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EXTREMAL SOLUTIONS OF $\Delta u=Pu$

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Given a Riemannian space or Riemann surface R and a function $P\geqq 0$, $P\not\equiv 0$ on R, we consider P-harmonic functions, that is, solutions of the partial differential equation $\Delta u=Pu$. In contrast with the case of harmonic functions, the P-harmonic Neumann's function p_0 and Green's function p_1 exist on every R. These functions play a fundamental role not only in abstract analysis on R, but also in fluid dynamics, heat conduction, and electro- and magnetostatics (see, e.g. Bergman-Schiffer [1]).

An important unsolved problem is to determine extremal properties of p_0 and p_1 and their linear combinations $\mu p_0+\nu p_1$. We shall give the solution as a special case of a general extremal theorem which will be established for the partial differential equation

$$\delta dTu+PTu=0. \tag{1}$$

Here T is an operator to be specified, d is the exterior derivative, and δ the coderivative. The identity operator T gives the P-harmonic equation $\Delta u=Pu$.

An essential part of our investigation is the extension of extremal properties from regular subregions to the entire space. In the process we establish some auxiliary results which are of interest in their own right.

Among the consequences of the extremal theorem for $\mu p_0+\nu p_1$ are simple characterizations of the class O_{PE} of Riemannian spaces R which do not carry P-harmonic functions with finite energy integrals.

Our presentation, given for Riemannian spaces, remains valid, mutatis mutandis, for Riemann surfaces.

§1. Extremal properties of principal solutions.

1. Examples of the operator T appearing in (1) are the Hodge star operator $*$, the exterior derivative d, and the coderivative δ defined by $\delta\varphi=(-1)^{np+n+1}*d*\varphi$ for differential p-forms φ. More generally, we choose for T various combinations of $*$ and d for which the existence of the functions to be considered is known. Several such operators were considered in Kawai-Sario [3], where extremal properties were obtained for solutions of various boundary value problems.

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In the present paper we allow the functions satisfying (1) to possess fundamental singularities, and apply the results to the case of the identity operator T. Since $\delta u=0$ for functions u (0-forms), the operator $\Delta=-(\delta d+d\delta)$ reduces to

$$\Delta=-\delta d=\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\frac{\partial u}{\partial x^j}\right)$$

and equation (1) becomes

$$\Delta u=Pu. \tag{2}$$

Here g_{ij} is the metric tensor of R, g is the determinant of (g_{ij}), and (g^{ij}) is the inverse matrix.

2. Given an open set G of a Riemannian space R, we denote by $S(G)$ the class of solutions of (1) on G.

Let $\zeta=\{\zeta_i\}$ be a finite set of points of R, $N=\cup N_i$ a union of disjoint open neighborhoods of the ζ_i, and $\alpha=\cup\alpha_i$ the boundary of a set $B=\cup B_i$ of geodesic balls B_i centered at the ζ_i with $\bar{B}\subset N$. A *fundamental singularity* σ at ζ is defined as

$$\sigma\in S(N-\zeta), \tag{3}$$

$$\lim_{\alpha\to\zeta}\int_\alpha *dT\sigma=-1. \tag{4}$$

We call a subregion Ω of R *regular* if it is relatively compact and has a C^∞-boundary $\beta=\beta(\Omega)$. Take an Ω which contains N, and consider solutions p_i $(i=0, 1)$ of (1) on $\Omega-\zeta$ satisfying the following conditions:

(a) $p_i\in C^1(\bar{\Omega}-\zeta)\cap S(\Omega-\zeta)$,

(b) $p_i-\sigma$ is extendable to a solution of (1) on N,

(c) $ndTp_0=0$ and $tTp_1=0$ on β,

where n and t stand for the normal and tangential components along β. For example, if T is the identity operator, then (c) becomes

$$\frac{\partial p_0}{\partial n}=0 \quad\text{and}\quad p_1=0 \quad\text{on}\quad \beta.$$

We call the p_i *principal solutions* of (1) on Ω with the singularity σ.

We shall consider operators T for which the existence of unique p_0 and p_1 is assured, and deduce extremal properties of linear combinations of p_0 and p_1. For real numbers μ and ν with $\lambda=\mu+\nu$, the function $p_{\mu\nu}=\mu p_0+\nu p_1$ has the singularity $\lambda\sigma$. For competing functions we choose all $p\in S(\Omega-\zeta)$ with the singularity $\lambda\sigma$ at ζ, and denote by e_i and e the regular parts of p_i and p on N:

$$p_i=\sigma+e_i,$$

$$p=\lambda\sigma+e.$$

We postulate the following conditions, trivially met in our applications:

$$\lim_{\alpha\to\zeta}\int_\alpha Te_i\wedge *dTe_j=\lim_{\alpha\to\zeta}\int_\alpha Te\wedge *dTe_i =\lim_{\alpha\to\zeta}\int_\alpha Te_i\wedge *dTe=0, \tag{5}$$

with $i, j=0, 1$. Let $\rho=\mu^2e_0-\nu^2e_1-(\mu-\nu)e$ and

$$A_{\mu\nu}(p)=\lim_{\alpha\to\zeta}\int_\alpha T\sigma\wedge *dT\rho-T\rho\wedge *dT\sigma.$$

