KŌDAI MATH. SEM. REP.
22 (1970), 385—400

ON THE TOTAL ABSOLUTE CURVATURE OF MANIFOLDS IMMERSED IN RIEMANNIAN MANIFOLDS, III

BY BANG-YEN CHEN

0. Introduction.

Before Willmore-Saleemi's paper [10] appeared, the total absolute curvature was defined only for the closed manifolds in euclidean space (see, for instances, [3, 4, 5, 6, 7, 9]).

In [10], Willmore and Saleemi defined the total absolute curvature for closed manifolds in riemannian manifolds and proved that if f and f' are two immersions of two closed manifolds into two riemannian manifolds, then the total absolute curvature of the product immersion $f\times f'$ satisfies the relation: $TA(f)\times TA(f')=TA(f\times f')$.

In the first paper of this series [1], the author used the Levi-Civita translation in riemannian manifold to define the generalized Gauss mapping and to generalize many results due to Chern, Lashof, and Kuiper [3, 4, 5, 6]. In the second paper of this series [2], the author studied the total absolute curvature for surfaces in real space forms.

In the present paper, we shall study the total absolute curvature for cornered manifolds and bounded manifolds in riemannian manifolds. This is the first paper we have so far which studies the total absolute curvature for cornered manifolds (also for bounded manifolds).

Thanks are due to Professor Tadashi Nagano who proposed the author to study the total absolute curvature of these kinds.

In section 1, we give the definitions of bounded manifolds, cornered manifolds, and the total absolute curvature. In section 2, we find some relations between the total absolute curvature and totally geodesic submanifolds of the ambient riemannian manifold. In section 3, we give some inequalities of the total absolute curvature for attached cornered manifolds and product cornered manifolds. In section 4, we get a fundamental inequality for cornered manifolds in euclidean space. From this inequality, we find some relations between the total absolute curvature and critical point theory. In particular, if the cornered manifold M^n is closed, then this fundamental inequality gives us the Chern-Lashof ineqality for total absolute curvature. In section 5, we consider the total absolute curvature for cornered surfaces in euclidean space. In this section, we prove that if M^2 is com-

Received March 30, 1970.

pact, then $TA(M^2:E^3)=0$ for the case $\mathrm{Bd}\, M^2 \neq \phi$. From this equation, we prove that every orientable cornered surface with non-empty boundary can be convexly immersed in E^3, and every cornered surface with non-empty boundary always admits a real-valued function on it without critical points. In section 6, we study the total absolute curvature for cornered surfaces in N-spheres. In this section, we also prove that every cornered surface with non-empty boundary has vanishing total absolute curvature. In the last section, we study the total absolute curvature for cornered surfaces in Kähler manifolds. Under some suitable conditions, we find that there are some relations between the total absolute curvature, the riemannian sectional curvature and the second fundamental form.

1. Preliminaries.

We recall that a (usual) smooth manifold structure Σ of M^n is a set of homeomorphisms λ: $U_\lambda \to V_\lambda$ between open sets U_λ and V_λ in E^n and M^n respectively, subject to the condition among others to the effect that $\theta^{-1}\circ\lambda$ is smooth for θ, λ in Σ. Now, a *bounded manifold* and a *cornered manifold* respectively are defined by the above with E^n replaced by $E^{n-1}\times[0,\infty)$ and $[0,\infty)^n$, where by a *smooth map* means a map defined on an open set U in $E^{n-1}\times[0,\infty)$ or $[0,\infty)^n$ into E^m which extends to a smooth map in the usual sense on a neighborhood of U in E^n. The *smooth maps* between bounded or cornered manifolds are defined in the obvious way. The boundary of a bounded or cornered manifold M^n, denoted by $\mathrm{Bd}\, M^n$, is by definition the subset of M^n consisting of all points which are in the image of the boundary $\mathrm{Bd}\,(E^{n-1}\times[0,\infty))$ or $\mathrm{Bd}\,([0,\infty)^n)$ in E^n under some (hence any) coordinate map λ. Every bounded or cornered manifold can "extend beyond its boundary" to a smooth manifold, which is not at all unique. Thus, a smooth map between bounded or cornered manifolds extends to a smooth map between usual smooth manifolds. A bounded manifold is a cornered one and its boundary is a smooth manifold. A smooth manifold M^n is a cornered manifold with $\mathrm{Bd}\, M^n=\phi$. The interior set $(M^n-\mathrm{Bd}\, M^n)$ of a cornered or bounded manifold M^n is a smooth manifold of the same dimension. A closed manifold is a compact smooth manifold.

In the following, we assume throughout that M^n is an n-dimensional cornered manifold unless otherwise stated. We assume throughout that Y^{n+N} is an oriented riemannian manifold of dimension $n+N$.

