complex coordinate system

(4. 26)
$$Z^1 = x^1 + ix^2, \cdots, Z^{n-1} = x^{2n-3} + ix^{2n-2},$$
$$Z^0 = x^{2n-1} + ix^{2n}.$$

Then we have a complex space $\boldsymbol{C}^n$. In $\boldsymbol{C}^n - \{0\}$ we can regard $(Z^0, Z^1, \cdots, Z^{n-1})$ as a system of homogeneous complex coordinates of the complex projective space $P^{n-1}(\boldsymbol{C})$. If we assume $Z^0 \neq 0$, we can introduce an inhomogeneous complex coordinate system by

(4. 27)
$$z^1 = \frac{Z^1}{Z^0}, \cdots, z^{n-1} = \frac{Z^{n-1}}{Z^0},$$

and, if we introduce real local coordinates $w^1, \cdots, w^{2n-2}$ in $P^{n-1}(\boldsymbol{C})$ by

(4. 28)
$$z^1 = w^1 + iw^2, \cdots, z^{n-1} = w^{2n-3} + iw^{2n-2},$$

then we obtain

(4. 29)
$$w^1 = \frac{x^1x^{2n-1} + x^2x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2},$$
$$w^2 = \frac{x^2x^{2n-1} - x^1x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$w^{2n-3} = \frac{x^{2n-3}x^{2n-1} + x^{2n-2}x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2},$$
$$w^{2n-2} = \frac{x^{2n-2}x^{2n-1} - x^{2n-3}x^{2n}}{(x^{2n-1})^2 + (x^{2n})^2}.$$

If the ordinary Kähler metric of $P^{n-1}(\boldsymbol{C})$ is multiplied by a suitable constant, the corresponding metric tensor has following components g^*_{cb} in real coordinates

$w^1, \cdots, w^{2n-2}$,

$$g^*_{cb} = \frac{\delta_{cb}}{1+w^a w^a} - \frac{w^c w^b + f_{ct} w^t f_{bs} w^s}{(1+w^a w^a)^2} \tag{4.30}$$

which will be easily proved by direct calculation.

The relation between $w^1, \cdots, w^{2n-2}$ and $y^1, \cdots, y^{2n-2}$ is obtained from (4.7) and (4.29) to be

$$w^a = \frac{1}{r} f_{at} y^t, \qquad w^a w^a = \frac{1}{r^2} - 1. \tag{4.31}$$

Hence we can write (4.30) in the form

$$g^*_{cb} = r^2(\delta_{cb} - y^c y^b - f_{ct} y^t f_{bs} y^s). \tag{4.32}$$

That the metric tensor whose components are g^*_{cb} in local coordinates (w^a) is identical with the metric tensor whose components are h^*_{cb} in local coordinates (y^a) is immediately shown since we have

$$g^*_{cb} w'^c w'^b = h^*_{cb} y'^c y'^b$$

because of (4.14), (4.31) and (4.32).

As f_{cb} satisfies

$$\nabla^*_c f_{ba} = \left\{ {e \atop c\ b} \right\}^* f_{ea} + \left\{ {e \atop c\ a} \right\}^* f_{be} = 0$$

on account of (4.16), (h^*_{cb}, f_{cb}) is a Kähler structure of $P^{n-1}(\boldsymbol{C})$.

4° Let

$$\alpha^0 Z^0 + \alpha^1 Z^1 + \cdots + \alpha^{n-1} Z^{n-1} = 0 \tag{4.33}$$

be the equation of a hyperplane of $P^{n-1}(\boldsymbol{C})$. If we use only real numbers, we can write (4.33) in the form

$$A^a y^a = Kr, \qquad A^a f_{at} y^t = Lr \tag{4.34}$$

where r is given by (4.8). Hence, to a complex hyperplane of $P^{n-1}(\boldsymbol{C})$ corresponds a subspace M' of codimension 2 in M^*. The subspace M' determined by (4.34) will be denoted by $M'(A^a, K, L)$.

