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NOTE ON THE LAW OF THE ITERATED LOGARITHM FOR STATIONARY PROCESSES SATISFYING MIXING CONDITIONS

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1. Strassen [6] presented a generalization of the law of the iterated logarithm for independent random sequences and Chover [1] gave a proof of Strassen's main result using Esseen's estimate for the central limit theorems. Recently Iosifescu [3] and Reznik [5] extended the classical law to some classes of strictly stationary processes satisfying mixing conditions, and in [4] the assumptions imposed in [3] and [5] have been considerably weakened. In this note we show that Strassen's version of the law of the iterated logarithm holds for the classes of stationary processes considered in [4]. We use Chover's approach [1] and our previous results [4].

2. Let $\{x_j\}$ be a strictly stationary process defined on a probability space $(\Omega, \mathcal{F}, P)$ with $Ex_j=0$, $Ex_j^2<\infty$, satisfying either uniform strong mixing (u.s.m.) condition:

$$\sup_{A\in\mathcal{M}_{-\infty}^{k},B\in\mathcal{M}_{k+n}^{\infty}} \frac{1}{P(A)}|P(A\cap B)-P(A)P(B)|=\varphi(n)\downarrow 0,\ n\to\infty,$$

or the strong mixing (s.m.) condition:

$$\sup_{A\in\mathcal{M}_{-\infty}^{k},\mathcal{M}B\in_{k+n}^{\infty}} |PA\cap B)-P(A)P(B)|=\alpha(n)\downarrow 0,\ n\to\infty,$$

where $\mathcal{M}_a^b$ denotes the σ-algebra generated by the random variales x_j, $j=a, a+1, \cdots, b$.

Let $S_0=0$, $S_n=\sum_{j=1}^{n}x_j$, $\sigma_n^2=ES_n^2$ and $\sigma^2=Ex_0^2+2\sum_{j=1}^{\infty}Ex_0x_j$, and assume that $0<\sigma^2<\infty$ and $\sigma_n^2=n\sigma^2(1+o(1))$.

Consider the space C of continuous functions on $[0, 1]$ vanishing at 0, with the usual maximum norm, and, for each $\omega\in\Omega$, define the functions $f_n(t, \omega)$, $n\geqq 3/\sigma^2$, in C as follows:

$$f_n(t,\omega)=\begin{cases} S_k(\omega)/\chi(n) & \text{for } t=k/n,\ k=0, 1, \cdots, n \\ \text{linearly interpolated} & \text{for } t\in[k/, (k+1)/n],\ k=0, \cdots, n-1, \end{cases}$$

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where $\chi(n)=(2n\sigma^2 \log\log n\sigma^2)^{1/2}$. We denote by K the subset of C consisting of all functions $h(t)$ absolutely continuous with respect to Lebesgue measure such that $\int_0^1 \dot{h}(t)^2 dt \leqq 1$, where $\dot{h}(t)$ stands for the Radon-Nikodym derivative of h. Further, for any integer m and function $h \in C$, let $\Pi_m h$ be the piecewise approximation to h defined by

$$(\Pi_m h)(t)=\begin{cases} h(\nu/m) & \text{for } t=\nu/m,\ \nu=0, 1, \cdots, m \\ \text{linearly interpolated} & \text{for } t\in[\nu/m, (\nu+1)/m],\ \nu=0, \cdots, m-1. \end{cases}$$

Our result is the following

THEOREM 1. *Suppose that* $\{x_j\}$ *is a strictly stationary process and satisfies one of the following conditions.*

Condition (I):

(I-1) $$\int_{|x|>N} x_0^2 dP = O((\log N)^{-5}) \quad \text{as } N\to\infty,$$

(I-2) *the u.s.m. condition with* $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$.

Condition (II):

(II-1) $E|x_j|^{2+\delta}<\infty$ *for some* $\delta>0$,

(II-2) *the u.s.m. condition with* $\varphi(n)=O(n^{-1-\varepsilon})$ *for some* $\varepsilon>(1+\delta)^{-1}$.

Condition (III):

(III-1) $|x_j|<$ *constant with probability one,*

(III-2) *the s.m. condition with* $\alpha(n)=O(n^{-1-\varepsilon})$ *for some* $\varepsilon>0$.

Condition (IV):

(IV-1) $E|x_j|^{2+\delta}<\infty$ *for some* $\delta>0$,

(IV-2) *the s.m. condition with* $\sum_{n=1}^{\infty}\{\alpha(n)\}^{\delta'/(2+\delta')}<\infty$ *for some* $0<\delta'<\delta$.