THEOREM 1. *Among* $p\in S(\Omega-\zeta)$ *with the singularity* $\lambda\sigma$ *at* ζ, *the function* $p_{\mu\nu}$ *gives*

$$\min\left\{\int_\beta Tp\wedge *dTp+A_{\mu\nu}(p)\right\}=0.$$

Proof. For $p, q\in S(\Omega-B)$ we introduce the inner product:

$$\begin{aligned}[p, q]_{\Omega-B}&=(dTp, dTq)_{\Omega-B}+(PTp, Tq)_{\Omega-B}\\ &=(dTp, dTq)_{\Omega-B}-(Tp, \delta dTq)_{\Omega-B}\\ &=\int_{\beta-\alpha}Tp\wedge *dTq.\end{aligned}$$

We write $|||p|||^2_{\Omega-B}=[p, p]_{\Omega-B}$. Then

$$\begin{aligned}|||p-p_{\mu\nu}|||^2_{\Omega-B}&=\int_{\beta-\alpha}(Tp-Tp_{\mu\nu})\wedge *d(Tp-Tp_{\mu\nu})\\ &=\int_{\beta-\alpha}Tp\wedge *dTp-Tp_{\mu\nu}\wedge *dTp-Tp\wedge *dTp+Tp_{\mu\nu}\wedge *dTp_{\mu\nu}.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_\beta Tp_{\mu\nu}\wedge *dTp&=\int_\beta \mu Tp_0\wedge *dTp\\ &=\int_\beta \mu Tp_0\wedge *dTp-\mu Tp\wedge *dTp_0.\end{aligned}$$

By Green's formula the path of integration β can be replaced by α. Hence

$$\int_\beta Tp_{\mu\nu}\wedge *dTp=\int_\alpha \mu Tp_0\wedge *dTp-\mu Tp\wedge *dTp_0.$$

Therefore

$$\int_{\beta-\alpha} Tp_{\mu\nu}\wedge *dTp=\int_{\alpha}\mu Tp_0\wedge *dTp-\mu Tp\wedge *dTp_0-\int_{\alpha}(\mu Tp_0+\nu Tp_1)\wedge *dTp$$
$$=-\int_{\alpha}\nu Tp_1\wedge *dTp+\mu Tp\wedge *dTp_0.$$

Similarly

$$\int_{\beta-\alpha} Tp\wedge *dTd_{\mu\nu}=-\int_{\alpha}\nu Tp_1\wedge *dTp+\mu Tp\wedge *dTp_0,$$

and

$$\int_{\beta-\alpha} Tp_{\mu\nu}\wedge *d\,Tp_{\nu\mu}=-\int_{\alpha}\mu^2 Tp_0\wedge *Tp_0+\nu^2 Tp_1\wedge *dTp_1+2\mu\nu Tp_1\wedge *dTp_0.$$

Therefore

$$|||p-p_{\mu\nu}|||^2{}_{\Omega-B}=\int_{\beta} Tp\wedge *dTp+A_{\mu\nu}(p;\alpha),$$

where

$$A_{\mu\nu}(p;\alpha)=\int_{\alpha}-\mu^2 Tp_0\wedge *dTp_0-\nu^2 Tp_1\wedge *dTp_1$$
$$-2\mu\nu Tp_1\wedge *dTp_0+2\mu Tp\wedge *dTp_0+2\nu Tp_1\wedge *dTp-Tp\wedge *dTp.$$

On substituting in $A_{\mu\nu}(p;\alpha)$ the local expressions of $p_i=\sigma+e_i$ and $p=\lambda\sigma+e$, we have

$$A_{\mu\nu}(p;\alpha)=\int_{\alpha} T\sigma\wedge *dT\rho-T\rho\wedge *dT\sigma$$
$$+\int_{\alpha}-\mu^2 Te_0\wedge *dTe_0-2\mu\nu Te_1\wedge *dTe_0$$
$$-\nu^2 Te_1\wedge *dTe_1+2\mu Te\wedge *dTe_0+2\nu Te_1\wedge *dTe-Te\wedge *dTe,$$

where $\rho=\mu^2e_0-\nu^2e_1+(\nu-\mu)e$. As α shrinks to ζ, using (5) we obtain

$$\lim_{\alpha\to\zeta} A_{\mu\nu}(p;\alpha)=\lim_{\alpha\to\zeta}\int T\sigma\wedge *dT\rho-T\rho\wedge *dT\sigma.$$

This limit is denoted by $A_{\mu\nu}(p)$.

