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AN APPLICATION OF GREEN'S FORMULA OF A DISCRETE FUNCTION: DETERMINATION OF PERIODICITY MODULI, I

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Introduction. Recently Opfer published a very interesting result [6] (also cf. [5]) in which he concerned himself with the problem of determining the modulus of a doubly connected domain by means of the difference method.

In the present paper we shall consider a corresponding problem for a general multiply connected domain. It is known that for a non-degenerated N-ply connected domain ($N \geqq 2$) there exist $N(N-1)/2$ quantities which are said to be periodicity moduli of the domain, which are conformally invariant, and which have an important meaning in the function theory. We shall concern ourselves with the problem of determining the system of periodicity moduli by means of the difference method (cf. Theorem 3. 1 and Corollaries 2. 4, 3. 1).

Our method making effective use of Green's formula of a discrete function admits to deal with our problem by a unified principle. Also for a harmonic function u and a discrete harmonic function U on a domain G and a lattice R respectively which are constant on each boundary component of G and R, the monotonicity of the Dirichlet integral $D_G(u)$ and the summation $S_R(U)$ (see § 2. 2) with respect to G and R is effectively utilized (cf. Lemmas 1. 1, 2. 4, 2. 5 and 2. 6, and Theorem 2. 1).

For $N=2$ our main results (Theorem 3. 1 and Corollary 3. 1) coincide to Opfer's (Satz 7 of [6]). However even such a special case our method is deferent from his and is more simplified.

§ 1. Periodicity moduli of multiply-connected domain.

1. Periodicity moduli. Let G be an N-ply connected bounded domain on a complex z-plane ($z=x+iy$), where $N \geqq 2$. If there exists a boundary component of G consisting of a point, then G is said to be *degenerated*. A domain G being not degenerated is said to be *non-degenerated*. Let $\Gamma_0, \cdots, \Gamma_{N-1}$ be boundary components of a non-degenerated domain G, and set $\Gamma = \cup_{j=0}^{N-1} \Gamma_j$.

Let u_j ($j=0, \cdots, N-1$) be a *harmonic measure of Γ_j on G* respectively which is defined as a harmonic function on G which has the boundary property

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$$u_j=\begin{cases}1 & \text{on } \Gamma_j,\\ 0 & \text{on } \Gamma-\Gamma_j.\end{cases}$$

Let u_j^* $(j=0, \cdots, N-1)$ be a conjugate harmonic function of u_j on G respectively which is multi-valued. Let γ_j $(j=0, \cdots, N-1)$ be a piecewise analytic Jordan curve in G homotopic to Γ_j respectively. We define

$$\tau_{jk}\equiv-\int_{\gamma_k}\frac{\partial u_j}{\partial n}ds=\int_{\gamma_k}du_j^* \qquad (j, k=0, \cdots, N-1),$$

which is independent of a particular choice of γ_k, where by $\partial/\partial n$ and ds we denote the inner normal derivative on γ_k and the line element of γ_k respectively.

It is easy to see the relations

$$\sum_{j=0}^{N-1}\tau_{jk}=0 \qquad (k=0, \cdots, N-1)$$

and

$$\tau_{jk}=\tau_{kj} \qquad (j, k=0, \cdots, N-1).$$

τ_{jk} $(j, k=1, \cdots, N-1)$ is said to be a *system of periodicity moduli of* G, and the matrix $(\tau_{jk})_{j,k=1,\cdots,N-1}$ is said to be a *matrix of periodicity moduli of* G, which is symmetric and positive definite.

The following theorem is well known.

THEOREM 1. 1. *Let G and G' be two non-degenerated N-ply connected bounded domains. Let Γ_j $(j=0, \cdots, N-1)$ and Γ_j' $(j=0, \cdots, N-1)$ be the boundary components of G and G' respectively. Then G is conformally equivalent to G' so that Γ_j corresponds to Γ_j' respectively if and only if*

$$\tau_{jk}=\tau_{jk}' \qquad (j, k=1, \cdots, N-1),$$

where τ_{jk} $(j, k=1, \cdots, N-1)$ and τ_{jk}' $(j, k=1, \cdots, N-1)$ are the systems of periodicity moduli of G and G' respectively.

The sufficiency in Theorem 1. 1 is called the Torelli theorem.

REMARK. It is known that for each non-degenerated N-ply connected domain there exists a system of 1 $(N=2)$ or $3N-6$ $(N\geqq 3)$ real parameters as follows:

Two domains are conformally equivalent each other if and only if the systems of real parameters for the domains coincide with each other.
Because a number of different periodicity moduli is $N(N-1)/2$, we see that for $N\geqq 5$ there exists yet a dependency among the periodicity moduli.

