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ANALYTIC MAPPING AND HARMONIC LENGTH

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The notion of harmonic length was introduced by Landau and Osserman [2]. They established the uniqueness of extremal functions of harmonic length problems for a Dirichlet domain i.e. a relatively compact subregion of a Riemann surface whose boundary is regular with respect to the Dirichlet problem. By using this uniqueness theorem they discussed an extremal problem of harmonic lengths under analytic mappings and conformal rigidity of planar Dirichlet domains.

Recently the author and Kato [3] proved the same uniqueness for an arbitrary Riemann surface. In the present paper we shall deal with the same extremal problem under analytic mappings of arbitrary Riemann surfaces. A similar result to Theorem 3.1 of [2] will be obtained for the extremal case. As a result of this it will be found that their main theorem of conformal rigidity of planar Dirichlet domains (Theorem 4.1 [2]) holds for arbitrary domains. The uniqueness theorem [3] will be stated for completeness.

Our proofs of the results obtained here are quite different from theirs [2]. In the extremal problem of harmonic lengths, we shall use the Lindelöf principle due to Heins [1].

1. Harmonic length. Let R be a Riemann surface. We denote by H the class of all functions u harmonic on R which satisfy $0\leqq u\leqq 1$. Let c be a cycle on R. We define

$$h(c)=\sup_{u\in H}\int_c du^*$$

and call it *harmonic length* of c.

Landau and Osserman [2] proved the following: *There exists always a function u such that*

$$h(c)=\int_c du^*,$$

and if R is a Dirichlet domain and if c is homologous to a level locus of a harmonic measure u then u is the unique extremal function in determining $h(c)$.

Let $\hat{R}$ be the Stoilow compactification of R which makes every boundary component a point. Let c be a dividing cycle which divides the boundary $\partial R=\hat{R}-R$

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into two sets A, B in such a way that the intersection number of c with any curve starting from A and ending at B is equal to 1. Clearly both A and B are closed. Conversely, if two non-void closed A, B give a partition of the boundary ∂R, which is denoted by (A, B) and called a *regular partition* of ∂R, we can easily construct such a cycle. We call the cycle *a dividing cycle relative to* (A, B). The cycle in the result of Landau and Osserman is just of this type.

We now improve the last part of the above statement and prove

THEOREM 1 (Suita and Kato [3]). *Let R be an arbitrary Riemann surface and let c be a dividing cycle relative to a regular partition (A, B) on R. If $h(c)>0$, the function u_0 satisfying*

$$h(c)=\int_c du_0^*, \qquad u_0 \in H$$

is unique and coincides with the harmonic measure of B.

2. Before proving Theorem 1, we shall explain some exhaustions of R. Every boundary component α is defined by a defining sequence $\{\varDelta_n\}$ such that the relative boundary of $\varDelta_n$ consists of a single Jordan curve, $\varDelta_n \supset \bar{\varDelta}_{n+1}$ and $\cap \bar{\varDelta}_n = \emptyset$, where $\bar{\varDelta}_n$ denotes the closure of $\varDelta_n$ in R. We can take $\hat{\varDelta}_n \cap \hat{R}$ as a basis of neighborhoods of α. Let A be closed subset of ∂R. Since A is compact, it can be covered by a finite number of members of defining sequences of elements of A. Then we can construct such an exhaustion of R, denoted by $\{R_n(A)\}$, that $\overline{R_n(A)} \subset R_{n+1}(A)$, $R-R_n(A)$ consists of only non compact components, each of whose relative boundaries is an analytic Jordan curve and $R=\cup R_n(A)$. Clearly we have $\cap \overline{(R-R_n(A))}=\emptyset$ and $\cap \mathrm{Cl}(R-R_n(A))=A$, where $\mathrm{Cl}(*)$ means the closure in the Stoilow compactification $\hat{R}$. The exhaustion $\{R_n(A)\}$ is referred to as an *exhaustion of R towards A*.

3. Proof of Theorem 1. We may suppose that the cycle c consists of a finite number of Jordan curves, each component of whose complement is non-compact. Let u_0 be the harmonic measure of B. $u_0 \not\equiv \mathrm{const}$ by $h(c)>0$. Let v be an extremal function of the harmonic length problem for c which satisfies

$$h(c)=\int_c dv^*.$$

We take two exhaustions $\{R_n(A)\}_{n=1}^{\infty}$ and $\{R_n(B)\}_{n=1}^{\infty}$ of R so that $R_1(A) \supset c$ and $R_1(B) \supset c$. Denote by A_n and B_n the relative boundaries of $R_n(A)$ and $R_n(B)$ respectively and set $R_{mn}=R_m(A) \cap R_n(B)$.