Let

$$f\colon M^n \to Y^{n+N} \tag{1}$$

be an immersion of M^n into Y^{n+N}. By a frame $x, e_1, \cdots, e_{n+N}$ in Y^{n+N} we mean a point x and an ordered set of mutually perpendicular unit vectors $e_1, \cdots, e_{n+N}$, such that their orientation is coherent with that of Y^{n+N}. Unless otherwise stated, we agree on the following ranges of the indices:

$$1\leqq i,j,k,\cdots\leqq n; \qquad 1+n\leqq r,s,t,\cdots\leqq n+N; \qquad 1\leqq A,B,C,\cdots\leqq n+N. \tag{2}$$

Let $F(Y^{n+N})$ be the bundle of the frames on Y^{n+N}. In $F(Y^{n+N})$ we introduce the linear differential forms θ_A, θ_{AB} by the equations:

(3) $$de_A=\sum_B \theta_{AB}e_B, \qquad df=\sum_A \theta_A e_A, \qquad \theta_{AB}+\theta_{BA}=0.$$

Their exterior derivative satisfy the equations of structure:

(4) $$d\theta_A=\sum_B \theta_B\wedge\theta_{BA}, \qquad d\theta_{AB}=\sum_C \theta_{AC}\wedge\theta_{CB}+\Omega_{AB}.$$

Let B be the set of elements $b=(p, e_1, \cdots, e_{n+N})$ such that $(f(p), e_1, \cdots, e_{n+N})$ $\in F(Y^{n+N})$, $p\in M^n, e_1, \cdots, e_n$ are tangent vectors and $e_{n+1}, \cdots, e_{n+N}$ are normal vectors at $f(p)$. Let ω_A, ω_{AB} be the 1-forms on B induced from the natural immersion $B\to F(Y^{n+N})$; $(p, e_1, \cdots, e_{n+N})\longmapsto(f(p), e_1, \cdots, e_{n+N})$. Then by the definition of B, we have

(5) $$\omega_r=0,$$

and $\omega_1, \cdots, \omega_n$ are linearly independent. Hence the first equation of (4) gives

(6) $$\sum_i \omega_i\wedge\omega_{ir}=0.$$

From this it follows that

(7) $$\omega_{ir}=\sum_j A_{rij}\omega_j, \qquad A_{rij}=A_{rji}.$$

We define the normal bundle B_v by

(8) $$B_v=\{(p, e): p\in M^n,\ e \text{ unit normal vector at } f(p)\}.$$

We call

(9) $$K(p, e_r)=(-1)^n \det(A_{rij}),$$

the Lipschitz-Killing curvature at $(p, e_r)\in B_v$. We call the integral

(10) $$TA(f)=\int_{B_v}|K(p, e)|dW/c_{n+N-1},$$

the *total absolute curvature of the immersion* f, if the right hand side of (10) exists, where c_{n+N-1} denotes the volume of the unit $(n+N-1)$-sphere and dW the volume element of the normal bundle B_v.

We call

(11) $$TA(M^n: Y^{n+N})=\inf\{TA(f): f: M^n\to Y^{n+N} \text{ immersions}\},$$

the *total absolute curvature of the cornered manifold* M^n *in* Y^{n+N}, if the right hand side of (11) exists.

If there exists no immersion of M^n into Y^{n+N} or for every immersion of M^n into Y^{n+N}, the right hand side of (10) doesn't exist, then we set $TA(M^n; Y^{n+N})=\infty$.

REMARK. In the special case, M^n is closed, the definitons of the total absolute curvature of [2, 10] and the present paper are equivalent.

2. Total absolute curvature and totally geodesic submanifold.

In this section, we want to seek some relations between the total absolute curvature and totally geodesic submanifold of the ambient manifold.

LEMMA 2. 1. *Let* f: $M^n \to Y^{n+N}$ *be an immersion of a cornered manifold* M^n *into a riemannian manifold* Y^{n+N}. *If* M^n *is immersed in a totally geodesic submanifold* $\bar{Y}$ *of* Y^{n+N}, *then*

$$TA(f)=TA(\bar{f}) \tag{12}$$

where $\bar{f}$: $M^n \to \bar{Y}$ *is defind by* $\bar{f}(p)=f(p)$ *for all* p *in* M^n.

Proof. If $\bar{Y}$ is a totally geodesic submanifold of Y^{n+N}, then we can easily derive from equations (7) and (9) that

$$K(p, e)=(\cos^n \theta)\bar{K}(p, \bar{e}), \tag{13}$$

where $K(p, e)$ and $\bar{K}(p, \bar{e})$ denote the Lipschitz-Killing curvatures of f and $\bar{f}$ respectively, and θ denotes the angle between e and the unit vector $\bar{e}$ which is in the direction of the projection of e into the tangent space $T_{f(p)}(\bar{Y})$. Using (13) we can easily deduce that equation (12) holds.

By the definition of total absolute curvature, we have easily the follows:

THEOREM 2. 2. *Let* f: $M^n \to Y^{n+N}$ *be an immersion of a cornered manifold* M^n *into a riemannian manifold* Y^{n+N}. *If* M^n *is immersed in an n-dimensional totally geodesic submanifold of* Y^{n+N}, *then* $TA(f)=0$.

