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INVARIANT IDEALS FOR AMENABLE SEMIGROUPS OF MARKOV OPERATORS

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1. Introduction.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach algebra of continuous real or complex valued functions on X, with supremum norm. We denote by $C(X)^*$ the strong dual of Banach space $C(X)$. A *Markov operator* on $C(X)$ is a continuous linear mapping of $C(X)$ into itself such that $Te=e$ and $Tf\geqq 0$ whenever $f\geqq 0$, where e denotes the constant 1 function on X. Let Σ be an amenable semigroup of Markov operators T of $C(X)$ into itself. Some properties on invariant ideals have been investigated by Schaefer [5], [6] and Sine [7] for the case when Σ is the semigroup generated by a single Markov operator T. These results can be extended in obvious way to an amenable semigroup of Markov operators on $C(X)$. For example, we can extend the notion of ergodicity of Markov operator T on $C(X)$, defined for the case of the semigroup generated by T in [5]; that is, an amenable semigroup $\Sigma=\{T\}$ is *ergodic* if and only if for each $f\in C(X)$, the convex closure $\overline{co}\{Tf: T\in\Sigma\}$ of $\{Tf: T\in\Sigma\}$ contains an invariant function g for all $T\in\Sigma$. In fact, this invariant function is unique in $\overline{co}\{Tf: T\in\Sigma\}$. Thus, in this paper, we generalize main results in [5] and [7] by a modification of their methods; that is, in § 2 we give a representation theorem for maximal ideals invariant under each element T of an amenable semigroup Σ. In § 3 we prove that an amenable semigroup $\Sigma=\{T\}$ is ergodic if and only if invariant functions under every T in $\Sigma=\{T\}$ separate invariant probabirities under every adjoint operator T^* of T in Σ and then prove the bijective correspondence, for ergodic amenable semigroup $\Sigma=\{T\}$, between the family of maximal ideals invariant under every T in Σ and the set of the extreme points of the set of probabilities on X invariant under every T^*.

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2. Representation theorem.

Let Σ be an abstract semigroup and $m(\Sigma)$ be the space of all bounded real

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valued functions of Σ, with supremum norm. An element $\mu \in m(\Sigma)^*$ (the dual space of $m(\Sigma)$) is said to be a *mean* on $m(\Sigma)$ if $\mu(e)=\|\mu\|=1$. A mean μ is *left* [*right*] *invariant* if $\mu(l_s f)=\mu(f)$ [$\mu(r_s f)=\mu(f)$] for all $f \in m(\Sigma)$ and $s \in \Sigma$, where the left [right] translation l_s [r_s] of $m(\Sigma)$ by s given by $(l_s f)(s')=f(ss')$ [$(r_s f)(s')=f(s's)$]. An *invariant mean* is a left and right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called *left amenable* [*right amenable*]. A semigroup that has a invariant mean is called *amenable*. The following Lemma is essentially contained in Day's fixed point theorem [2]:

LEMMA 1. *Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators on $C(X)$, then there exists $\phi \in C(X)^*$ such that $\|\phi\|=1$, $\phi \geqq 0$ and $T^*\phi=\phi$ for each $T \in \Sigma$, where T^* is the adjoint of T.*

Proof. $K=\{\psi \in C(X)^* : \|\psi\|=1, \psi \geqq 0\}$ is a compact and convex set. Since $\Sigma^*=\{T^*\}$ is an amenable semigroup of affine weakly*-continuous mappings of K into itself, from Day's fixed point theorem, there exists $\phi \in C(X)^*$ such that $\|\phi\|=1$, $\phi \geqq 0$ and $T^*\phi=\phi$ for all $T \in \Sigma$.

Under a modification of Schaefer [5], we obtain the following two Definitions and two Lemmas.

DEFINITION 1. Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators on $C(X)$. A $\{T\}$-*ideal* is a closed proper ideal in $C(X)$ which is invariant under each $T \in \Sigma$. A $\{T\}$-ideal is said to be *maximal* if it is not properly contained in any other $\{T\}$-ideal.

DEFINITION 2. $\Sigma=\{T\}$ is said to be *irreducible* if there exist no $\{T\}$-ideals distinct from (O).

LEMMA 2. *$\Sigma=\{T\}$ possesses at least one maximal $\{T\}$-ideal, and each $\{T\}$-ideal is contained in some maximal $\{T\}$-ideal.*

LEMMA 3. *Let J be a $\{T\}$-ideal and denote by q the canonical mapping of $C(X)$ onto $C(X)/J$. Then $I \to q(I)$ is a bijective map of the set of all $\{T\}$-ideals containing J onto the set of all $\{T_J\}$-ideals, where T_J is an operator induced by T on $C(X)/J$. A $\{T\}$-ideal I is maximal if and only if $\{T_J\}$ is irreducible.*

Since the above two Lemmas are clear, we do not give the proofs. If ϕ is an element of $C(X)^*$, I_ϕ denotes the ideal $\{f \in C(X) : \phi(|f|)=0\}$, where $|f|(x)=|f(x)|$ for $x \in X$.