Since $|||p-p_{\mu\nu}|||^2{}_{\Omega-B}$ tends to $|||p-p_{\mu\nu}|||^2{}_{\Omega}$ as $\alpha\to\zeta$, we have the following explicit formula:

$$\int_{\beta} Tp\wedge *dTp+A_{\mu\nu}(p)=|||p-p_{\mu\nu}|||^2{}_{\Omega}. \tag{6}$$

We conclude that $p_{\mu\nu}$ minimizes the left-hand side of the expression, the minimum being 0, with the deviation from the minimum given by the right-hand side.

§ 2. *P*-harmonic principal functions on a regular region.

1. Hereafter we restrict our consideration to the case of the identity operator T, that is, to equation (2). We call the principal solutions p_i of (2) on Ω the *P-harmonic principal functions* on Ω.

In this section we shall show the existence of unique P-harmonic principal functions p_i on Ω with prescribed singularities. Toward this end, we use the linear operator method for P-harmonic functions given in Sario-Nakai [8]. For the reader's convenience we restate it here briefly.

2. Let R_0 be a regular subregion of a Riemannian space R with $\alpha=\partial R_0$. For an open set G of R, denote by $P(G)$ the class of P-harmonic functions on G. We consider an operator L which assigns to each $f\in C(\alpha)$ a unique $Lf\in P(R-\bar{R}_0)\cap C(R-R_0)$ such that

$$Lf|\alpha=f, \tag{7}$$

$$(\min_\alpha f)\cap 0\leqq Lf\leqq(\max_\alpha f)\cup 0, \tag{8}$$

$$L(cf)=cL(f), \qquad L(f_1+f_2)=L(f_1)+L(f_2). \tag{9}$$

The following existence theorem, due to Nakai [4] (cf. Sario-Nakai [8]), provides the key step in the linear operator method.

THEOREM 2. *Given $s\in P(R-\bar{R}_0)\cap C(R-R_0)$ and L as above. Then there exists a unique $u\in P(R)$ satisfying*

$$L(u-s|\alpha)=(u-s)|R-R_0. \tag{10}$$

Proof. We may assume $s|\alpha=0$, for otherwise we replace s by $s_0=s-Ls$ and have

$$\begin{aligned} u|R-R_0&=s_0|R-R_0+L(u-s_0|\alpha)\\ &=s|R-R_0-Ls|R-R_0+L(u|\alpha). \end{aligned}$$

Therefore $u|R-R_0=s|R-R_0+L(u-s|\alpha)$.

Take a regular subregion R_1 of R such that $R_1\supset\bar{R}_0$ and $P\not\equiv 0$ on R_1. Let $\alpha_1=\partial R_1$ and let K be the operator which assigns to each $f\in C(\alpha_1)$ the unique function $Kf\in P(R_1)\cap C(\bar{R}_1)$ with the boundary value f on α_1. Set $\lambda=s|\alpha_1$.

We may reduce equation (10) to the following equation on $C(\alpha_1)$:

$$\varphi-L(K\varphi|\alpha)|\alpha_1=\lambda. \tag{11}$$

If we obtain a solution φ for a given λ, we may define:

$$u_1=K\varphi, \qquad u_0=s+L(K\varphi|\alpha). \tag{12}$$

By (11),

$$u_1|\alpha_1 = \varphi \quad \text{and} \quad u_0|\alpha_1 = \lambda + L(K\varphi|\alpha)|\alpha_1 = \varphi;$$

$$u_1|\alpha = K\varphi|\alpha \quad \text{and} \quad u_0|\alpha = s|\alpha + L(K\varphi|\alpha)|\alpha = K\varphi|\alpha.$$

Hence u_1 and u_0 coincide on $\partial(\bar{R}_1 - R_0)$ and consequently on $\bar{R}_1 - R_0$. If we set $u|R_1 = u_1$ and $u|R - R_0 = u_0$, u is well-defined and $u \in P(R)$. Moreover, u satisfies (10): $u|R - R_0 = u_0 = s + L(u|\alpha)$. Thus it suffices to solve for the function φ in (11).

Let $\Lambda\varphi = L(K\varphi|\alpha)|\alpha_1$. Then Λ is a linear operator from $C(\alpha_1)$ into itself. Consider $C(\alpha_1)$ as a Banach space with $||\varphi||_\infty = \sup_{\alpha_1} |\varphi|$. Then property (8) and the maximum principle assure that $||\Lambda||_\infty = \sup(||\Lambda\varphi||_\infty / ||\varphi||_\infty)$ is bounded and actually dominated by 1. Since (11) is $(I - \Lambda)\varphi = \lambda$, φ is obtained as $\varphi = \sum_{n=0}^{\infty} \Lambda^n \lambda$ if $||\Lambda||_\infty < 1$.