We may consider conventionally the case $N=0$ in the above, hence we have

COROLLARY 2. 3. *If a cornered manifold* M^n *can be immersed into a riemannian manifold* Y^n *of dimension n, then* $TA(M^n : Y^n)=0$.

3. Total absolute curvature for attached cornered manifolds and product cornered manifolds.

Let M^n and $\bar{M}^n$ be two n-dimensional cornered manifolds. Then we say that a cornered manifold U^n is an *attached cornered manifold* of M^n and $\bar{M}^n$ if U^n can be obtained by identified some boundaries of M^n and $\bar{M}^n$ from the (abstract) union of M^n and $\bar{M}^n$.

THEOREM 3.1. *Let U^n be an attached cornered manifold of M^n and $\bar{M}^n$. Then we have*

$$TA(U^n : Y^{n+N}) \geqq TA(M^n : Y^{n+N}) + TA(\bar{M}^n : Y^{n+N}). \tag{14}$$

There exists an attached cornered manifold which satisfies the inequality sign of (14).

Proof. Let U^n be an attached cornered manifold of M^n and $\bar{M}^n$ denoted by

$$U^n = M^n \underset{g}{\cup} \bar{M}^n. \tag{15}$$

Let f: $U^n \rightarrow Y^{n+N}$ be an immersion of U^n into Y^{n+N}. Then the restriction immersions $f|M^n$ and $f|\bar{M}^n$ can be regarded as two immersions of M^n and $\bar{M}^n$ into Y^{n+N} respectively. From (9) and (10), we can easily deduce that

$$TA(f) = TA(f|M^n) + TA(f|\bar{M}^n). \tag{16}$$

Hence, by (16), we get (14).

Now, suppose that M^n and $\bar{M}^n$ are the upper and lower semi-spheres of a euclidean n-sphere S^n in E^{n+1}, and U^n is the euclidean n-sphere S^n itself. Then U^n is an attached cornered manifold of M^n and $\bar{M}^n$ by identified the boundaries of M^n and $\bar{M}^n$ in a usual way. Since M^n and $\bar{M}^n$ can be immersed into a euclidean n-space E^n. Hence, by Theorem 2.2, we have

$$TA(M^n : E^{n+1}) = TA(\bar{M}^n : E^{n+1}) = 0. \tag{17}$$

On the other hand, by a result of Chern and Lashof, we have $TA(U^n : E^{n+1}) = 2$. Hence the inequality sign of (14) holds for this example. This completes the proof of the theorem.

EXAMPLE 3.1. Let

$$M^n = S^{n-1} \times [0, 1], \qquad \bar{M}^n = S^{n-1} \times [2, 3], \qquad U^n = S^{n-1} \times [1, 2].$$

The U^n can be regarded as an attached cornered manifold of M^n and $\bar{M}^n$ by identified $(x, 1) \in M^n$ with $(x, 2) \in \bar{M}^n$, $x \in S^{n-1}$. Since M^n, $\bar{M}^n$ and U^n can be immersed into E^n as shells, hence we have

$$TA(U^n : E^{n+N}) = TA(M^n : E^{n+N}) + TA(\bar{M}^n : E^{n+N}) = 0, \qquad N \geqq 0.$$

This shows that there exists an attached cornered manifold such that the equality sign of (14) holds.

THEOREM 3. 2. *Let M^n and $\bar{M}^m$ be two cornered manifolds, and let Y, $\bar{Y}$ be two riemannian manifolds. Then we have*

$$TA(M^n \times \bar{M}^m : Y \times \bar{Y}) \leqq TA(M^n : Y) \times TA(\bar{M}^m : \bar{Y}). \tag{18}$$

Proof. Let f: $M^n \to Y$ and $\bar{f}$: $\bar{M}^m \to \bar{Y}$ be two immersions of the cornered manifolds M^n and $\bar{M}^m$ into Y and $\bar{Y}$, respectively. Let $f \times \bar{f}$: $M^n \times \bar{M}^m \to Y \times \bar{Y}$ be the product immersion of f and $\bar{f}$. Then by a direct computation, we can prove that

$$TA(f \times \bar{f}) = TA(f) \times TA(\bar{f}). \tag{19}$$

From this equation, we get (18).

EXAMPLE 3. 2. Let $M^n = S^n$ and $\bar{M}^m = S^m$ be two euclidean spheres of dimension n and m respectively. Then we have

$$TA(M^n \times \bar{M}^m : E^{n+m+2}) = TA(M^n : E^{n+1}) \times TA(\bar{M}^m : E^{m+1}) = 4.$$

EXAMFLE 3. 3. Let M^2 be the real projective plane P^2 and $\bar{M}^m$ the euclidean m-sphere S^m. Then, by a result due to Kuiper [5, 7], we have

$$TA(M^2 : E^3) > TA(M^2 : E^4) \quad \text{and} \quad TA(\bar{M}^m : E^{m+2}) = TA(\bar{M}^m : E^{m+1}) = 2. \tag{20}$$

Therefore, by (18) and (20), we have the following:

$$TA(M^2 \times \bar{M}^m : E^{m+5}) < TA(M^2 : E^3) \times TA(\bar{M}^m : E^{m+2}).$$

Examples 3. 2 and 3. 3 show that the equality sign and the inequality sign of (18) hold for some product cornered manifolds.