Let v_{mn} be a function harmonic in R_{mn}, continuous on $\bar{R}_{mn}$, and satisfying $v_{mn}=v$ on A_m and $v_{mn}=1$ on B_n. It is easily verified that $\{v_{mn}\}$ is increasing with m. Since $v_{mn} \leqq 1$, we can set

$$v_n = \lim_{m \to \infty} v_{mn}.$$

The sequence $\{v_n\}$ is decreasing with n and $v_n \geqq 0$. We have a limit function

$$v_0 = \lim_{n\to\infty} v_n.$$

We shall prove $v_0 = v$. In fact, since $v_{mn} - v \geqq 0$, we have

$$\int_c d(v_{mn}-v)^* = \int_{A_m} \frac{\partial}{\partial \nu}(v_{mn}-v)\, ds \geqq 0,$$

where $\partial/\partial\nu$ denotes the inner normal differential operator with respect to R_{mn}. By letting $m \to \infty$ and $n \to \infty$, we get

$$\int_c dv_0^* \geqq \int_c dv^* = h(c)$$

and by the extremality

$$\int_c dv_0^* = h(c).$$

Set

$$\mu = \inf_{z\in c} (v_0(z) - v(z)).$$

To show $\mu = 0$, for every $\varepsilon > 0$ there exist an n and m such that $\mu - \varepsilon \leqq v_{mn} - v$ on c. Let ω_n be the harmonic measure of c with respect to the set of regions bounded by A_m and c. Then we have $(\mu-\varepsilon)\omega_m \leqq v_{mn} - v$ in each component, and as before

$$(\mu-\varepsilon)\int_c d\omega_m^* \leqq \int_c d(v_{mn}-v)^*.$$

Since ω_m is the extremal function of the harmonic length problem for each component of c in the respective region [2], we have

$$\int_c d\omega_m^* \geqq \int_c dv^* = h(c) > 0.$$

By lettting $m \to \infty$ and $n \to \infty$, we get $\mu - \varepsilon \leqq 0$ and thus $\mu = 0$. By the minimum principle we have $v = v_0$. Obviously $v = v_0 \geqq u_0$.

By taking the cycle $-c$ and an extremal $1 - v$, we have $1 - v \geqq 1 - u_0$ which implies $v = u_0$.

4. Analytic mappings. Let $f(z)$ be an analytic mapping of a Riemann surface R into a Riemann surface S and let c be a cycle on R. Then an inequality $h_R(c) \geqq h_S(f(c))$ holds in general [2], where h_R, h_S mean harmonic lengths on R, S respectively. Landau and Osserman obtained the following further result [3]: *Suppose that R is a Dirichlet domain and that c is homologous to a level locus of a harmonic measure. If $h_R(c) = h_S(f(c))$, f maps R k-to-1 onto S, for some integer k.*

We shall discuss this extremal case for an arbitrary Riemann surface and prove

THEOREM 2. *Let c be a dividing cycle relative to a regular partition. If $h_R(c)=h_S(f(c))>0$, then f covers S exactly k times except for a closed set of capacity zero, where k is an integer.*

5. To prove this theorem we shall need some of Heins' results about analytic mappings [1]. From the assumption $h_R(c))=h_S(f(c))>0$ we can deduce that both R and S have Green's functions G_R and G_S. We shall use the following Lindelöf principle [1].

$$G_S(f(z), \omega)=\sum_{\omega=f(\zeta)} n(\zeta; f)G_R(z, \zeta)+U_\omega(z), \tag{1}$$

where $U_\omega(z)$ is a positive harmonic function on R and $n(\zeta; f)$ is the order of f at ζ. Furthermore, U_ω has the following canonical decomposition

$$U_\omega(z)=V_\omega(z)+P_\omega(z), \tag{2}$$

where V_ω is quasi-bounded and P_ω is singular. It is known that either $V_\omega(z)>0$ for all $\omega\in S$ or else that $V_\omega=0$, $\omega\in S$. The function f is said to be *of type Bl* if $V_\omega=0$, $\omega\in S$. A remarkable property of a mapping f of type Bl is as follows: f covers S exactly the same number of times (may be infinite) except for an F_σ set of capacity zero and if the valence of f is finite, the exceptional set is closed. As to these facts, the readers will be referred to Heins [1].