To show $||\Lambda||_\infty < 1$, take $w = K1$, which belongs to $P(R_1) \cap C(\bar{R}_1)$ and has boundary value 1. Since $P \not\equiv 0$ on R_1 and $w \not\equiv 0$ on R_1, the maximum principle implies that $0 < q = \max_\alpha w < 1$. From $\pm\varphi \leqq ||\varphi||_\infty$, it follows that

$$\pm K\varphi \leqq ||\varphi||_\infty K1.$$

On α we have

$$\pm K\varphi|\alpha \leqq q||\varphi||_\infty,$$

hence

$$\pm L(K\varphi|\alpha) \leqq q||\varphi||_\infty L1.$$

Since $0 < L1 < 1$ on $R - \bar{R}_0$, $|L(K\varphi|\alpha)|\alpha_1| \leqq q||\varphi||_\infty$, that is, $||\Lambda\varphi|| \leqq q||\varphi||_\infty$. This means $||\Lambda||_\infty \leqq q$, which completes the proof.

From the above proof we can deduce a theorem which demonstrates the continuous dependence of u upon s.

THEOREM 3. *Let $Q = (1-q)^{-1}$. The principal function u of Theorem 2 satisfies*

$$||u||_{R_1} \leqq Q||s||_{\alpha_1}, \tag{13}$$

$$||u - s||_{R - \bar{R}_0} \leqq Q||s||_{\alpha_1}, \tag{14}$$

$$||(u' - s') - (u'' - s'')||_{R - \bar{R}_0} \leqq Q||s' - s''||_{\alpha_1}, \tag{15}$$

where u' and u'' are solutions corresponding to s' and s'' with the same operator L.

Proof. For (13), we note that

$$||u||_{R_1} \leqq ||u||_{\alpha_1} = ||\varphi||_{\alpha_1} \leqq \sum_{n=0}^{\infty} q^n ||\lambda||_\infty = (1-q)^{-1}||\lambda||_\infty.$$

Similarly (14) follows from

$$||u - s||_{R - R_0} \leqq ||L(u|\alpha)||_{R - R_0} \leqq ||u||_\alpha \leqq ||u||_{\alpha_1} \leqq (1-q)^{-1}||\lambda||_\infty.$$

Recall that $s'|\alpha=0$, $s''|\alpha=0$. Thus

$$(u'-s')-(u''-s'')|R-R_0=L(u'-u''|\alpha).$$

Denote by φ' and φ'' the solutions of (11) corresponding to $\lambda'=s'|\alpha_1$ and $\lambda''=s''|\alpha_1$. Since $|Lf|\leqq\max_\alpha|f|$,

$$\|(u'-s')-(u''-s'')\|_{R-R_0}\leqq\|L(u'-u'')\|_{R-R_0}\leqq\max_\alpha|u'-u''|$$
$$\leqq\max_{\alpha_1}|u'-u''|=\|\varphi'-\varphi''\|_{\alpha_1}.$$

From $\varphi'-\varphi''=\sum_{n=0}^\infty A^n(\lambda'-\lambda'')$,

$$\|\varphi'-\varphi''\|_\infty\leqq\sum q^n\|\lambda'-\lambda''\|_{\alpha_1}=\frac{1}{1-q}\|s'-s''\|_{\alpha_1}.$$

3. Let D be regular subregion of R with a partition of its boundary into components α, β.

We define operators L_0 and L_1 on $C(\alpha)$ which assign to each $f\in C(\alpha)$ a unique P-harmonic function $u_0=L_0f$ on D with $u_0|\alpha=f$ and $(\partial u_0/\partial n)|\beta=0$, and a unique P-harmonic function $u_1=L_1f$ on D with $u_1|\alpha=f$ and $u_1|\beta=0$, respectively. Since the existence of unique u_0 and u_1 is guaranteed by the solvability of the Dirichlet boundary value problem and the mixed boundary value problem for $\Delta u=Pu$ (see Itō [2]), the operators L_0 and L_1 are well-defined.

We claim that L_0 and L_1 are operators satisfying conditions (7), (8), and (9). Properties (7) and (9) are obvious; to show (8), we will use the following lemma.

LEMMA 1. *Let u be a P-harmonic function on a regular region D. Suppose that $u\leqq M$ on D, where M is a nonnegative constant, and that $u=M$ at a boundary point x_0. If u is continuous on $D\cup\{x_0\}$ and nonzero, the outward normal derivative $\partial u/\partial n$ is positive at x_0.*

The proof is given in Protter-Weinberger [6] for the case in which D is in a Euclidean space. Since it is not difficult to extend the proof to the present case we omit the details.