EXAMPLE 3. 4. Let M and $\bar{M}$ be two unit 1-spheres. Let Y be a unit 3-sphere and i_1, i_2 the inclusion mappings of M and $\bar{M}$ into Y respectively. Then by Theorem 2. 2, we have

$$TA(i_1) = TA(i_2) = 0. \tag{21}$$

On the other hand, by a result due to Chen [2], we know that every immersion of $M \times \bar{M}$ into Y has positive total absolute curvature.

By Theorem 3. 2, we have the following:

COROLLARY 3. 3. *Let M^n and $\bar{M}^m$ be two cornered manifolds, Y and $\bar{Y}$ two riemannian manifolds. If $TA(M^n : Y) = 0$ and $TA(\bar{M}^m : \bar{Y})$ is finite, then we have*

$TA(M^n \times \bar{M}^m : Y \times \bar{Y}) = 0.$

In particular, we have

COROLLARY 3.4. *If a cornered manifold M^n can be immersed in a riemannian manifold Y with finite total absolute curvature, then we have*

$$TA(M^n \times I : Y \times E^1) = 0,$$

where I denotes an interval of 1-dimensional euclidean space E^1.

REMARK. It is easy to see that the ambient riemannian manifold Y^{n+N} can be replaced by a cornered riemannian manifold for almost all propositions.

4. An inequality for cornered manifolds in euclidean space.

In the following, let $\mathcal{F}(M^n)$ be the set of real-valued functions on a cornered manifold M^n with only non-degenerate critical points on $M^n - \mathrm{Bd}\, M^n$. For any $f \in \mathcal{F}(M^n)$, let $m_i(f)$ denote the number of critical points of index i of f on M^n. Let

$$m(f) = \sum_i m_i(f), \tag{23}$$

$$m(M^n) = \inf \{m(f) : f \in \mathcal{F}(M^n)\}. \tag{24}$$

THEOREM 4.1. *Let M^n be a compact cornered manifold. Then we have*

$$TA(M^n : E^{n+N}) \geqq m(M^n), \qquad N \geqq 1. \tag{25}$$

In particular, if M^n is closed, then we have

$$TA(M^n : E^{n+N}) \geqq \sum_i b_i(M^n), \tag{26}$$

where $b_i(M^n)$ denotes the i-th betti number of M^n; $i = 0, 1, \cdots, n$.

Proof. Let f: $M^n \to E^{n+N}$ ($N \geqq 0$) be an immersion of a cornered manifold M^n into E^{n+N}. Let S_0^{n+N-1} denote the unit sphere with center at the origin O in E^{n+N}. For any e in S_0^{n+N-1}, we define the height function:

$$h_e \colon M^n \to R \tag{26}$$

by $h_e(p) = f(p) \cdot e$ for every e in S_0^{n+N-1}. Then it is clear that for almost all e in S_0^{n+N-1}, the height function h_e have only non-degenerate critical points.

Let

$$\tilde{\nu} \colon B_\nu \to S_0^{n+N-1} \tag{27}$$

be the mapping defined by $\tilde{v}(p,\bar{e})=\bar{e}$. Then, by the definition of h_e, we know that for every $(p,\bar{e})\in B_v$ with $\tilde{v}(p,\bar{e})=e$, we have

$$dh_e(p)=df(p)\cdot e=0. \tag{28}$$

Hence, p is a critical point of the height function h_e.

Conversely, if p is a critical point of the height function h_e, then, by the definition of the critical points, we have

$$df(p)\cdot e=dh_e(p)=0.$$

Hence (p,e) belongs to B_v. For almost all e in S_0^{n+N-1}, the number of all critical points of h_e is equal to the number of points $(p,\bar{e})$ in B_v with $\tilde{v}(p,\bar{e})=e$. Therefore, we have the following equation:

$$\int_{B_v}|\tilde{v}^*d\Sigma_{n+N-1}| = \int_{S_0^{n+N-1}} m(h_e)d\Sigma_{n+N-1}, \tag{29}$$

where $d\Sigma_{n+N-1}$ is the volume element of S_0^{n+N-1}, and $\tilde{v}^*$ the dual mapping of $\tilde{v}$. Thus, by (23), (24) and (29), we have

$$\int_{B_v}|\tilde{v}^*d\Sigma_{n+N-1}|\geqq c_{n+N-1}m(M^n). \tag{30}$$

On the other hand, since the ambient riemannian manifold is euclidean, hence, by (7), (8), (9) and (10), we find that

$$TA(f)=\int_{B_v}|\tilde{v}^*d\Sigma_{n+N-1}|/c_{n+N-1}. \tag{31}$$

By (30) and (31), we get the inequality (25), This completes the proof of the theorem.

