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ON INTRINSIC STRUCTURES SIMILAR TO THOSE INDUCED ON S^{2n}

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1. In [1] Yano and the authors studied submanifolds of codimension 2 of almost complex manifolds and hypersurfaces of almost contact manifolds. In both cases the structure on the ambient space induced the same structure on the submanifold. The induced structure consists of a tensor field f of type (1,1), vector fields E, A, 1-forms η, α and a function λ satisfying

$$
\begin{aligned}
&f^2=-I+\eta\otimes E+\alpha\otimes A,\\
&\eta\circ f=\lambda\alpha, \qquad \alpha\circ f=-\lambda\eta,\\
&fE=-\lambda A, \qquad fA=\lambda E,\\
&\eta(E)=1-\lambda^2, \qquad \alpha(E)=0,\\
&\eta(A)=0, \qquad \alpha(A)=1-\lambda^2.
\end{aligned}
\tag{1}
$$

Moreover the metric g induced from a metric compatible with the structure on the ambient space satisfies

$$
\begin{aligned}
&g(X,E)=\eta(X), \qquad g(X,A)=\alpha(X),\\
&g(fX,fY)=g(X,Y)-\eta(X)\eta(Y)-\alpha(X)\alpha(Y).
\end{aligned}
\tag{2}
$$

It is well known that on an almost complex manifold or an almost contact manifold there exists a metric compatible with the given structure, i.e. we have an almost Hermitian structure or an almost contact metric structure. However given a $2n$-dimensional manifold M^{2n} with tensors $(f,E,A,\eta,\alpha,\lambda)$ satisfying equations (1), we show in section 2 that there does not in general exist a Riemannian metric on M^{2n} satisfying equations (2). Thus to study manifolds with an intrinsically defined $(f,E,A,\eta,\alpha,\lambda)$-structure from the standpoint of Riemannian geometry it is necessary to assume the existence of a Riemannian metric satisfying equations (2).

The even-dimensional spheres are clearly examples of manifolds with an $(f,E,A,\eta,\alpha,\lambda)$-structure and a compatible metric g, the structure being induced from the natural structure on the ambient Euclidean space. If ∇ denotes the Riemannian connexion of g, then for the sphere example the structure tensors satisfy

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$$\begin{aligned} &\nabla_X E = -fX, \qquad \nabla_X A = -\lambda X, \\ &(\nabla_X f)Y = -\eta(Y)X + g(X, Y)E \end{aligned} \tag{3}$$

for any vector fields X, Y on the sphere [1]. Note also that from equations (1) and (2), $g(A, A) = g(E, E) = 1 - \lambda^2$.

In [3] Yano and Okumura obtained some characterizations of even-dimensional spheres by imposing some conditions on the tensors $f, E, A, \eta, \alpha, \lambda$. Here in section 3 we study the role that the equations (3) play in characterizing spheres.

THEOREM 3. 2. *Let M^n be a compact Riemannian manifold (of any dimension $n \geqq 2$) admitting a vector field A and a non-constant function λ satisfying*

$$\nabla_X A = -\lambda X, \qquad g(A, A) = 1 - \lambda^2$$

for every vector field X. Then M^n is globally isometric to the unit sphere in R^{n+1}.

THEOREM 3. 3. *Let M^{2n} be an even-dimensional manifold with an $(f, E, A, \eta, \alpha, \lambda)$-structure and compatible metric g satisfying*

$$\lambda \textit{ non-constant}, \ \nabla_X E = -fX,$$

$$(\nabla_X f)Y = -\eta(Y)X + g(X, Y)E.$$

Then $\nabla_X A = -\lambda X$; in particular if M^{2n} is compact it is globally isometric to the unit sphere in R^{2n+1}.

2. Let M be an almost complex manifold with almost complex structure J and let Z be a vector field on M that is not the zero vector field. Let $\tilde{M} = M \times R^2$, where R is the real line. Define a tensor f of type (1,1), vector fields E and A, and 1-forms η and α on $\tilde{M}$ in the following way:

$$\begin{aligned} f(X, t, s) &= (-JX - sZ, s, -t), \\ E &= (Z, 0, 0), \\ A &= (JZ, 0, 0), \\ \eta(X, t, s) &= t, \\ \alpha(X, t, s) &= s \end{aligned} \tag{4}$$

where X is any vector field on M and $t, s \in R$. Then we have that

$$\begin{aligned} f^2(X, t, s) &= f(-JX - sZ, s, -t) \\ &= (J^2X + sJZ + tZ, -t, -s) \\ &= -(X, t, s) + s(JZ, 0, 0) + t(Z, 0, 0) \\ &= -(X, t, s) + \eta(X, t, s)E + \alpha(X, t, s)A \end{aligned}$$

and hence $f^2=-I+\eta\otimes E+\alpha\otimes A$. Also, we see that $\eta(E)=\eta(A)=\alpha(E)=\alpha(A)=0$, $f(E)=f(Z,0,0)=(-JZ,0,0)=-A$, $f(A)=f(JZ,0,0)=(-J^2Z,0,0)=E$, $\eta\circ f(X,t,s)=\eta(-X-sZ,s,-t)=s=\alpha(X,t,s)$ and $\alpha\circ f(X,t,s)=\alpha(-X-sZ,s,-t)=-t=-\eta(X,t,s)$. Thus, (4) gives an $(f,E,A,\eta,\alpha,\lambda)$-structure on $\tilde{M}$ with $\lambda=1$. If there exists a Riemannian metric $\tilde{g}$ on $\tilde{M}$ satisfying (2), then we have that $\tilde{g}(E,E)=\eta(E)=0$, contradicting the fact that Z is *not* the zero vector and hence E is not the zero vector.