To show that L_0 meets (8), we suppose, to the contrary, that u_0 violates the condition. Hence there exists a point $x\in D$ such that $u(x)>(\max_\alpha f)\cup 0$. Since the nonnegative maximum of a P-harmonic function is attained only on the boundary, u_0 must take its nonnegative maximum at some boundary point x_0. Therefore $u(x_0)>u(x)>(\max_\alpha f)\cup 0$, and x_0 belong to β. Lemma 1 gives $\partial u_0/\partial n(x_0)>0$, in violation of the condition $(\partial u_0/\partial n)|\beta=0$.

The inequality $(\min_\alpha f)\cap 0\leqq u_0$ follows from the above argument, applied to $-u_0$.

For L_1, property (8) is immediate from the condition $u_1=0$ on β, and the P-harmonic maximum principle.

4. We are ready to construct p_0 and p_1 on Ω. Let σ, ζ, N, and Ω be as in §1. Take a subregion D of Ω such that $\Omega-\bar{D}$ is a regular region in Ω and

$\Omega-\bar{D}\supset\bar{N}$. Note that D is a boundary neighborhood of $\beta(\Omega)$, and let $\partial D=\alpha_1\cup\beta$.

Remove ζ from Ω and consider $\Omega-\zeta$ as R in **2**, and $\Omega-\bar{D}-\bar{N}$ as R_0. Then $\partial N\cup\alpha_1$ plays the role of α in **2**. We apply Theorem 2 to following s and $L: s=\sigma$ on $N-\zeta$ with $s=0$ on D; L is the operator acting on $C(\partial N\cup\alpha_1)$ such that L on $N-\zeta$ is the Dirichlet operator assigning to each $f\in C(\partial N)$ the solution of the Dirichlet problem on N, and L on D is the operator L_i $(i=0, 1)$ defined as in **3**.

From Theorem 2 we have p_i $(i=0, 1)$ on $\Omega-\zeta$ such that $p_i-\sigma$ on $N-\zeta$ is P-harmonically extendable to ζ, and $L_ip_i=p_i$ on D, that is, $(\partial p_0/\partial n)|\beta=0$ and $p_1|\beta=0$. Thus it is assured that p_0 and p_1 exist on Ω.

§ 3. Extremal theorom for *P*-harmenic principal functions on a regular region.

1. Application of Theorem 1 to the P-harmonic principal functions $p_{\mu\nu}=\mu p_0+\nu p_1$ yields the following result:

THEOREM 4. *Let* $A_{\mu\nu}(p)=\lim_{\alpha\to\zeta}\int_\alpha \sigma*d\rho-\rho*d\sigma$, *where* $\rho=\mu^2e_0-\nu^2e_1-(\mu-\nu)e$. *Among P-harmonic functions p on* $\Omega-\zeta$ *with singularity* $\lambda\sigma$ *at* ξ, *the function* $p_{\mu\nu}$ *gives*

$$\min\left\{\int_\beta p*dp+A_{\mu\nu}(p)\right\}=0.$$

2. We next seek a more explicit expression of $A_{\mu\nu}(p)$. Toward this end we reconsider the singularities σ.

Let g_i be the P-harmonic Green's function on N_i with pole at ζ_i, and ε_i any nonzero constant assigned to ζ_i. We redefine σ by setting $\sigma=\varepsilon_ig_i$ on N_i. This σ meets condition (3) but not (4). Thus we must enlarge the class of singularities by revising condition (4) so as not to affect the discussion thus far. This is accomplished by requiring simply the following:

$$\lim_{\alpha_i\to\zeta_i}\int_{\alpha_i}*dT\sigma \text{ is a finite constant at each } \zeta_i. \tag{16}$$

A σ which equals ε_ig_i on N_i is a singularity in this new sense. Hereafter we consider only σ of this type.

In view of

$$\lim_{\alpha_i\to\zeta_i}\int_{\alpha_i}*dg_i=-1$$

and

$$\lim_{\alpha_i\to\zeta_i}\int_{\alpha_i}g_i*df=0 \quad\text{for}\quad f\in C^1(N_i),$$

$$A_{\mu\nu}(p)=\lim_{\alpha_i\to\zeta_i}\int\sigma*d\rho-\rho*d\sigma=\sum_i\rho(\zeta_i)\varepsilon_i$$

$$=\sum_i\varepsilon_i(\mu^2e_0(\zeta_i)-\nu^2e_1(\zeta_i)-(\mu-\nu)e(\zeta_i)).$$

Hence we have an improved result:

THEOREM 5. *Among P-harmonic functions p on $\Omega-\zeta$ with singularity $\lambda\sigma$ such that $\sigma=\varepsilon_i g_i$ on N_ι, $p_{\mu\nu}$ minimizes the functional $\{\int_\beta p*dp-\sum_\iota \varepsilon_i(\mu-\nu)e(\zeta_i)\}$, and the minimum is $\sum_\iota \varepsilon_i(\nu^2 e_1(\zeta_i)-\mu^2 e_0(\zeta_\iota))$. Explicitly,*

$$\int_\beta p*dp-\sum_\iota \varepsilon_i(\mu-\nu)e(\zeta_i)=\sum_\iota \varepsilon_i(\nu^2 e_1(\zeta_i)-\mu^2 e_0(\zeta_i))+E_\Omega(p-p_{\mu\nu}), \tag{17}$$

where $E_\Omega(u)$ is the energy integral of the function $u\in C^1(\Omega)$.