Then, for almost every $\omega\in\Omega$, *the sequence of functions* $\{f_n(t, \omega),\ n\geqq 3/\sigma^2\}$ *is precompact in C and its derived set is the set K.*

REMARK. As we shall see below, it is sufficient for the conclusion of Theorem 1 that the following requirements be fulfilled: for some $\rho>0$ and sufficiently large n,

(i) $P(\max_{1\leqq j\leqq n}|S_j|>b\chi(n))=O((\log n)^{-1-\rho})$ for any $b>1$

and either

(ii-1) $\sup_{-\infty<z<\infty}|P(S_n<z\sqrt{n})-\Phi(z)|=O((\log n)^{-1-\rho})$

and

(ii-2) $\sum \varphi(n) < \infty$

or

(iii-1) $\sup_{-\infty<z<\infty} |P(S_n < z\sigma\sqrt{n}) - \Phi(z)| = O((\log n)^{-2-\rho})$

and

(iii-2) $\sum \alpha(n) < \infty$,

where

$$\Phi(z) = (1/\sqrt{2\pi})\int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

Now, it is shown in [4] that (i) holds under any of Conditions (I)-(IV), (ii-1) under Conditions (I) or (II), and (iii-1) under Condition (IV). Also, by changing the argument used in [4] slightly, it can be shown that (iii-1) holds under Condition (III).

3. In view of the above remark, Theorem 1 follows from Theorems 2-5 below. The proofs can be carried out by the method of Chover, and hence we shall give only proofs of the points where some changes are required, referring to [1] for the rest.

THEOREM 2. *If* (i) *is satisfied, then, for almost every* $\omega \in \Omega$, *the sequence of functions* $\{f_n(t, \omega), n \geqq 3/\sigma^2\}$ *is equicontinuous.*

Proof. Only obvious change is needed in the proof of Theorem 2 in [1].

We note that Corollaries 1 and 2 of [1] can be carried over to the present case without any change.

THEOREM 3. *Suppose that* (ii-1) *and* (ii-2) *hold. Then, for almost every* $\omega \in \Omega$, *the derived set of* $\{f_n(t, \omega)\}$ *is contained in* K.

Proof. It suffices to show (see, [1]) that

$$\sum P(A_r) < \infty, \tag{1}$$

where

$$A_r = \left\{\omega \,\middle|\, (2\log\log n_r\sigma^2)\left\{m\sum_{\nu=0}^{m-1}\left[\Pi_m f_{n_r}\left(\frac{\nu+1}{m}, \omega\right) - \Pi_m f_{n_r}\left(\frac{\nu}{m}, \omega\right)\right]^2\right\} > (1+\varepsilon)^2(2\log\log n_r\sigma^2)\right\}$$

and $n_r=[c^r]$ with some suitably chosen $c=c(\varepsilon)>1$.

The increment of $\Pi_m f_{n_r}(t)$ over $[\nu/m, (\nu+1)/m]$ is given by

$$\Pi_m f_{n_r}\left(\frac{\nu+1}{m}\right)-\Pi_m f_{n_r}\left(\frac{\nu}{m}\right)=\{1/\chi(n_r)\}\sum_{k=i}^{j} x_k+y_{r,\nu},$$

where i is the smallest integer such that $i/n_r \geqq \nu/m$ and j is the largest integer such that $j/n_r<(\nu+1)/m$. Let

$$\begin{aligned}\xi_{r,\nu}&=(2m \log\log n_r\sigma^2)^{1/2}\left\{(1/\chi(n_r))\sum_{k=i}^{j} x_k+y_{r,\nu}\right\}\\&=\{1/(n_r/m)^{1/2}\sigma\}\sum_{k=i}^{j} x_k+(2m \log\log n_r\sigma^2)y_{r,\nu},\quad \nu=0, 1, \cdots, m-1.\end{aligned}$$