THEOREM 1. *Let $\Sigma=\{T\}$ be an amenable semigroup of Markov operators on $C(X)$ and let I be a maximal $\{T\}$-ideal, then there exists a nomalized positive* *measure* $\phi \in C(X)^*$ *such that* $I=I_\phi$ *and* $T^*\phi=\phi$ *for all* $T\in\Sigma$.

Proof. Since $\{T_I\}$ is the amenable semigroup of Markov operators on $C(S_I)$ where S_I denotes the support of I, it follows from Lemma 1 that there exists a nomalized positive measure $\hat{\phi}\in C(S_I)^*$ such that $T_I^*\hat{\phi}=\hat{\phi}$ for $T\in\Sigma$. Now, since I is maximal and hence $\{T_I\}$ is irreducible, $I_{\hat{\phi}}=(O)$. Therefore, if q denotes the canonical mapping of $C(X)$ onto $C(X)/I$, $\phi=\hat{\phi}\circ q$ is a positive measure on X such that $I=I_\phi$. The fact $T^*\phi=\phi$ for $T\in\Sigma$ follows from that for all $f\in C(X)$

$$T^*\phi(f)=\phi(Tf)=\hat{\phi}\circ q(f)=\hat{\phi}(Tf+I)$$

$$=\hat{\phi}(T_J(f+I))=T_J^*\hat{\phi}(f+I)$$

$$=\hat{\phi}(f+I)=\hat{\phi}\circ q(f)=\phi(f).$$

3. Ergodic amenable semigroup of Markov operators.

Schaefer in [5] defined that a bounded operator T on a Banach space E is called *ergodic* if for each $x\in E$, the convex closure $K(x)$ of the orbit $(x, Tx, T^2x, \cdots)$ contains a fixed vector x_0 of T. We extend this and give the following Definition.

DEFINITION 3. Let $\Sigma=\{T\}$ be a semigroup of bounded operators on $C(X)$. $\Sigma=\{T\}$ is said to be *ergodic* if for each $f\in C(X)$, the convex closure $\overline{co}\{Tf: T\in\Sigma\}$ of $\{Tf: T\in\Sigma\}$ contains an invariant function g for all $T\in\Sigma$.

It is known (e.g. [2]) that if a semigroup $\Sigma=\{T\}$ is amenable and $\{Tf: T\in\Sigma\}$ weakly compact then $\Sigma=\{T\}$ is ergodic.

THEOREM 2. *Let* $\Sigma=\{T\}$ *be an ergodic and amenable semigroup of Markov operators on* $C(X)$. *Then there exists a positive projection* P *from* $C(X)$ *onto closed subspace*

$$F=\{f\in C(X): Tf=f \textit{ for each } T\in\Sigma\}$$

such that $Pe=e$ *and* $PT=TP=P$ *for all* $T\in\Sigma$.

Proof. Since each $T\in\Sigma$ transforms real function in $C(X)$ to real one, for the present purpose, we can restrict the domain of each $T\in\Sigma$ to real function in $C(X)$. Since $\Sigma=\{T\}$ is ergodic, there exists $g\in\overline{co}\{Tf: T\in\Sigma\}$ such that $Tg=g$ for all $T\in\Sigma$. For $\varepsilon>0$, there exists $T_1, T_2, \cdots, T_n\in\Sigma$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $\alpha_i>0$, $\Sigma\alpha_i=1$ such that

$$||g-\Sigma\alpha_i T_i f||<\varepsilon$$

and we have

$$|g(x)-\Sigma\alpha_i TT_i f(x)|<\varepsilon$$

for $x\in X$ and $T\in\Sigma$.

If μ is an invariant mean on $m(\Sigma)$ and we denote $\mu_T(h(T))=\mu(h)$ where $h\in m(\Sigma)$, we obtain

$$\begin{aligned}\varepsilon &> \sup_T |g(x)-\Sigma\alpha_i TT_i f(x)|\\ &\geqq |\mu_T(g(x)-\Sigma\alpha_i TT_i f(x))|\\ &=|g(x)-\Sigma\alpha_i\mu_T(TT_i f(x))|\\ &=|g(x)-\Sigma\alpha_i\mu_T(Tf(x))|\\ &=|g(x)-\mu_T(Tf(x))|.\end{aligned}$$

Therefore, $g(x)=\mu_T(Tf(x))$ for each $x\in X$.

Defining $(Pf)(x)=\mu_T(Tf(x))$, we obtain Theorem. In fact, for $T_0\in\Sigma$,

$$PT_0 f(x)=\mu_T(TT_0 f(x))=\mu_T(Tf(x))=Pf(x)$$

and hence $PT_0=P$.

We call the above projection $\{T\}$-*projection.*

COROLLARY 1. *Let P be $\{T\}$-projection, then P^* is the mapping of $C(X)^*$ onto $\{\phi\in C(X)^*: T^*\phi=\phi$ for each $T\in\Sigma\}$. Moreover, $\phi\in C(X)^*$ is invariant under each $T^*\in\Sigma^*$ if and only if it is invariant under P^*.*

Proof. Since $PT=TP=P$ for each $T\in\Sigma$ and $Pf\in\overline{co}\{Tf: T\in\Sigma\}$ for each $f\in C(X)$, we obtain Corollary.