Proof. The first part is deduced from Theorem 4, and the second part from (6), in view of $u\in C^1(\Omega)$, $|||u|||^2_\Omega=(du, du)_\Omega+(Pu, u)_\Omega=E_\Omega(u)$.

For $\mu=1$, $\nu=-1$, ζ consisting of one point $a\in\Omega$, and $\varepsilon=1$, we have:

COROLLARY. *Among regular P-harmonic functions u on Ω, the function p_0-p_1 gives* $\min\{E_\Omega(u)-2u(a)\}$, *and the minimum is $e_1(a)-e_0(a)$. Explicitly,*

$$E_\Omega(u)-2u(a)=e_1(a)-e_0(a)+E_\Omega(u-p_0+p_1). \tag{18}$$

Note that for regular P-harmonic functions u,

$$\int_\beta u*du=E_\Omega(u).$$

We remark that p_0 is the P-harmonic Neumann's function and p_1 is the P-harmonic Green's function on Ω with pole at a. We call the quantity $e_0(a)-e_1(a)$ the *P-span of* Ω and denote it by $S_P=S_P(\Omega, a)$.

The P-span has a significant relation to the norm of p_0-p_1:

$$S_P=E_\Omega(p_0-p_1).$$

This follows by setting $u\equiv 0$ in the Corollary.

§ 4. Convergence theorem.

1. In order to generalize Theorem 4 from a subregion $\Omega\subset R$ to the entire space R, we shall first establish a convergence theorem. This argument is analogous to the one in Sario-Schiffer-Glasner [9] which deals with harmonic functions and the Dirichlet norm.

LEMMA 2. *For P-harmonic functions u on Ω,*

$$u(a)^2\leqq S_P\cdot E_\Omega(u).$$

Proof. An application of (18) to tu, where t is a real parameter, yields

$$t^2E(u)-2tu(a)+E_\Omega(p_0-p_1)=E_\Omega(tu-p_0+p_1)\geqq 0.$$

Since the discriminant of the quadratic on the left must be nonpositive,

$$u(a)^2-E_\Omega(p_0-p_1)\cdot E_\Omega(u)\leqq 0.$$

2. Essential to the proof of the convergence theorem is the following boundedness property of the P-span.

LEMMA 3. *The P-span $S_P=e_0(a)-e_1(a)$, as a function of a, is uniformly bounded on compact subsets of Ω.*

Proof. We show that $S_P=e_0(a)-e_1(a)$ is a continuous function for a fixed Ω.

Denote by N_a a neighborhood of a and let $g_a(x)$ be the P-harmonic Green's function on N_a with pole at a. We change a slightly but keep the original N_a fixed. Let a' stand for a varying point near a and let $g_{a'}(x)$ be the P-harmonic Green's function corresponding to a'. Denote by p_i' and e_i' the principal functions and their regular parts corresponding to a'.

Take a hypersphere $\alpha_1\subset N_a$. By Theorem 3,

$$||(p_i'-g_{a'})-(p_i-g_a)||_{N_a}\leqq Q||g_{a'}(x)-g_a(x)||_{\alpha_1}. \tag{19}$$

Since $g_a(x)$ is jointly continuous in x and a for $x\neq a$, the compactness of α_1 in N_a implies that

$$\lim_{a'\to a}\max_{x\in\alpha_1}|g_{a'}(x)-g_a(x)|=0. \tag{20}$$

From (19), (20) and the continuity of the function e_i', we conclude that $e_i(a)$ is continuous. Hence S_P is continuous in a.

3. The following convergence theorem is the P-harmonic counterpart of the theorem asserting that a sequence of harmonic functions which is Cauchy in Dirichlet norm has a harmonic limit.

THEOREM 6. *Suppose that to each regular region Ω of a Riemannian space R a unique P-harmonic function u_Ω is assigned. If for $\Omega\subset\Omega'\subset R$,*

$$\lim_{\Omega,\Omega'\to R} E_\Omega(u_\Omega-u_{\Omega'})=0,$$

then u_Ω converges to a P-harmonic function on R, uniformly on compact sets.