COROLLARY 4. 2. *Let M^n be a cornered manifold. If $TA(M^n:E^{n+N})=0$, then there exists a real-valued function on M^n which has no critical points.*

This Corollary follows immediately from Theorem 4. 1.

5. Total absolute curvature for cornered surfaces in E^3.

In this section, we assume throughout that M^2 is compact and of dimension 2. Put

$$k=\text{number of components of } \mathrm{Bd}\, M^2. \tag{32}$$

THEOREM 5. 1. *Let M^2 be a compact orientable cornered surface. Then we* *have the following*:

(I) $\qquad TA(M^2:E^2)=0$ *if and only if* $k\geqq 1$ *and* $b_1(M^2)=k-1$,

(II) $\qquad TA(M^2:E^3)=0$ *if* $k>0$.

Proof. By the assumption, M^2 is a compact orientable cornered manifold of dimension 2. Hence each component of $\mathrm{Bd}\,M^2$ is diffeomorphic to a 1-sphere i.e., a circle. If $\mathrm{Bd}\,M^2$ is not empty, then by the Mayer-Victoris sequence on homology groups, we can deduce that the betti numbers of M^2 are given by the following:

(33) $$b_0(M^2)=1, \qquad b_1(M^2)=2g+k-1, \qquad b_2(M^2)=0,$$

where g is a non-negative integer which is called the *genus* of the cornered surface M^2.

(I) If $TA(M^2:E^2)=0$, then by (11) we know that M^2 can be immersed as a compact subset of E^2. Hence we can attach k 2-cells to M^2 as a 2-sphere. Thus, by a direct computation on the betti numbers of M^2, we find that the genus $g=0$ and $k\neq 0$. Thus we get $b_1(M^2)=k-1$.

Conversely, if $k\neq 0$ and $b_1(M^2)=k-1$, then by (33), we get $g=0$. Hence, M^2 is a proper subset of a 2-sphere. This shows us that M^2 can be immersed into E^2.

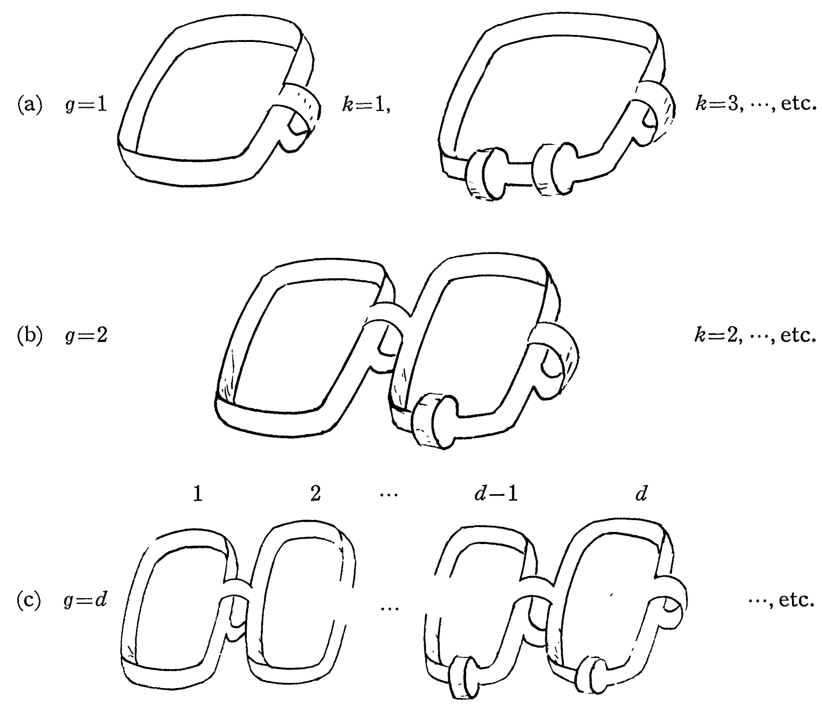

Fig. 5. 1.

By Corollary 2. 3, we get $TA(M^2:E^2)=0$.

(II) If $k\neq 0$ and $g\neq 0$, then we can immerse M^2 into E^3 as a cornered surface which is the union of some cylinders[1)] in E^3 shown in Fig. 5. 1. Since the Gaussian curvature of each point of a cylinder is zero. Hence, by (7), (9), (10) and (24), we get $TA(M^2:E^3)=0$. This completes the proof of the theorem.

If an immersion f: $M^n \rightarrow E^{n+N}$ of a cornered manifold M^n into E^{n+N} has the total absolute curvature

$$TA(f)=m(M^n),$$

then it is called to be *convex*.

COROLLARY 5. 2. *If M^2 is a compact orientable cornered surface, then there exists a convex immersion of M^2 into E^3. In particular, if $k\neq 0$ and $b_1(M^2)=k-1$, then there exists a convex immersion of M^2 into E^2.*

These convex immersions have been constructed in the proof of theorem 5. 1 and in [6].