Let us define

$$(1.\ 1) \qquad \sigma_{jk}\equiv D_G(u_j+u_k)=\int_{\gamma_j+\gamma_k}d(u_j^*+u_k^*)=\tau_{jj}+2\tau_{jk}+\tau_{kk} \qquad (j, k=1, \cdots, N-1),$$

where by $D_G(u)$ we denote the Dirichlet integral of a function u over G. Obviously $\sigma_{jk}>0$, $\sigma_{jk}=\sigma_{kj}$ and $\sigma_{jj}=4\tau_{jj}$ $(j, k=1, \cdots, N-1)$. σ_{jk} $(j, k=1, \cdots, N-1)$ is said to be a *system of modified periodicity moduli.* Obviously the system σ_{jk} $(j, k=1, \cdots, N-1)$ is found from the system τ_{jk} $(j, k=1, \cdots, N-1)$, and vice versa.

2. Monotonicity. With the notations in **1**, let $\{G_n\}_{n=0}^{\infty}$ be an exhaustion of a non-degenerated N-ply connected bounded domain G $(N\geqq 2)$ such that a boundary component Γ_j^n $(j=0, \cdots, N-1)$ of each G_n consists of a piecewise analytic Jordan curve and Γ_j^n is homotopic to Γ_j on G respectively. Let u_j^n $(j=0, \cdots, N-1)$ be the harmonic measure of Γ_j^n on G_n $(n=0, 1, \cdots)$ respectively. Let τ_{jk}^n $(j, k=1, \cdots, N-1)$ be the system of periodicity moduli of G_n $(n=0, 1, \cdots)$ respectively, and σ_{jk}^n $(j, k=1, \cdots, N-1)$ be the system of modified periodicity moduli of G_n $(n=0, 1, \cdots)$ respectively.

LEMMA 1.1. *Let $c_1, \cdots, c_{N-1}$ be a system of real numbers being not simultaneously zero. Then*

$$\sum_{j,k=1}^{N-1} c_j c_k \tau_{jk}^m > \sum_{j,k=1}^{N-1} c_j c_k \tau_{jk}^n \qquad (n>m) \tag{1.2}$$

and

$$\sum_{j,k=1}^{N-1} c_j c_k \tau_{jk}^n \searrow \sum_{j,k=1}^{N-1} c_j c_k \tau_{jk} \qquad (n\to\infty). \tag{1.3}$$

Proof. It is due to a standard method. Set

$$u^n=\sum_{j=1}^{N-1} c_j u_j^n \qquad \text{and} \qquad u=\sum_{j=1}^{N-1} c_j u_j.$$

Then

$$D_{G_n}(u^n)=\sum_{j,k=1}^{N-1} c_j c_k \tau_{jk}^n \qquad \text{and} \qquad D_G(u)=\sum_{j,k=1}^{N-1} c_j c_k \tau_{jk}.$$

The equality

$$D_{G_m}(u^m, u^n)=-\int_{\Gamma^m} u^m \frac{\partial u^n}{\partial n}\, ds=-\int_{\Gamma^n} u^n \frac{\partial u^n}{\partial n}\, ds=D_{G_n}(u^n)$$

$$(n>m;\ \Gamma^n=\cup_{j=0}^{N-1}\Gamma_j^n)$$

implies

$$D_{G_m}(u^n-u^m)=D_{G_m}(u^m)-D_{G_n}(u^n)-D_{G_n-G_m}(u^n),$$

which implies (1.2) and the strong convergence of u^n to u; $\lim_{n\to\infty} D_{G_n}(u-u^n)=0$, where by $D_{G_m}(u^m, u^n)$ we denote the mixed Dirichlet integral of u^m and u^n over G_m. Analogously we see that

$$D_{G_n}(u-u^n)=D_{G_n}(u^n)-D_G(u)-D_{G-G_n}(u).$$

Hence

$$D_{G_n}(u^n)\searrow D_G(u) \qquad (n\to\infty).$$

When we set $c_j=c_k=1$ and $c_l=0$ $(l\neq j, k)$ in Lemma 1.1, we obtain the corollary.

COROLLARY 1.1.

(i) $$\sigma^m_{jk}>\sigma^n_{jk} \qquad (n>m);$$

(ii) $$\sigma^n_{jk}\searrow\sigma_{jk} \qquad (n\to\infty);$$

(iii) $$\tau^n_{jk}\to\tau_{jk} \qquad (n\to\infty).$$

§ 2. Monotone convergence of summation $S_R(U)$.