3. The proof of Theorem 3. 2 is by means of a well known result of Obata [2] which states that a compact Riemannian manifold M^n admits a non-trivial solution λ of $(D_s d\lambda)(X,Y)=-k\lambda g(X,Y)$ for some real number $k>0$ if and only if M^n is globally isometric to a Euclidean sphere of radius $1/\sqrt{k}$. Here D_s denotes the symmetric covariant derivative, for example for a 1-form θ,

$$(D_s\theta)(X,Y)=\frac{1}{2}((\nabla_X\theta)(Y)+(\nabla_Y\theta)(X)).$$

LEMMA 3. 1. *Let M^n be a Riemannian manifold admitting a vector field A and a non-constant function λ satisfying $\nabla_X A=-\lambda X$, $g(A,A)=1-\lambda^2$. Let $\alpha(X)=g(X,A)$, then $\alpha(X)=X\lambda$.*

Proof. $\nabla_X\lambda A=(X\lambda)A-\lambda^2 X$, therefore

$$g(\nabla_X\lambda A,A)=(X\lambda)(1-\lambda^2)-\lambda^2\alpha(X).$$

On the other hand

$$g(\nabla_X\lambda A,A)=-g(\lambda A,\nabla_X A)+Xg(\lambda A,A)$$
$$=\lambda^2\alpha(X)+(X\lambda)(1-\lambda^2)+\lambda(-2\lambda X\lambda).$$

Comparing we have $2\lambda^2\alpha(X)=2\lambda^2 X\lambda$ and hence $\alpha(X)=X\lambda$ for $\lambda\neq 0$. Let $\varphi(m)=(\alpha(X)-X\lambda)(m)$, $m\in M^n$ and suppose $\varphi(m)\neq 0$. Then there exists a neighborhood of m on which φ is non-zero. Therefore $\lambda=0$ on this neighborhood contradicting the non-constancy of λ.

THEOREM 3. 2. *Let M^n be a compact Riemannian manifold admitting a vector field A and a non-constant function λ satisfying*

$$\nabla_X A=-\lambda X, \qquad g(A,A)=1-\lambda^2.$$

Then M^n is globally isometric to the unit sphere in R^{n+1}.

Proof. Using the Lemma and the result of Obata the proof is a short computation, the first equality holding since $d\lambda$ is an exact form.

$$\begin{aligned}(D_s d\lambda)(X, Y) &= X(Y\lambda)-(\nabla_X Y)\lambda \\ &= X\alpha(Y)-\alpha(\nabla_X Y) \\ &= Xg(Y, A)-g(\nabla_X Y, A) \\ &= g(Y, \nabla_X A) \\ &= -\lambda g(X, Y).\end{aligned}$$

THEOREM 3. 3. *Let M^{2n} be a manifold with an $(f, E, A, \eta, \alpha, \lambda)$-structure and compatible metric g satisfying*

$$\lambda \text{ non-constant}, \quad \nabla_X E = -fX,$$

$$(\nabla_X f)Y = -\eta(Y)X + g(X, Y)E.$$

Then $\nabla_X A = -\lambda X$; in particular if M^{2n} is compact it is globally isometric to the unit sphere in R^{2n+1}.

Proof. We first show that $\alpha(X)=X\lambda$. Since $g(E, E)=1-\lambda^2$ we have $2g(\nabla_X E, E) = -2\lambda X\lambda$ and hence $\eta(-fX) = -\lambda X\lambda$ so that by equations (1) $-\lambda\alpha(X) = -\lambda X\lambda$. Now proceeding as in the proof of Lemma 3. 1 we have $\alpha(X)=X\lambda$. Thus, $\nabla_X \lambda A = \alpha(X)A + \lambda\nabla_X A$, while on the other hand

$$\begin{aligned}\nabla_X \lambda A &= -\nabla_X fE = -f\nabla_X E - (\nabla_X f)E \\ &= f^2 X + \eta(E)X - g(X, E)E \\ &= -X + \alpha(X)A + (1-\lambda^2)X.\end{aligned}$$

Therefore $\lambda\nabla_X A = -\lambda^2 X$ and $\nabla_X A = -\lambda X$ for $\lambda \neq 0$. Now set $V = \nabla_X A + \lambda X$ and suppose $V(m) \neq 0$ for some $m \in M^{2n}$. Then there exists a neighborhood of m on which $V \neq 0$ and hence $\lambda = 0$, contradicting the non-constancy of λ. Thus $\nabla_X A = -\lambda X$ and the second statement follows from Theorem 3. 2.

REMARK. The normality of an $(f, E, A, \eta, \alpha, \lambda)$-structure has been defined and studied in [1] and [3]. In particular, Yano and Okumura [3] have shown that if M is a complete manifold with a normal $(f, E, A, \eta, \alpha, \lambda)$ metric structure such that $\lambda(1-\lambda^2)$ is almost everywhere non-zero and $\nabla_X E = fX$ then M is isometric to a sphere. It can easily be seen that Theorem 3. 3 implies this theorem.

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