KŌDAI MATH. SEM. REP.
22 (1970), 244—249

AN APPLICATION OF GREEN'S FORMULA OF A DISCRETE FUNCTION: DETERMINATION OF PERIODICITY MODULI, II

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Introduction. In the present paper, we shall briefly deal with a problem corresponding to the previous paper I in the case of the Hermitian method (Mehrstellenverfahren) (cf. p. 384 of Collatz [1] & Opfer [6]).[1]

§ 4. Determination of periodicity moduli by Mehrstellenverfahren.

1. Definition. We preserve the notations in § 2. **1**.[2] Let R be a lattice with mesh width h, and let V be a real function on R. Let z_0 be an inner point of R, and set $z_1=z_0+h$, $z_2=z_0+h+ih$, $z_3=z_0+ih$, $z_4=z_0-h+ih$, $z_5=z_0-h$, $z_6=z_0-h-ih$, $z_7=z_0-ih$ and $z_8=z_0+h-ih$. If the equation

$$20V_{(0)}-\sum_{j=1}^{4}(4V_{(2j-1)}+V_{(2j)})=0 \tag{4. 1}$$

holds for every $z_0\in R^\circ$, then V is said to be *discrete harmonic* on R with respect to Mehrstellenverfahren, where $V_{(j)}=V(z_j)$ $(j=0, \cdots, 8)$. Throughout § 4, the terms "discrete harmonic" is taken with respect to Mehrstellenverfahren.

2. Green's formula. We preserve the notations in § 2. **2**. Let V and V' be functions on R, and set $V_{(n)}=V(z_n)$ and $V'_{(n)}=V'(z_n)$ $(n=1, \cdots, \nu)$. We consider bilinear forms

$$\mathfrak{S}_R(V, V')=\frac{4}{6}\sum_{|z_m-z_n|=h, m<n}(V_{(m)}-V_{(n)})(V'_{(m)}-V'_{(n)})$$
$$+\frac{1}{6}\sum_{|z_m-z_n|=\sqrt{2}h, m<n, \overline{z_mz_n}\subset\bar{G}}(V_{(m)}-V_{(n)})(V'_{(m)}-V'_{(n)})$$

and

$$\mathfrak{S}_R^\circ(V, V')=\frac{4}{6}\sum_{|z_m-z_n|=h, m<n, \overline{z_mz_n}\not\subset\Gamma}(V_{(m)}-V_{(n)})(V'_{(m)}-V'_{(n)})$$
$$+\frac{1}{6}\sum_{|z_m-z_n|=\sqrt{2}h, m<n, \overline{z_mz_n}\subset\bar{G}}(V_{(m)}-V_{(n)})(V'_{(m)}-V'_{(n)}).$$

Received December 26, 1969.

1) See References of I.

2) § 2. **1** indicates one of I.

If V or V' is constant on each boundary component A_j $(j=0, \cdots, N-1)$ of R, then we see immediately that

$$\mathfrak{S}_R^\circ(V, V')=\mathfrak{S}_R(V, V').$$

We set $\mathfrak{S}_R(V)\equiv\mathfrak{S}_R(V, V)$ and $\mathfrak{S}_R^\circ(V)\equiv\mathfrak{S}_R^\circ(V, V)$.

LEMMA 4.1. *Let V and V' be two functions on a lattice R. Then the formula*

$$\begin{aligned}(4.2)\qquad &\mathfrak{S}_R^\circ(V, V')+\sum_{n=1}^{\mu} V_{(n)}\left(\frac{1}{6}\left(\sum_{j=1}^{4}(4V'_{(n_{2j-1})}+V'_{(n_{2j})})-20V'_{(n)}\right)\right)\\ &=\sum_{n=\mu+1}^{\nu} V_{(n)}\left(\frac{2}{3}\sum_{nl}(V'_{(n)}-V'_{(nl)})+\frac{1}{6}\sum_{nd}(V'_{(n)}-V'_{(nd)})\right)\end{aligned}$$

holds. Here z_{n_j} $(j=1, \cdots, 8)$ is the point z_j in **1** *respectively on taking z_0 in* **1** *in place of the present z_n, z_{nl} is a point of R neighboring to z_n which lies on the left of z_n with respect to the oriented curve Γ and which is not neighboring to z_n along Γ, z_{nd} is a point of R with $|z_n-z_{nd}|=\sqrt{2}h$ which lies on the left of z_n with respect to Γ, and thus if a number of z_{nl} for some n $(n=\mu+1, \cdots, \nu)$ is κ $(\kappa=0, 1, 2$ or $3)$ then a number of z_{nd} is $\kappa+1$ respectively.*

Proof.