COROLLARY 5. 3. *Let M^2 be a compact orientable cornered surface with $k>0$. Then there exists a real-valued function on M^2 which has no critial points.*

This Corollary follows immediately from Corollary 4. 2 and Theorem 5. 1.

THEOREM 5. 4. *Let M^2 be a compact non-orientable cornered surface with non-empty boundary. Then we have the following equation*:

$$TA(M^2:E^3)=0. \tag{35}$$

Proof. Let M^2 be a compact non-orientable cornered surface with non-empty boundary. Then every component of Bd M^2 is diffeomorphic to a 1-sphere. Now, we go to construct the immersions of M^2 into E^3 as follows:

Case I. If M^2 is a Möbius band, i.e., M^2 is obtained from a real projective plane by taking off a 2-cell. Then we can immerse M^2 into E^3 shown in Fig. 5. 2.

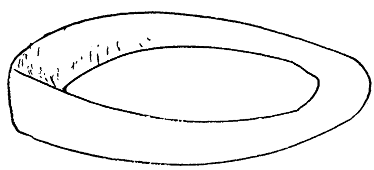

Fig. 5. 2.

1) In this paper, a cylinder in E^3 means that it is a cornered surface which is given by the following: Through each point of a curve in E^3, there passes a straight line segment which has the constant direction, and the curve is not equal to one of these line segments.

Thus, by taking an immersion of the Möbius band in Fig. 5. 2 into a Möbius band shown in Fig. 5. 4 of the next page we get $TA(M^2:E^3)=0$.

Case II. If M^2 is not a Möbius band, then, by the assumption, we know that M^2 is the union of a compact orientable cornered surface with a Möbius band. Hence, we can immerse M^2 into E^3 as a cornered surface shown in Fig. 5. 3.

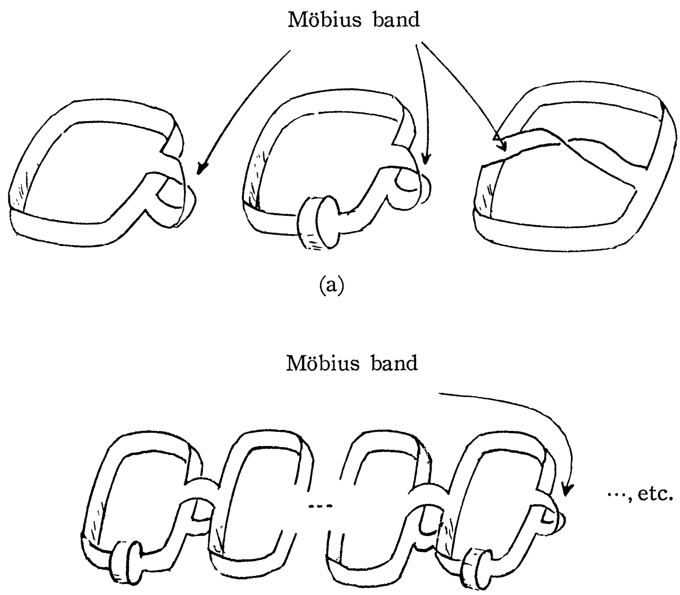

Fig. 5. 3.

Hence, by Theorem 5. 1 and adjoint the part of Möbius band as given in Fig. 5. 4, we know that there exists an immersion of M^2 into E^3 such that the total absolute curvature of this immersion equal to zero. Consequently, we have $TA(M^2:E^3)=0$. This completes the proof of the theorem.

REMARK. There exists a Möbius band in E^3 with vanishing Gaussian curvature as the figure illustrated below: C is a closed curve passing through $P_1\in\overline{A_2A_3}$, $P_2\in\overline{A_3A_1}$, $P_3\in\overline{A_1A_2}$, $P_4\in\mathrm{Int}\,\Delta A_1A_2A_3$. Each point P_i, $i=1,2,3$, has a small neighborhood arc of C which is a straight line segment. Make conical surface pieces projecting from the vertices A_1, A_2, A_3 for the subarcs $C(P_2,P_3)$, $C(P_3,P_1)$, $C(P_1,P_2)$, respectively as shown in Fig. 5. 4. Then, we get a Möbius band composed of three conical surface pieces. Hence, its Gaussian curvature is zero everywhere. (The above construction of Möbius band in E^3 is due to Dr. Hideki Ohmori.) This example tells us that there is a convex immersion of a Möbius band in E^3.

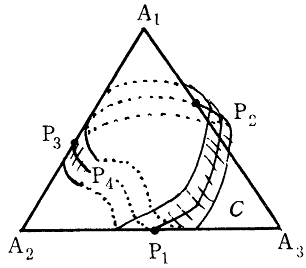

Fig. 5. 4.

6. Total absolute curvature for cornered surfaces in S^N.

In the following, let S^N denote the euclidean N-sphere of radius a (or of curvature $1/a^2$).