If we define functions $X(s)$ and $Y(s)$ by

$$\begin{aligned} X(s) &= A^a y^a(s) - Kr(s), \\ Y(s) &= A^a f_{at} y^t(s) - Lr(s) \end{aligned} \tag{4.35}$$

along a curve C^*, these satisfy

$$X''=\left\{-(1+C^2)+\frac{k^2}{r^4}\right\}X-\frac{2kr'}{r^3}Y+\frac{2k}{r^2}Y',$$

$$Y''=\left\{-(1+C^2)+\frac{k^2}{r^4}\right\}Y+\frac{2kr'}{r^3}Y-\frac{2k}{r^2}X',$$

for we get

$$y''^a=\left\{-(1+C^2)+\frac{k^2}{r^4}\right\}y^a-\frac{2kr'}{r^3}f_{at}y^t+\frac{2k}{r^2}f_{at}y'^t \tag{4.36}$$

from (4.18), (4.23) and (4.25). Hence we get $X(s)=Y(s)=0$ if $X(s)$ and $Y(s)$ satisfy $X(0)=Y(0)=X'(0)=Y'(0)=0$.

This proves the following lemma.

LEMMA 4.1. *Let $\mathcal{C}^*$ be a curve of M^* which is the projection of a critical $\mathcal{D}$-curve $\mathcal{C}$ in S^{2n-1}. If, in the corresponding curve in $P^{n-1}(\boldsymbol{C})$, which will also be denoted by $\mathcal{C}^*$, a point* P *and the tangent of $\mathcal{C}^*$ at* P *lie in a complex hyperplane, then $\mathcal{C}^*$ lies completely in this complex hyperplane.*

From (4.20) we observe that a curve $\mathcal{C}^*$ where $C=0$ is a geodesic of M^* and that any geodesic of M^* is a curve $\mathcal{C}^*$. Hence $M'(A^a, K, L)$ is a totally geodesic subspace. Notice that $M'(f_{at}A^t, -L, K)$ is the same subspace as $M'(A^a, K, L)$.

A subspace $M'(A^a, K, L)$ tangent to a given curve $\mathcal{C}^*$ at the point $s=0$ is obtained if we take A^a, K, L satisfying

$$\begin{aligned} &A^a y^a(0)-Kr(0)=0, \qquad A^a y'^a(0)-Kr'(0)=0,\\ &A^a f_{at}y^t(0)-Lr(0)=0, \qquad A^a f_{at}y'^t(0)-Lr'(0)=0. \end{aligned} \tag{4.37}$$

If we define M by

$$M=\begin{pmatrix} y^1 & y^2 \cdots y^{2n-3} & y^{2n-2} & r & 0\\ y'^1 & y'^2 \cdots y'^{2n-3} & y'^{2n-2} & r' & 0\\ y^2 & -y^1 \cdots y^{2n-2} & -y^{2n-3} & 0 & r\\ y'^2 & -y'^1 \cdots y'^{2n-2} & -y'^{2n-3} & 0 & r' \end{pmatrix},$$

the rank of M is 4, since we have

$$MM^{\mathrm{T}}=\begin{pmatrix} 1 & 0 & 0 & -\rho\\ 0 & 1+\rho^2 & \rho & 0\\ 0 & \rho & 1 & 0\\ -\rho & 0 & 0 & 1+\rho^2 \end{pmatrix}, \ \det(MM^{\mathrm{T}})=1$$

because of (4. 22) and (4. 23). Hence we have $2n-4$ linearly independent solutions of (4. 37). We also observe that, if (A^a, K, L) is a solution of (4. 37), $(f_{at}A^t, -L, K)$ is also a solution.

Suppose that $(\underset{(\xi)}{A^a}, \underset{(\xi)}{K}, \underset{(\xi)}{L})$ $(\xi=1, \cdots, 2p)$ are $2p$ linearly independent solutions of (4. 37) where

$$\underset{(2u)}{A^a}=f_{at}\underset{(2u-1)}{A^t} \qquad (u=1, \cdots, p).$$

If (A^a, K, L) is a solution of (4. 37) such that

$$A^a=\underset{(1)}{k}\underset{(1)}{A^a}+\cdots+\underset{(2p)}{k}\underset{(2p)}{A^a},$$

then we find immediately that

$$K=\underset{(1)}{k}\underset{(1)}{K}+\cdots+\underset{(2p)}{k}\underset{(2p)}{K},$$

$$L=\underset{(1)}{k}\underset{(1)}{L}+\cdots+\underset{(2p)}{k}\underset{(2p)}{L},$$

hence (A^a, K, L) is a linear combination of $(\underset{(1)}{A^a}, \underset{(1)}{K}, \underset{(1)}{L}), \cdots, (\underset{(2p)}{A^a}, \underset{(2p)}{K}, \underset{(2p)}{L})$. Then we also find that the $2p$ $(2n-2)$-tuples $\underset{(1)}{A^a}, \cdots, \underset{(2p)}{A^a}$ are linearly independent, for a solution (A^a, K, L) must satisfy $K=L=0$ if $A^a=0$.