$$\begin{aligned}6\mathfrak{S}_R^\circ(V, V')=&\frac{1}{2}\Bigg(\sum_{n=1}^{\mu}\Bigg(\sum_{j=1}^{4}(4(V_{(n)}-V_{(n_{2j-1})})(V'_{(n)}-V'_{(n_{2j-1})})\\ &\qquad+(V_{(n)}-V_{(n_{2j})})(V'_{(n)}-V'_{(n_{2j})}))\Bigg)\\ &+\sum_{n=\mu+1}^{\nu}\Bigg(4\sum_{nl}(V_{(n)}-V_{(nl)})(V'_{(n)}-V'_{(nl)})\\ &\qquad+\sum_{nd}(V_{(n)}-V_{(nd)})(V'_{(n)}-V'_{(nd)})\Bigg)\Bigg)\\ =&\frac{1}{2}\Bigg(\sum_{n=1}^{\mu}V_{(n)}\sum_{j=1}^{4}(5V'_{(n)}-4V'_{(n_{2j-1})}-V'_{(n_{2j})})\\ &+\sum_{n=1}^{\mu}\sum_{j=1}^{4}(4V_{(n_{2j-1})}(V'_{(n_{2j-1})}-V'_{(n)})+V_{(n_{2j})}(V'_{(n_{2j})}-V'_{(n)}))\\ &+\sum_{n=\mu+1}^{\nu}V_{(n)}\left(4\sum_{nl}(V'_{(n)}-V'_{(nl)})+\sum_{nd}(V'_{(n)}-V'_{(nd)})\right)\\ &+\sum_{n=\mu+1}^{\nu}\left(\sum_{nl}4V_{(nl)}(V'_{(nl)}-V'_{(n)})+\sum_{nd}V_{(nd)}(V'_{(nd)}-V'_{(n)})\right)\Bigg)\\ =&\frac{1}{2}(\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4),\end{aligned}$$

where by $\sum_j$ $(j=1, 2, 3, 4)$ we denote the j-th summation with respect to n of the third side respectively. We note that a summation of all terms with $z_{n_{2j-1}}, z_{n_{2j}}, z_{nl}, z_{nd} \in R^\circ$ $(z_{n_{2j-1}}, z_{n_{2j}}, z_{nl}, z_{nd} \in \Lambda)$ of $\sum_2$ and $\sum_4$ is equal to $\sum_1$ ($\sum_3$ resp.). Then

$$6\mathfrak{S}^\circ_R(V, V')=\sum\nolimits_1+\sum\nolimits_3.$$

COROLLARY 4. 1. *If V' in Lemma 4. 1 is discrete harmonic, then*

$$\mathfrak{S}^\circ_R(V, V')=\sum_{n=\mu+1}^{\nu} V_{(n)}\left(\frac{2}{3}\sum_{nl}(V'_{(n)}-V'_{(nl)})+\frac{1}{6}\sum_{nd}(V'_{(n)}-V'_{(nd)})\right).$$

COROLLARY 4. 2. *If V is a function on R with the boundary property $V(z)=0$ for $z\in\Lambda$, and V' is a discrete harmonic function on R, then*

$$\mathfrak{S}_R(V, V')=\mathfrak{S}^\circ_R(V, V')=0. \tag{4. 3}$$

Conversely, if a function V' on R satisfies the relation (4. 3) *for every function V on R with the boundary property $V(z)=0$ for $z\in\Lambda$, then V' is discrete harmonic on R.*

COROLLARY 4. 3. *If V is a discrete harmonic function on R, then*

$$\sum_{n=\mu+1}^{\nu}\left(\frac{2}{3}\sum_{nl}(V_{(n)}-V_{(nl)})+\frac{1}{6}\sum_{nd}(V_{(n)}-V_{(nd)})\right)=0.$$

3. Boundary value problem, minimum problem, monotonicity. The following lemmas are quite analogous to Lemmas 2. 2, 2. 3 and 2. 4 respectively.

LEMMA 4. 2. (Cf. pp. 212–213 of Milne [4].) *Let f be an arbitrarily given function on the boundary Λ of a lattice R. Then there exists one and only one discrete harmonic function V on R which has the boundary property $V(z)=f(z)$ for $z\in\Lambda$.*

LEMMA 4. 3. (Cf. p. 213 of Milne [4].) *Let W be a function on a lattice R, and let V be a discrete harmonic function on R with the boundary property $V(z)=W(z)$ for $z\in\Lambda$. Then the inequality*

$$\mathfrak{S}_R(V)\leqq\mathfrak{S}_R(W)$$

holds, where the equality appears if and only if $W\equiv V$.