Proof. Let Ω_0 be a regular region with $\Omega_0\subset\Omega$ and $S_P=S_P(\Omega_0, a)$. By Lemma 2,

$$(u_\Omega(a)-u_{\Omega'}(a))^2\leqq S_P\cdot E_{\Omega_0}(u_\Omega-u_{\Omega'}).$$

By Lemma 3 and the hypothesis, we see that $\{u_\Omega\}$ is uniformly Cauchy on compact subsets of Ω_0. In view of the completeness property of P-harmonic functions, $\{u_\Omega\}$ converges to a P-harmonic function, uniformly on compact subsets of Ω_0.

Since Ω_0 is arbitrary, the theorem is proved.

§ 5. Extension to the space R.

1. We have principal functions p_i on each regular subregion $\Omega\subset R$. We proceed to extend the p_i to the space R.

Fix ζ and N, and consider all regular regions Ω containg N. Let $\Omega\subset\Omega'$, and denote by p_i' and e_i' the functions corresponding to Ω'. For simplicity, we use the abbreviated notation $e(\zeta)=\sum_\iota \varepsilon_i e(\zeta_i)$, $e_0(\zeta)=\sum_\iota \varepsilon_i e_0(\zeta_\iota)$, $e_1(\zeta)=\sum_\iota \varepsilon_i e_1(\zeta_\iota)$.

Letting $p_{\mu\nu}$ in (17) equal first p_0 and then p_1, we obtain

$$\int_\beta p*dp-e(\zeta)=-e_0(\zeta)+E_\Omega(p-p_0), \tag{21}$$

$$\int_\beta p*dp+e(\zeta)=e_1(\zeta)+E_\Omega(p-p_1). \tag{22}$$

Substitution of $p_0'|\Omega$ for p in (21) and of $p_1'|\Omega$ for p in (22) yields

$$\int_\beta p_0'*dp_0'-e_0'(\zeta)=-e_0(\zeta)+E_\Omega(p_0'-p_0), \tag{23}$$

$$\int_\beta p_1'*dp_1'+e_1'(\zeta)=e_1(\zeta)+E_\Omega(p_1'-p_1), \tag{24}$$

while substitution of p_1 for p in (21) gives

$$\int_\beta p_1*dp_1-e_1(\zeta)=-e_0(\zeta)+E_\Omega(p-p_0). \tag{25}$$

Let $\beta'=\beta(\Omega')$. Since $\int_{\beta'} p_i'*dp_i'=0$ and $\int_{\beta'-\beta} p_i'*dp_i'=E_{\Omega'-\Omega}(p_i)\geqq 0$, we have $\int_\beta p_i'*dp_i'\leqq 0$. Consequently (23) implies $e_0'(\zeta)\leqq e_0(\zeta)$, and (24) yields $e_1'(\zeta)\geqq e_1(\zeta)$. Finally, $e_1(\zeta)\leqq e_0(\zeta)$ follows from (25).

We conclude that the quantity $e_0(\zeta)$ is decreasing and bounded below, whereas $e_1(\zeta)$ is increasing and bounded above. Hence these quantities converge as Ω tends to R.

2. We write p_i on Ω as $p_{i\Omega}$ and p_i' on Ω' as $p_{i\Omega'}$. From (23) and (24) we obtain

$$\lim_{\Omega,\Omega'\to R} E_\Omega(p_{i\Omega}-p_{i\Omega'})=0.$$

Thus by Theorem 6, there exist functions $p_i=\lim p_{i\Omega}$ $(i=0, 1)$, which are P-harmonic on $R-\zeta$. Set $p_{\mu\nu}=\mu p_0+\nu p_1$ for real numbers μ and ν.

We substitute p_0 for p in (21), and p_1 for p in (22) to obtain

$$\int_{\beta(\Omega)} p_0*dp_0-e_0(\zeta)=-e_{0\Omega}(\zeta)+E_\Omega(p_0-p_{0\Omega}), \tag{26}$$

$$\int_{\beta(\Omega)} p_1*dp_1+e_1(\zeta)=e_{1\Omega}(\zeta)+E_\Omega(p_1-p_{1\Omega}). \tag{27}$$

Since $\int_{\beta(\Omega)} p_i * dp_i \leqq 0$, we have on letting Ω tend to R in (26) and (27),

$$\lim_{\Omega\to R} E_\Omega(p_i - p_{i\Omega}) = 0.$$

Hence by the triangle inequality $\lim_{\Omega\to R} E_\Omega(p_{\mu\nu} - p_{\mu\nu\Omega}) = 0$. This and the definition $E(p-p_{\mu\nu}) = \lim_{\Omega\to R} E_\Omega(p-p_{\mu\nu})$ give

$$\lim_{\Omega\to R} E_\Omega(p-p_{\mu\nu\Omega}) = E(p-p_{\mu\nu}).$$

By (17),

$$\int_{\beta\ \Omega)} p * dp - \sum_i \varepsilon_i(\mu-\nu)e(\zeta_i) = \sum_i \varepsilon_i(\mu^2 e_{1\Omega}(\zeta_i) - \nu^2 e_{0\Omega}(\zeta_i)) + E_\Omega(p-p_{\mu\nu\Omega}),$$