THEOREM 6. 1. *Let M^2 be a compact orientable cornered surface. Then $TA(M^2:S^2)=0$ if and only if either M^2 is diffeomorphic to a 2-sphere or M^2 has non-empty boundary and $b_1(M^2)=k-1$.*

Proof. If $TA(M^2:S^2)=0$, then there exists an immersion of M^2 into S^2. Hence, if $k=0$, then M^2 is diffeomorphic to a 2-sphere. If $k\neq 0$, then we can attach k 2-cells to M^2 to become a 2-sphere. Thus, by a direction computation on the betti numbers of M^2, we find that $b_1(M^2)=k-1$.

Conversely, if $k\neq 0$ and $b_1(M^2)=k-1$, then by (33), we get $g=0$. Hence, M^2 is a proper subset of a 2-sphere. Thus, we can immerse M^2 into S^2. Therefore by Corollary 2. 3, we get $TA(M^2:S^2)=0$. If M^2 is diffeomorphic to S^2, then $TA(M^2:S^2)=0$ is trivial. This completes the proof of the theorem.

If f: $M^n \rightarrow S^{n+N}$ is an immersion of M^n into S^{n+N} with total absolute curvature $TA(f)=TA(M^n:S^{n+N})$, then f is called to be convex.

THEOREM 6. 2. *Let M^2 be a compact orientable cornered surface. Then we have the following*:

(I) *$k=0$ and $TA(M^2:S^3)=0$ if and only if M^2 is either diffeomorphic to a 2-sphere or diffeomorphic to a torus. In the second case, there exists no convex immersion of M^2 into S^3.*

(II) *If $k\neq 0$ and $g=0,1$, then $TA(M^2:S^3)=0$.*

Proof. (I), Case (a). If M^2 is diffeomorphic to a 2-sphere, then $TA(M^2:S^3)=0$ follows immediately from Lemma 2. 1 and Theorem 6. 1.

(I), Case (b). If M^2 is diffeomorphic to a torus, then for any pair (c, d) with $c>0$, $d>0$ and $c^2+d^2=a^2$, we construct an immersion

$$f_{(c,d)}: M^2 \rightarrow S^3$$

of M^2 into S^3 as follows:

$$(c\cos u,\ c\sin u,\ d\cos v,\ d\sin v). \tag{36}$$

By a direct computation, we can find that

$$TA(f_{(c,d)})=2cd\pi/a^2, \qquad c^2+d^2=a^2. \tag{37}$$

Hence, by (37), we can choose a sequence of immersions of M^2 into S^3 such that the total absolute curvature of these immersions converge to zero. Thus, by (11), $TA(M^2:S^3)=0$.

Conversely, by Theorem 2 of [2], we know that if $k=0$ and $TA(M^2:S^3)=0$, then M^2 is either a 2-sphere or a torus. Furthermore, by Theorem 5 of [2], we know that if M^2 is a torus, then the total absolute curvature for every immersion of a torus into S^3 never vanishes. Thus there exists no convex immersion of M^2 into S^3.

(II), Case (c). If $g=0$, and $k\neq 0$, then M^2 is a proper subset of a 2-sphere. Hence we can immerse M^2 into a great 2-sphere of S^3. Hence, by Theorem 2. 2, we get $TA(M^2:S^3)=0$.

(II), Case (d). If $k\neq 0$ and $g=1$, then M^2 can be immersed into E^3 as a cornered surface which is the union of a compact cylinder, say $\bar{C}$, with k handles (each handle is a small compact cylinder) shown in Fig. 6. 1.

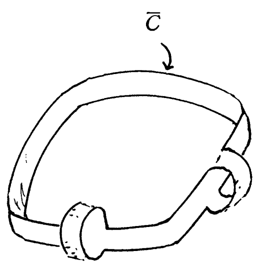

Fig. 6. 1.

Furthermore, since there exists an immersion of M^2 into S^3 which maps the part $\bar{C}$ into a great 2-sphere of S^3 and map these k handles onto some strips in 2-spheres in S^3. Hence we can construct an immersion of M^2 into S^3 such that the total absolute curvature of these immersion equal to zero. Hence, we get $TA(M^2:S^3)=0$.

This completes the proof of the theorem.

7. Cornered surfaces in a Kähler manifold with $TA(f)=0$.

In this section, let Y^{2N} be a Kähler manifold with complex dimension N. Let $f: M^n\to Y^{2N}$ be an immersion of a cornered manifold of dimension n into Y^{2N}. In the following, let S' denote the riemannian sectional curvature of Y^{2N}, and let S

the riemannian sectional curvature at the points of M^n with respect to the induced riemannian metric. An immersion f: $M^n \to Y^{2N}$ immerses M^n into Y^{2N} as a complex submanifold if the interior set of M^n is immersed as a complex submanifold of Y^{2N}. The aim of this section is to prove the following theorem:

THEOREM 7.1. *Let f: $M^2 \to Y^{2N}$ be an immersion of a cornerd surface M^2 into a Kähler manifold Y^{2N} as a complex submanifold of Y^{2N}. Then the following three statements are equivalent*:

(I) $TA(f)=0$,

(II) $S \geqq S'$,

(III) *f is totally geodesic, i.e., the second fundamental from $II=0$.*

Proof. In order to prove the theorem, we first prove the following lemma:

LEMMA 7.2. *Let f: $M^n \to Y^{2N}$ be an immersion of a cornered manifold M^n into Y^{2N} such that M^n is immersed as a complex submanifold of Y^{2N}. If the holomorphic sectional curuature H of M^n (with the induced structure) and the holomorphic sectional curvatuie H' of Y^{2N} satisfy the following inequality*:

$$H \geqq H', \tag{41}$$

then the immersion f: $M^n \to Y^{2N}$ is totally geodesic.