LEMMA 4. 4. *Let R_1 and R_2 be the lattices defined in* § 2. 4. *Let c_j $(j=1, \cdots, N-1)$ be a system of real numbers being not simultaneously zero. Let V^k $(k=1, 2)$ be a discrete harmonic function on R_k respectively which has the boundary property*

$$V^k(z)=c_j \quad \text{for} \quad z\in\Lambda^k_j=\Gamma^k_j\cap R_k \qquad (j=0, \cdots, N-1;\ c_0=0).$$

Then the inequality

$$\mathfrak{S}_{R_1}(V^1)\geqq\mathfrak{S}_{R_2}(V^2)$$

holds.

4. Monotone convergence theorem of $\mathfrak{S}_{R_n}(V^n)$. We preserve the notations in § 2. 5. Let V be a discrete harmonic function on R with the boundary property $V(z)=c_j$ for $z\in A_j$ $(j=0, \cdots, N-1; c_0=0)$. Then by Opfer's method (see pp. 293–294 of [6]) we see that

$$S_R(U)>\mathfrak{S}_R(V)>S_{R'}(U'). \tag{4. 4}$$

By Lemmas 1. 1, 4. 4, 2. 6 and (4. 4) we can easily conclude the theorem.

THEOREM 4. 1. *With the notations of Theorem* 2. 1, *let* V^n $(n=0,1,\cdots)$ *be a discrete harmonic function on* R_n *respectively with respect to Mehrstellenverfahren which has the boundary property* $V^n(z)=c_j$ *for* $z\in A_j^n=\Gamma_j^n\cap R_n$ $(j=0,\cdots,N-1; c_0=0)$. *Then*

$$S_{R_n}(U^n)>\mathfrak{S}_{R_n}(V^n)>D_G(u) \qquad (n=0,1,\cdots),$$

and if $R_n\nearrow G$ $(n\to\infty)$,

$$\mathfrak{S}_{R_n}(V^n)\searrow D_G(u) \qquad (n\to\infty).$$

Let R be an N-ply connected lattice $(N\geqq 2)$, and let A_j $(j=0,\cdots,N-1)$ be its boundary components. A discrete harmonic function V_j $(j=0,\cdots,N-1)$ on R which has the boundary property

$$V_j(z)=\begin{cases}1 & \text{for } z\in A_j\\ 0 & \text{for } z\in A-A_j \qquad (A=\cup_{j=0}^{N-1}A_j),\end{cases}$$

is said to be a *discrete harmonic measure of* A_j *on* R (*with respect to Mehrstellenverfahren*) respectively.

COROLLARY 4. 4. *With the notations of Theorem* 2. 1, *let* V_j^n $(j=1,\cdots,N-1)$ *be a discrete harmonic measure of* A_j^n *on* R_n $(n=0,1,\cdots)$ *respectively with respect to Mehrstellenverfahren, and* σ_{jk} $(j,k=1,\cdots,N-1)$ *be the system of modified periodicity moduli of G. Then*

$$S_{R_n}(U_j^n+U_k^n)>\mathfrak{S}_{R_n}(V_j^n+V_k^n)>\sigma_{jk} \qquad (j,k=1,\cdots,N-1;\ n=0,1,\cdots),$$

and if $R_n\nearrow G$ $(n\to\infty)$,

$$\mathfrak{S}_{R_n}(V_j^n+V_k^n)\searrow\sigma_{jk} \qquad (n\to\infty;\ j,k=1,\cdots,N-1).$$

5. Period of conjugate discrete harmonic function. We preserve the notations in § 3. 1. Let V be a discrete harmonic function on R. We set $z_{j_r\pm1}=z_{j_r}\pm(z_j-z_{j-1})$ and $z_{j_l\pm1}=z_{j_l}\pm(z_j-z_{j-1})$, respectively. Furthermore, we set

$$\begin{aligned}\delta\mathfrak{t}_{(j)}=\frac{2}{3}(V_{(j_r)}-V_{(j_l)})+\frac{1}{12}((V_{(j_r-1)}-V_{(j_l)})+(V_{(j_r)}-V_{(j_l-1)})\\ +(V_{(j_r)}-V_{(j_l+1)})+(V_{(j_r+1)}-V_{(j_l)})) \qquad (j=1,\cdots,\iota)\end{aligned} \tag{4. 5}$$

and

$$\mathfrak{t}_\gamma = \sum_{j=1}^{t} \delta \mathfrak{t}_{(j)},$$

where $V_{(k)} = V(z_k)$.

LEMMA 4. 5. (Cf. Satz 8 of Opfer [6].) *If γ and γ' are two Jordan curves defined in Lemma* 3. 1, *then* $\mathfrak{t}_\gamma = \mathfrak{t}_{\gamma'}$.

Proof. It is easily shown by making use of Corollary 4. 3.

$\mathfrak{t}_\gamma$ is said to be a *period of the conjugate discrete harmonic function of V along* γ. We can easily verify that our definition of the period $\mathfrak{t}_\gamma$ is equivalent to Opfer's S_γ ((23) of [6]).