We recall that K is the set of positive nomalized elements of $C(X)^*$ and F is the set of functions invariant under each $T\in\Sigma$. An element of K is called *probability measure.*

COROLLARY 2. *Let P be $\{T\}$-projection and let*

$$\Phi=\{\phi\in K: T^*\phi=\phi \text{ for each } T\in\Sigma\}.$$

Then, for distinct elements $\phi,\psi\in\Phi$, there exists an invariant function $g\in F$ such that $\phi(g)\neq\psi(g)$.

Proof. If $\phi \neq \psi$, there exists $f \in C(X)$ such that $\phi(f) \neq \psi(f)$. From

$$\phi(Pf) = P^*\phi(f) = \phi(f) \neq \psi(f) = P^*\psi(f) = \psi(Pf),$$

if $Pf = g$, we obtain Corollary.

We can prove the converse of Corollary 2.

THEOREM 3. *Let $\Sigma = \{T\}$ be an amenable semigroup of Markov operators $C(X)$. Suppose that for distinct elements $\phi, \psi \in \Phi$, there exists an invariant function $f \in F$ such that $\phi(f) \neq \psi(f)$. Then $\Sigma = \{T\}$ is ergodic.*

Proof. For $x \in X$, δ_x denotes the point measure at x. The set $\{T^*\delta_x : T \in \Sigma\}$ is invariant under each $T_0^* \in \Sigma^*$ and so is the weak*-closed convex hull $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$ of $\{T^*\delta_x : T \in \Sigma\}$. Since $\Sigma^* = \{T^*\}$ is amenable, by Day's fixed point theorem [2], $\Sigma^* = \{T^*\}$ has an invariant probability measure ϕ_x in $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$. That the invariant probability measure ϕ_x is unique in $w^*\overline{co}\{T^*\delta_x : T \in \Sigma\}$ is clear from $T^*\delta_x(f) = \delta_x(Tf) = \delta_x(f) = f(x)$ for $f \in F$ and that invariant functions separate invariant probability measures. The weak*-continuity of the mapping $x \to \phi_x$ follows from the facts that $f(x) = T^*\delta_x(f) = \phi_x(f)$ for $f \in F$ and that invariant functions separate invariant probability measures. Defining $Pf(x) = \phi_x(f)$ for each $f \in C(X)$, we obtain $Pf \in C(X)$. Now, we show that for each $f \in C(X)$, Pf is a $\{T\}$-invariant function and Pf is contained in $\overline{co}\{Tf : T \in \Sigma\}$. In fact, for $\phi \in C(X)^*$, let $Q\phi$ be a unique invariant measure in $w^*\overline{co}\{T^*\phi : T \in \Sigma\}$. Then we obtain $P^*\phi = Q\phi$ from that invariant functions separate invariant probability measures. On the other hand, we obtain $T^*Q = QT^* = Q$ for all $T \in \Sigma$. Hence, we have $T^*P^* = P^*T^* = P^*$. By using this, it follows that for each $f \in C(X)$, Pf is a $\{T\}$-invariant function. If Pf is not contained in $\overline{co}\{Tf : T \in \Sigma\}$, there exists $\phi \in C(X)^*$ such that $\phi(Pf) > \sup[\phi(g) : g \in \overline{co}\{Tf : T \in \Sigma\}]$. From

$$\sup[\phi(g) : g \in \overline{co}\{Tf : T \in \Sigma\}] \geqq \phi(Pf),$$

we obtain a contradiction. Hence for each $f \in C(X)$, Pf is contained in $\overline{co}\{Tf : T \in \Sigma\}$.

The following Theorem is an extension of the theorem 2 in [5].

THEOREM 4. *Let $\Sigma = \{T\}$ be an ergodic and amenable semigroup of Markov operators and let*

$$\Phi = \{\phi \in K : T^*\phi = \phi \text{ for each } T \in \Sigma\}.$$

Then, $\phi \to I_\phi$ is a bijective mapping of the set $ex\Phi$ of extreme points of Φ onto the set of maximal $\{T\}$-ideals and also every $\{T\}$-ideal of the form I_φ ($\phi \in \Phi$) is the intersection of all maximal $\{T\}$-ideals containing it. Moreover, Φ is simplex in the sense of [4] *and $ex\Phi$ is weakly*-closed.*

Proof. To show that Φ is simplex in the sense of [4], it is sufficient that ϕ^+ is P^*-invariant whenever so is ϕ. Since $\phi^+ \geqq 0$ and $\phi^+ \geqq \phi$, $P^*\phi^+ \geqq 0$ and $P^*\phi^+ \geqq P^*\phi = \phi$. Hence $P^*\phi^+ \geqq \phi^+$. Therefore, we obtain $P^*\phi^+ = \phi^+$ from $(P^*\phi^+ - \phi^+)(e) = 0$.

Since an element of Φ is invariant under P^*, it follows that $ex\Phi$ is weakly*-compact. The remainder is obvious from Theorem 1, Corollary 1 and theorem 2 in [5].

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