Let $N_{r,\nu}$ denote the number of summands of the first term, $j-i$, which is $\sim n_r/m$. Put $q_r=[N_{r,\nu}^{1-\beta}]$, with some $0<\beta<1$, and let

$$\eta_{r,\nu}=\{1/(N_{r,\nu}-q_r)^{1/2}\sigma\}\sum_{k=i}^{j-q_r} x_k,\quad \nu=0, 1, \cdots, m-1,$$

and

$$\zeta_{r,\nu}=\xi_{r,\nu}-\eta_{r,\nu},\quad \nu=0, 1, \cdots, m-1.$$

An easy calculation shows that $E|\zeta_{r,\nu}|^2=O(n_r^{-\beta})$, and hence

$$E\left|\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2-\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right|\leqq 2\sum_{\nu=0}^{m-1}E|\eta_{r,\nu}\cdot\zeta_{r,\nu}|+\sum_{\nu=0}^{m-1}E|\zeta_{r,\nu}|^2,$$

$$2\sum_{\nu=0}^{m-1}\{E|\eta_{r,\nu}|^2\}^{1/2}\{E|\zeta_{r,\nu}|^2\}^{1/2}+\sum_{\nu=0}^{m-1}E|\zeta_{r,\nu}|^2=O(n_r^{-\beta/2}).$$

Therefore, by Chebyshev's inequality, we have, for sufficiently large r,

$$\begin{aligned}P(A_r)&=P\left(\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2>(1+\varepsilon)^2(2\log\log n_r\sigma^2)\right)\\&\leqq P\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2>(1+\varepsilon)^2(2\log\log n_r\sigma^2)-n_r^{-\beta/4}\right)\\&\quad+P\left(\left|\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2-\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right|\geqq n_r^{-\beta/4}\right)\\&\leqq P\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2>(1+\varepsilon')(\log\log n_r\sigma^2)\right)+O(n_r^{-\beta/4}),\end{aligned}\tag{2}$$

where $\varepsilon'>0$ with $1+\varepsilon'<(1+\varepsilon)^2$.

Let, now, $\eta'_{r,\nu}$, $\nu=0, 1, \cdots, m-1$, be independent random variables distributed in the same way as $\eta_{r,\nu}$'s and also independent of $\eta_{r,\nu}$'s. Then we have (cf. Lemma 4, [3]),

$$(3)\qquad \sup_z \left|P\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\leqq z\right)-P\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}'^2\leqq z\right)\right|=(m-1)\varphi(q_r).$$

It follows easily from the assumption (iii-1) that

$$(4)\qquad \sup_z \left|P\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}'^2\leqq z\right)-\Psi_m(z)\right|=O((\log n_r)^{-1-\rho}),$$

where $\Psi_m(z)$ is the distribution function of the χ^2-distribution with m degree of freedom. (2)-(4) and (iii-2) together imply (1), completing the proof.

THEOREM 4. *The assumptions* (ii-1) *and* (ii-2) *of Theorem* 3 *can be replaced by* (iii-1) *and* (iii-2).

Proof. It is enough to prove that for some $\rho'>0$

$$(5)\qquad \sup_z \left|P\left(\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2\leqq z\right)-\Psi_m(z)\right|=O((\log n_r)^{-1-\rho'}).$$

In what follows K_i's will denote some positive constants. By a theorem of Esseen [2], we have

$$(6)\qquad \begin{aligned} &\left|P\left(\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2\leqq z\right)-\Psi_m(z)\right| \\ &\leqq K_1\int_{-T_r}^{T_r}\left|\frac{E\left(\exp\left(it\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2\right)\right)-(1-2it)^{-m/2}}{t}\right|dt+K_2/T_r \\ &=K_1(I_1+I_2+I_3)+K_2/T_r, \end{aligned}$$

where

$$I_1=\int_{-T_r}^{T_r}\left|\frac{E\left(\exp\left(it\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2\right)\right)-E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right)\right)}{t}\right|dt,$$

$$I_2=\int_{-T_r}^{T_r}\left|\frac{E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right)\right)-E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}'^2\right)\right)}{t}\right|dt,$$

$$I_3=\int_{-T_r}^{T_r}\left|\frac{\prod_{\nu=0}^{m-1}E(\exp(it\eta_{r,\nu}'^2))-(1-2it)^{-m/2}}{t}\right|dt,$$

and we put $T_r=(\log n_r)^{1+(\rho/4)}$. Firstly we note that

$$\left|E\left(\exp\left(it\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2\right)\right)-E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right)\right)\right|$$

$$\leqq |t|\cdot E\left|\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2-\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right|$$

$$=|t|\cdot O(n_r^{-\beta}),$$

and hence

$$I_1=o(n_r^{-\gamma}) \qquad \text{for some} \quad 0<\gamma<\beta. \tag{7}$$

Secondly, by (iii-2),

$$\left|E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right)\right)-E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}'^2\right)\right)\right|\leqq 16m\cdot\alpha(q_r),$$