6. Proof of Theorem 2. Let u_0 be the extremal function of the harmonic length problem for c, which is a harmonic measure from Theorem 1. Let v denote an extremal function for the cycle $f(c)$. Then the function $v\circ f$ is an extremal function for c. From Theorem 1, we get a functional equation $u_0=v\circ f$.

To show that f is of type Bl, we use an inequality from (1) and (2):

$$G_S(f(z), \omega)\geqq V_\omega(z).$$

Since V_ω is quasi-bounded, we have

$$V_\omega(z)=\lim_{N\to\infty}(V_\omega(z)\wedge N),$$

where $V_\omega\wedge N$ means the greatest harmonic minorant of $\min(V_\omega(z), N)$. We select so large N that the open set $O_N=\{w|G_S(w, \omega)>N\}$ is simply connected and relatively compact. Set

$$\mu_N=\inf_{w\in O_N} v(w), \qquad \mu'_N=\inf_{w\in O_N}(1-v(w))$$

and

$$K_N=\max\left(\frac{N}{\mu_N}, \frac{N}{\mu'_N}\right).$$

We have

$$K_N \min(v(w), 1-v(w))\geqq \min(G_S(w, \omega), N)\geqq \min(V_\omega(z), N)$$

with $w=f(z)$, $z\in R$ and by the equation $u_0=v\circ f$

$$K_N(u_0(z)\wedge(1-u_0(z))\geqq V_\omega(z)\wedge N.$$

It is well-known that $u_0\wedge(1-u_0)\equiv 0$ for any harmonic measure u_0. We get $V_\omega(z)=0$ and thus f is of type Bl.

7. Valence of f.[1] To investigate the valence of f, we consider the Dirichlet norms $||du_0||_R$, $||dv||_S$ of u_0, v. It is well-known that $||du_0||^2=h_R(c)$, since u_0 is the harmonic measure of B of a regular partition (A, B). Let $\nu_f(w)$ be the valence function of f on S. Then $\nu_f(w)\equiv$const except for at most an F_σ set of capacity zero, say E, by the properties of a mapping of type Bl mentioned in No. **5**. Let l be any positive integer not greater than the constant value of ν_f. We have

$$\int_R du_0\wedge du_0^*\geqq l\int_{S-E} dv\wedge dv^*.$$

Since E is of area zero, this in turn means $||dv||_S\leqq||du_0||_R$ and that ν_f is dominated by $||du_0||^2/||dv||^2$. Clearly $||dv||^2>0$ and thus f is of finite, valence, which completes the proof.

8. Conformal rigidity. In this section we shall discuss a property of a plane region referred to as "conformal rigidity". A plane region Ω is said to be *rigid* if every analytic mapping of Ω into itself is either one-to-one onto, or else takes some cycle not homologous to zero onto one which is homologous to zero [2]. By definition every non-compact simply-connected region is not rigid. We now show that a criterion of conformal rigidity of Dirichlet regions due to Landau and Osserman [2] is valid for arbitrary plane regions.

THEOREM 3. *Let Ω be a plane region. Suppose that there exists a (positive) minimum of the harmonic lengths of all dividing cycles c relative to regular partitions. Then Ω is rigid.*

Proof. Let c_0 be a dividing cycle which attains the minimum. It is known that $h(c_0)$ is the minimum of harmonic lengths of all cycles not homologous to zero in Ω [2]. Let $f(z)$ be au analytic mapping of Ω into itself. Then we have $h(c_0)=h(f(c_0))$, which implies, from Theorem 2, that f is m-to-1 onto itself except for at most a closed set of capacity zero. Let u_0 and v_0 be the extremal functions for c_0 and $f(c_0)$ respectively. We have also $v_0=u_0\circ f$ and similarly as in No. **7** $||dv_0||^2=m||du_0||^2$. It follows that $m=1$ since $||du_0||^2=h(c_0)$ and $||dv_0||^2=h(f(c_0))$. It suffices to show $f(\Omega)=\Omega$. Suppose $f(\Omega)\subsetneqq\Omega$ and set $E=\Omega-f(\Omega)$. Since E is totally disconneted, there exists a dividing cyle c relative to a regular partition (C, D) of $\partial f(\Omega)$ such that $C\subset E$ in the Stoilow compactification of $f(\Omega)$. The inverse image of c, denoted by c', is

1) The author is indebted to Mr. M. Sakai for this shorter proof than his original.

dividing cycle relative to the partition $(f^{-1}(C), f^{-1}(D))$, since f is conformal. We get $h(c')=0$ which is a contradiction to the existence of a positive minimum.

References

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