REMARK. Our definition admits to define the *conjugate discrete harmonic function* V^* of a discrete harmonic function V by a relation

$$V^*(z_j) - V^*(z_{j-1}) = \delta \mathfrak{t}_{(j)}$$

with the notation of (4. 5). It is easily verified that the function V^* is discrete harmonic on the set of middle points of meshes of R. The detailed argument is omitted.

6. Periodicity moduli of N-ply connected lattice. We preserve the notations in § 3. 2. Let V_j $(j=0, \cdots, N-1)$ be the discrete harmonic measure of Λ_j on R respectively. By $\mathfrak{t}_{jk}$ $(j, k=0, \cdots, N-1)$ we denote the period of the conjugate discrete harmonic function of V_j along γ_k respectively. By Lemma 4. 5, $\mathfrak{t}_{jk}$ is independent of a particular choice of γ_k. It is immediately seen that

$$\sum_{j=1}^{N-1} \mathfrak{t}_{jk} = 0 \qquad (k=0, \cdots, N-1).$$

Furthermore by Corollary 4. 1 we see that

$$\mathfrak{S}_R(V_j, V_k) = \mathfrak{S}_R^\circ(V_j, V_k) = \sum_{z_n \in \Lambda_j} \left(\frac{2}{3} \sum_{nl} (V'_{k(n)} - V'_{k(nl)}) + \frac{1}{6} \sum_{nd} (V'_{k(n)} - V'_{k(nd)}) \right) = \mathfrak{t}_{kj} \tag{4. 6}$$

$$(j, k=0, \cdots, N-1),$$

which implies

$$\mathfrak{t}_{jk} = \mathfrak{t}_{kj} \qquad (j, k=0, \cdots, N-1),$$

where $V_{k(m)} = V_k(z_m)$. The collection of $\mathfrak{t}_{jk}$ $(j, k=1, \cdots, N-1)$ is said to be a *system of periodicity moduli of R* with respect to Mehrstellenverfahren. Furthermore a *system of modified periodicity moduli of R* with respect to Mehrstellenverfahren is defined by a collection of quantities

$$\mathfrak{s}_{jk} \equiv \mathfrak{S}_R(V_j + V_k) = \mathfrak{t}_{jj} + 2\mathfrak{t}_{jk} + \mathfrak{t}_{kk} \qquad (j, k=1, \cdots, N-1).$$

By Corollary 4. 1 we see that $\mathfrak{s}_{jk}$ is a period of the conjugate discrete harmonic function of V_j+V_k along $\gamma_j+\gamma_k$ respectively.

7. Monotone convergence theorem of periodicity moduli. By Theorem 4. 1 and (4. 6) we obtain the following results analogous to Theorem 3. 1 and Corollary 3. 1.

THEOREM 4. 2. *With the notations of Theorem* 2. 1, *the following hold*:

(i) $$\sum_{j,k=1}^{N-1} c_j c_k t_{jk}^n > \sum_{j,k=1}^{N-1} c_j c_k \mathfrak{t}_{jk}^n > \sum_{j,k=1}^{N-1} c_j c_k \tau_{jk} \qquad (n=0, 1, \cdots);$$

(ii) *If* $R_n \nearrow G$ $(n\to\infty)$, *then*

$$\sum_{j,k=1}^{N-1} c_j c_k \mathfrak{t}_{jk}^n \searrow \sum_{j,k=1}^{N-1} c_j c_k \tau_{jk} \qquad (n\to\infty),$$

where by t_{jk}^n, $\mathfrak{t}_{jk}^n$ *and* τ_{jk} $(j, k=1, \cdots, N-1)$ *we denote the systems of periodicity moduli of* R_n, R_n *with respect to Mehrstellenverfahren and* G *respectively.*

COROLLARY 4. 5. *With the notations of Theorem* 4. 1, *let* s_{jk}^n, $\mathfrak{s}_{jk}^n$ *and* σ_{jk} $(j, k=1, \cdots, N-1)$ *be the systems of modified periodicity moduli of* R_n, R_n *with respect to Mehrstellenverfahren and* G *respectively. Then the following hold*:

(i) $$s_{jk}^n > \mathfrak{s}_{jk}^n > \sigma_{jk} \qquad (j, k=1, \cdots, N-1;\ n=0, 1, \cdots);$$

(ii) *If* $R_n \nearrow G$ $(n\to\infty)$, *then*

$$\mathfrak{s}_{jk}^n \searrow \sigma_{jk} \qquad (n\to\infty;\ j, k=1, \cdots, N-1),$$

and thus

$$\mathfrak{t}_{jk}^n \to \tau_{jk} \qquad (n\to\infty;\ j, k=1, \cdots, N-1).$$

If $N=2$, then Theorem 4. 2 and Corollary 4. 5 coincide to Satz 14 of Opfer [6].

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