From the above result we can deduce that there exists a set of $2n-4$ linearly independent solutions $(\underset{(\xi)}{A^a}, \underset{(\xi)}{K}, \underset{(\xi)}{L})$ $(\xi=1, \cdots, 2n-4)$ of (4. 37) where

$$\underset{(2u)}{A^a}=f_{at}\underset{(2u-1)}{A^t}, \quad \underset{(2u)}{K}=-\underset{(2u-1)}{L}, \quad \underset{(2u)}{L}=\underset{(2u-1)}{K} \qquad (u=1, \cdots, n-2)$$

and such that the $(2n-2)$-tuples $\underset{(1)}{A^a}, \cdots, \underset{(2n-4)}{A^a}$ are linearly independent.

We can interpret this result geometrically as follows.

LEMMA 4. 2. *For any curve C^* there exists in M^* a totally geodesic subspace of dimension 2 which contains C^* and is determined by a system of equations*

$$\underset{(\xi)}{A^a}y^a=\underset{(\xi)}{K}r \qquad (\xi=1, \cdots, 2n-4)$$

where

$$\underset{(2u)}{A^a}=f_{at}\underset{(2u-1)}{A^t} \qquad (u=1, \cdots, n-2).$$

This subspace is common to all curves C^* passing a point P *and having a common tangent vector at the point* P.

The contents of §4 can be resumed in the following theorem.

THEOREM 4. 3. *According to Sasaki and Hatakeyama an S^{2n-1} in E^{2n} can be treated as a normal contact metric manifold. According to Yano and Ishihara S^{2n-1}*

can also be treated as a fibred space with invariant Riemannian metric. The base space M^* *is a Kähler space of constant holomorphic sectional curvature. By virtue of these structures an* S^{2n-1} *becomes a space equipped with a distribution* $\mathcal{D}$ *of dimension* $2n-2$ *where the* $\mathcal{D}$*-curves are horizontal curves of the fibred space. If* C *is a critical* $\mathcal{D}$*-curve, the projection* C^* *on* M^* *of* C *has following properties.* (I) C^* *is a Riemannian circle of* M^*. (II) *Let* $\{M'\}$ *be the set of* $(2n-4)$*-dimensional totally geodesic subspaces of* M^* *such that each subspace* M' *is a complex hypersurface if* M^* *is regarded as a complex projective space. Then any curve* C^* *passing a point* P *of a subspace* M' *and tangent at* P *to this* M' *is contained completely in this subspace* M'. (III) *For any curve* C^* *there exists in* M^* *a totally geodesic subspace of dimension* 2 *which contains* C^* *and is obtained as an intersection of* $n-2$ *elements of* $\{M'\}$. *This subspace is common to all curves* C^* *passing a common point* P *and having a common tangent at* P.

Appendix. The exceptional cases.

In this appendix we study critical $\mathcal{D}$-curves C of S^{2n-1} where $k=0$, $\rho=C$.
In this case the differential equation of r is reduced to the form

$$r''=-(1+C^2)r \tag{A. 1}$$

and the general solution $r=r_0\cos(\sqrt{1+C^2}(s-s_0))$ does not obey the restriction $r>0$. Hence, for the study of global properties of such exceptional critical $\mathcal{D}$-curves C, we use rectangular coordinates $x^1, \cdots, x^{2n}$ of E^{2n}.

Since the equations (4.18) of the projection curve C^* are written in local coordinates $y^1, \cdots, y^{2n-2}$, we must use (4.7) and (4.8) to return to the coordinates $x^1, \cdots, x^{2n}$. C is obtained by the process of lifting in which we use

$$\phi_a y'^a+\phi_{2n-1}z'=0, \tag{A. 2}$$

which becomes

$$z'=-C \tag{A. 3}$$

because of (4.13) and $\rho=C$.