and on letting Ω tend to R, we obtain the theorem stated below. We write symbolically

$$\int_\beta p * dp = \lim_{\Omega\to R} \int_{\beta(\Omega)} p * dp.$$

THEOREM 7. *Among P-harmonic functions p on $R-\zeta$ with singularity such that $\sigma=\varepsilon_i g_i$ on N_i, the function $p_{\mu\nu}$ minimizes the functional $\{\int_\beta p * dp - \sum_i \varepsilon_i(\mu-\nu)e(\zeta_i)\}$ and the minimum is $\sum \varepsilon_i(\nu^2 e_1(\zeta_i) - \mu^2 e_0(\zeta_i))$. Explicitly,*

$$\int_\beta p * dp - \sum_i \varepsilon_i(\mu-\nu)e(\zeta_i) = \sum_i \varepsilon_i(\nu^2 e_1(\zeta_i) - \mu^2 e_0(\zeta_i)) + E(p-p_{\mu\nu}). \tag{28}$$

In particular,

$$\int_\beta p_i * dp_i = 0 \qquad (i=0, 1). \tag{29}$$

§ 6. Applications.

1. In this final section we give some applications of the general extremal theorem and discuss a result in classification theory of Riemannian spaces.

We consider only one singular point denoted by a. In this case the corresponding functions p_0 and p_1, with the singularity g_a, are precisely the P-harmonic Neumann's and Green's functions on R, as remarked previously.

First take $\mu=1$, $\nu=-1$. Then the competing functions, which we denote by $\{u\}$, have no singularity. From Theorem 7 we immediately obtain:

THEOREM 8. *Among all regular P-harmonic functions u on R the function p_0-p_1 minimizes the functional $\{E(u)-2u(a)\}$, and the minimum is $e_1(a)-e_0(a)$. Explicitly,*

$$E(u) - 2u(a) = e_1(a) - e_0(a) + E(u-p_0+p_1). \tag{30}$$

We define the *P-span* $S_P=S_P(a)$ of R as $S_P=e_0(a)-e_1(a)$. The choice $u\equiv 0$ in

(30) gives the following:

COROLLARY. $S_P=E(p_0-p_1)$.

2. We next deduce an interesting extremal property of the P-harmonic Neumann's and Green's functions.

In (28), we take $\mu=1$, $\nu=0$, and then $\mu=0$, $\nu=1$:

$$\int_\beta p*dp-e(a)=-e_0(a)+E(p-p_0),$$

$$\int_\beta p*dp+e(a)=e_1(a)+E(p-p_1).$$

If the competing functions p are restricted to those with $\int_\beta p*dp\leqq 0$, we then have:

THEOREM 9. *Among P-harmonic functions p with singularity g_a and such that $\int_\beta p*dp\leqq 0$, the functon $e(a)$ is maximized by the regular part of the Neumann's function p_0 and minimized by that of the Green's function p_1.*

Since $e_0(a)=\max e(a)$ and $e_1(a)=\min e(a)$, we infer:

COROLLARY. $S_P=\max e(a)-\min e(a)$.

3. Take $\mu=\nu=1/2$. Then $p_3=(1/2)(p_0+p_1)$ has the same singularity as p_0 and p_1.

THEOREM 10. *Among P-harmonic functions p with singularity g_a, the function $(1/2)(p_0+p_1)$ minimizes $\int_\beta p*dp$, and the minimum is $(1/4)(e_1(a)-e_0(a))$. Explicitly,*

$$\int_\beta p*dp=\frac{1}{4}(e_1(a)-e_0(a))+E\left(p-\frac{1}{2}(p_0+p_1)\right).$$

4. The P-span of R turns out to be useful in the classification theory of Riemannian spaces:

THEOREM 11. *A Riemannian space $R\in O_{PE}$ if and only if $S_P=0$ for some $a\in R$.*

Proof. We note that $p_0-p_1\in PE(R)$. Suppose $R\notin O_{PE}$. Then there exists a nonzero function $u\in PE(R)$ and thus a point $a\in R$ with $u(a)\neq 0$. If $S_P=0$, then $E(p_0-p_1)=0$. Hence $p_0-p_1=0$. By (30), $E(u)-2u(a)=E(u)$ implies $u(a)=0$, a contradiction.

Suppose $S_P\neq 0$ for some point in R. Then $E(p_0-p_1)\neq 0$ and thus $p_0-p_1\in PE(R)$ is not zero. This completes the proof.

Since the vanishing of S_P at a point a is equivalent to $p_0=p_1$, we may characterize O_{PE} as follows (for Riemann surfaces see Ozawa [5]):

COROLLARY. *A Riemannian space $R\in O_{PE}$ if and only if the P-harmonic Green's and Neumann's functions on R coincide.*

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