and, for sufficiently small $|t|$,

$$\left|E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right)\right)-E\left(\exp\left(it\sum_{\nu=0}^{m-1}\eta_{r,\nu}'^2\right)\right)\right|$$

$$\leqq |t|\cdot\left\{E\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2\right)+E\left(\sum_{\nu=0}^{m-1}\eta_{r,\nu}'^2\right)\right\}$$

$$\leqq K_3\cdot|t|.$$

Hence we get, with any $\delta>0$ and for some $\varepsilon>0$,

$$\begin{aligned} I_2 &\leqq K_3\int_{0\leqq|t|\leqq n_r^{-\delta}}dt+16\cdot\alpha(q_r)\int_{n_r^{-\delta}\leqq|t|\leqq T_r}\frac{1}{|t|}dt \\ &=o(n_r^{-\varepsilon}). \end{aligned} \tag{8}$$

Thirdly we have

$$\left|\prod_{\nu=0}^{m-1}E(\exp(it\eta_{r,\nu}'^2))-(1-2it)^{-m/2}\right|$$

$$\leqq m\cdot|E(\exp(it\eta_{r,0}'^2))-(1-2it)^{-1/2}|$$

$$\leqq m\cdot\left\{\left|\int_0^{(\log n_r)^{\rho/4}}e^{itx}[dF_{\eta_{r,0}'^2}(x)-d\Psi_1(x)]\right|+\left|\int_{(\log n_r)^{\rho/4}}^{\infty}dF_{\eta_{r,0}'^2}(x)\right|+\left|\int_{(\log n_r)^{\rho/4}}^{\infty}d\Psi_1(x)\right|\right\}$$

$$\leqq m\cdot\left\{|F_{\eta_{r,0}'^2}((\log n_r)^{\rho/4})-\Psi_1((\log n_r)^{\rho/4})|+|F_{\eta_{r,0}'^2}(0)-\Psi_1(0)|\right.$$

$$\left.+|t|\int_0^{(\log n_r)^{\rho/4}}|F_{\eta_{r,0}'^2}(x)-\Psi_1(x)|\,dx+(1-F_{\eta_{r,0}'^2}((\log n_r)^{\rho/4}))+(1-\Psi_1((\log n_r)^{\rho/4})\right\}$$

by integration by parts, and so, using (iii-1) and noting that

$$1-\Psi_1((\log n_r)^{\rho/4})\leqq K_4(\log n_r)^{-\rho/8}\exp(-(\log n_r)^{\rho/4}/2),$$

we get

$$\left|\prod_{\nu=0}^{m-1}E(\exp(it\eta'^2_{r,\nu}))-(1-2it)^{-m/2}\right|$$

$$\leqq m\cdot|t|\cdot(\log n_r)^{\rho/4}\cdot O((\log n_r)^{-2-\rho})+O((\log n_r)^{-2-\rho}).$$

We have also, for sufficiently small $|t|$,

$$\left|\prod_{\nu=0}^{m-1}E(\exp(it\eta'^2_{r,r}))-(1-2it)^{-m/2}\right|$$

$$\leqq|t|\cdot m\cdot\{E\eta'^2_{r,0}+1\}\leqq K_5\cdot|t|.$$

Therefore,

$$\begin{aligned}I_3\leqq & K_5\int_{0\leqq|t|\leqq n_r^{-\delta}}dt\\ &+m(\log n_r)^{\rho/4}\cdot O((\log n_r)^{-2-\rho})\int_{n_r^{-\delta}\leqq|t|\leqq T_r}dt\\ &+O((\log n_r)^{-2-\rho})\int_{n_r^{-\delta}\leqq|t|\leqq T_r}\frac{1}{|t|}dt\\ =&O((\log n_r)^{-1-(\rho/2)}).\end{aligned}\tag{9}$$

(7)-(9), together with (6), yield (5) with $\rho'=\rho/4$, which concludes the proof.

THEOREM 5. *If* (ii-1) (*or* (iii-1)) *and* (iii-2) (*or* (ii-2)) *hold, then K is contained in the derived set of* $\{f_n(t,\omega)\}$.

Proof. We need only observe (see [1] for the notation) that $C_r^{(\nu)}$ and $C_{r+1}^{(\nu)}$ are separated from each other by at least $[m^r/2]$ and that under the assumption (iii-2), if $\sum_r P(C_r^{(\nu)})=\infty$, then $P(\limsup_r C_r^{(\nu)})=1$, which can be shown in the same manner as in the proof of Lemma 5 in [3].

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