Differentiating (4.7) and using (A.3) we obtain

$$\begin{aligned} x'^1&=y'^1\cos z+y'^2\sin z-Cx^2,\\ x'^2&=-y'^1\sin z+y'^2\cos z+Cx^1,\\ &\cdots\cdots\cdots\cdots,\\ x'^{2n-1}&=r'\sin z-Cx^{2n},\\ x'^{2n}&=r'\cos z+Cx^{2n-1}, \end{aligned} \tag{A. 4}$$

and

$$
\begin{aligned}
&x''^1 = y''^1 \cos z + y''^2 \sin z + 2C(y'^1 \sin z - y'^2 \cos z) - C^2 x^1,\\
&x''^2 = -y''^1 \sin z + y''^2 \cos z + 2C(y'^1 \cos z + y'^2 \sin z) - C^2 x^2,\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots,\\
&x''^{2n-1} = r'' \sin z - 2Cr' \cos z - C^2 x^{2n-1},\\
&x''^{2n} = r'' \cos z + 2Cr' \sin z - C^2 x^{2n}.
\end{aligned} \tag{A. 5}
$$

Since r and y^a satisfy

$$r'' = -(1+C^2)r, \qquad y''^a = -(1+C^2)y^a$$

along C, we obtain

$$
\begin{aligned}
&x''^1 + 2Cx'^2 + x^1 = 0,\\
&x''^2 - 2Cx'^1 + x^2 = 0,\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots,\\
&x''^{2n-1} + 2Cx'^{2n} + x^{2n-1} = 0,\\
&x''^{2n} - 2Cx'^{2n-1} + x^{2n} = 0.
\end{aligned} \tag{A. 6}
$$

C satisfies moreover

$$
\begin{aligned}
&(x^1)^2 + \cdots + (x^{2n})^2 = 1,\\
&(x'^1)^2 + \cdots + (x'^{2n})^2 = 1,\\
&x^1 x'^1 + \cdots + x^{2n} x'^{2n} = 0,\\
&x^1 x'^2 - x^2 x'^1 + \cdots + x^{2n-1} x'^{2n} - x^{2n} x'^{2n-1} = 0.
\end{aligned} \tag{A. 7}
$$

The fourth equation of (A. 7) is obtained from (A. 4) and $\rho = C$.

If F_{ji} is defined by

$$F_{ji} = 0 \text{ except } F_{12} = F_{34} = \cdots = F_{2n-1,2n} = -F_{21} = -F_{43} = \cdots = -F_{2n,2n-1} = 1,$$

then we can write the fourth equation of (A. 7) in the form

$$F_{ji} x'^j x^i = 0. \tag{A. 8}$$

Now we can write (A. 6) in the form

$$x''^h + 2CF_{hi} x'^i + x^h = 0.$$

If in E^{2n} the vector x^h is denoted by X and the vector $F_{hi}x^i$ by FX, (A. 6) is written

(A. 9) $$X''+2CFX'+X=0.$$

Differentiating repeatedly and eliminating FX, FX' we get

(A. 10) $$X^{(4)}+2(2C^2+1)X''+X=0.$$

Let us put

(A. 11) $$\alpha=\sqrt{1+C^2}+|C|, \qquad \beta=\sqrt{1+C^2}-|C|.$$

Assuming $C\neq 0$, we have $\alpha>\beta>0$. $-\alpha^2$ and $-\beta^2$ are the roots of $\lambda^2+2(2C^2+1)\lambda+1=0$. Hence

(A. 12) $$X=A_1 \cos \alpha s+A_2 \sin \alpha s+B_1 \cos \beta s+B_2 \sin \beta s$$

is the general solution of (A. 10).

Substituting (A. 12) into (A. 7) we can deduce

$$(A_1, A_1)=(A_2, A_2)=\frac{1}{2}-\frac{|C|}{2\sqrt{1+C^2}},$$

$$(B_1, B_1)=(B_2, B_2)=\frac{1}{2}+\frac{|C|}{2\sqrt{1+C^2}}$$

and that A_1, A_2, B_1, B_2 are mutually orthogonal.

Substituting (A. 12) into (A. 9) we can deduce

$$FA_1=-\frac{|C|}{C}A_2, \qquad FB_1=\frac{|C|}{C}B_2.$$

Thus we have

(A. 13) $$X=A \cos \alpha s-\varepsilon FA \sin \alpha s+B \cos \beta s+\varepsilon FB \sin \beta s$$

where $\varepsilon=\pm 1$ and

$$(A, A)=\frac{1-\beta^2}{\alpha^2-\beta^2}, \qquad (B, B)=\frac{\alpha^2-1}{\alpha^2-\beta^2}, \qquad (A, B)=0.$$

If $C=0$ we have the simplest case,

(A. 14) $$X=A \cos s+B \sin s$$

where $(A, A)=(B, B)=1$, $(A, B)=0$.

Thus we have the following result.

The equations of the exceptional critical $\mathcal{D}$-curves are (A. 13) or (A. 14) according as $C\neq 0$ or $C=0$.

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