Proof. In the following, let J denote the complex structure of the Kähler manifold Y^{2N}. By the assumption, we have the equation of Gauss:

$$R'(W, Z, X, Y) = R(W, Z, X, Y) + g(II(X, Z), II(Y, W)) - g(II(Y, Z), II(X, W)), \tag{42}$$

where R and R' denote the riemannian sectional curvature tensor field of $M^n - \mathrm{Bd}\, M^n$ and Y^{2N} respectively, and g is the riemannian metric of Y^{2N}.

Furthermore, the second fundamental form II of the immersion f: $M^n \to Y^{2N}$ satisfies the following equations:

$$II(JX, Y) = II(X, JY) = J(II(X, Y)). \tag{43}$$

By (42) and (43), we get

$$R(X, JX, X, JX) = R'(X, JX, X, JX) - 2g(II(X, X), II(X, X)), \tag{44}$$

for all vector fields X on $M^n - \mathrm{Bd}\, M^n$. Thus, by (41) and (44), we have

$$g(II(X, X), II(X, X)) = 0,$$

for all vector fields X on $M^n - \mathrm{Bd}\, M^n$. Hence, the second fundamental form $II=0$

Now, we return to prove theorem 7.1:

(II)$\Rightarrow$(III)$\Rightarrow$(I): These follow immediately from the definitions and Lemma 7. 2.

(I)$\Rightarrow$(III): Since M^2 is immersed as a complex submanifold of Y^{2N}. Hence f is minimal, that is the normal vector field

$$N=\sum_i II(e_i, e_i)=\sum_{r,i} A_{rii}e_r \tag{45}$$

vanishes. Thus, by (I) and (45), we have

$$\det(A_{rij})=\operatorname{trace}(A_{rij})=0.$$

Thus we get

$$A_{rij}=0, \qquad r=3, \cdots, 2+N, \qquad i, j=1, 2.$$

This means that the second fundamental from $II=0$.

(III)$\Rightarrow$(II): If the second fundamental form vanishes, then by equation (44), we have

$$R(X, JX, X, JX)=R'(X, JX, X, JX)$$

for all vector fields X on $M^2-\mathrm{Bd}\, M^2$. Thus, by $\dim M^2=2$, we get (II). This completes the proof of the theorem.

REMARK. From the proof of Theorem 7. 1, we know that statement (II) can be replaced by

(II)′: $S=S'$.

ACKNOWLEDGEMENTS. The author would like to express his deep appreciation to Professor T. Nagano for his kind help during the preparation of this paper, and also thanks to Professor T. Ōtsuki for the valuable improvement and encouragement.

REFERENCES

[1] CHEN, B.-Y., On the total absolute curvature of manifolds immersed in riemannian manifolds. Kōdai Math Sem. Rep. **19** (1967), 299–311.

[2] CHEN, B.-Y., On the total absolute curvature of manifolds immersed in riemannian manifolds, II. Kōdai Math. Sem. Rep. **22** (1970), 89–97.

[3] CHERN, S. S., AND R. K. LASHOF, On the total curvature of immersed manifolds. Amer. J. Math. **79** (1957), 306–318.

[4] CHERN, S. S., AND R. K. LASHOF, On the total curvature of immersed manifolds, II. Michigan Math. J. **5** (1958), 5–12.

[5] KUIPER, N. H., Immersions with minimal absolute curvature. Colloque de Géométrie Différentielle Globale, Bruxelles (1958), 75–87.

[6] KUIPER, N. H., On surfaces in euclidean three-space. Bull. Soc. Math. Belg. **12** (1960), 5–22.

[7] KUIPER, N. H., Convex immersions of closed surfaces in E^3. Non-orientable

closed surfaces in E^3 with minimal total absolute curvature. Comm. Math. Helv. **35** (1961), 85–92.

[8] KUIPER, N. H., Der Satz von Gauss-Bonnet für Abbildungen in E^N und damit verwandte Probleme. Jber. Deut. Math. Ver. **69** (1967), 77–88.

[9] ŌTSUKI, T., On the total curvature of surfaces in euclidean spaces. Japanese J. Math. **35** (1966), 61–71.

[10] WILLMORE, T. J., AND B. A. SALEEMI, The total absolute curvature of immersed manifolds. J. London Math. Soc. **41** (1966), 153–160.

MICHIGAN STATE UNIVERSITY, U.S.A.