1. Definitions. By L_h we denote the set $\{h(m+in)\,|\,m, n$: integers$\}$ $(h>0)$ on the z-plane $(z=x+iy)$. By a *mesh* M in L_h we call a set $\{z, z+h, z+ih, z+h(1+i)\}$ for a point $z\in L_h$. Let G be a non-degenerated bounded domain on the z-plane of which the boundary consists of the segments each of which joins two points of L_h and is parallel to one of the coordinate axes. Then G is said to be a *lattice domain with mesh width h.* Obviously a lattice domain with mesh width h is one with mesh width h/n for each positive integer n.

Let G be an N-ply connected lattice domain with mesh width h, Γ_j $(j=0,\cdots,N-1)$ be boundary components of G and set $\Gamma=\cup_{j=0}^{N-1}\Gamma_j$. We set $R=\bar{G}\cap L_h$, $\bar{G}$ being the closure of G. The sets $\Lambda=\Gamma\cap R$, $\Lambda_j=\Gamma_j\cap R$ $(j=0,\cdots,N-1)$ and $R^\circ=R-\Lambda$ are said to be the *boundary* of R, the *boundary components* of R and the *interior* of R respectively. Here we agree that a point of R, Λ and Λ_j $(j=0,\cdots,N-1)$ respectively through which Γ runs for k-times, is counted for k-times. A point $z\in R^\circ$ is said to be an *inner point* of R and a point $z\in\Lambda$ is said to be a *boundary point* of R. When R° is connected (see p. 345 of Collatz [1] for the definition), R is said to be a *lattice with mesh width h.* If G is N-ply connected, then R is said to be *N-ply connected.* A point $z\in L_h$ is said to be *neighboring* to a point $z'\in L_h$ or is said to be a *neighboring point* of z', if $|z-z'|=h$.

Let R be a lattice with mesh width h, and let U be a real function on R. Let z_0 be an inner point of R, and z_j $(j=1,2,3,4)$ be four neighboring points of z_0. If the equation

$$4U_{(0)}-(U_{(1)}+U_{(2)}+U_{(3)}+U_{(4)})=0 \tag{2.1}$$

holds for every $z_0\in R^\circ$, then U is said to be *discrete harmonic* on R, where $U_{(j)}=U(z_j)$ $(j=0,\cdots,4)$.

2. Green's formula. Let R be an N-ply connected lattice with mesh width h, let Λ be its boundary, and let Γ be the boundary of the domain G which defines R. Let $\{z_n\}_{n=1}^{\nu}$ be the set of points of R, and let $\{z_n\}_{n=1}^{\mu}$ $(\mu<\nu)$ be the set of points of R°. Let U and U' be functions on R, and set $U_{(n)}=U(z_n)$ and $U'_{(n)}=U'(z_n)$ $(n=1,\cdots,\nu)$. We consider a bilinear form

$$S_R(U, U')=\sum_{|z_m-z_n|=h, m<n}(U_{(m)}-U_{(n)})(U'_{(m)}-U'_{(n)}).$$

Furthermore, we consider the partial sum $S_R^\circ(U, U')$ of $S_R(U, U')$ which is obtained by elimination of the terms with respect to two points neighboring along the boundary Γ. Here a point $z_m \in A$ is said to be neighboring to a point $z_n \in A$ *along* Γ if and only if $|z_m-z_n|=h$ and the segment $\overline{z_m z_n}\subset\Gamma$. If U or U' is constant on each boundary component A_j $(j=0, \cdots, N-1)$ of R, then we see immediately that

$$S_R^\circ(U, U')=S_R(U, U').$$

Furthermore by $S_R(U)$ and $S_R^\circ(U)$ we denote $S_R(U, U)$ and $S_R^\circ(U, U)$ respectively.

LEMMA 2. 1. (Cf. pp. 34–36 of Courant, Friedrichs and Lewy [2].) *Let U and U' be two functions on a lattice R. Then the formula*

$$\begin{aligned}&S_R^\circ(U, U')+\sum_{n=1}^{\mu}U_{(n)}(U'_{(n_1)}+U'_{(n_2)}+U'_{(n_3)}+U'_{(n_4)}-4U'_{(n)})\\ &=\sum_{n=\mu+1}^{\nu}U_{(n)}\left(\kappa U'_{(n)}-\sum_{k=1}^{\kappa}U'_{(nl_k)}\right)\end{aligned} \tag{2. 2}$$

holds. Here z_{n_j} $(j=1, 2, 3, 4)$ are four neighboring points of z_n, z_{nl_k} $(k=1, \cdots, \kappa$; $\kappa=0, 1, 2$, or $3)$ are the points of R neighboring to z_n which lie on the left of z_n with respect to the oriented curve Γ and which are not neighboring to z_n along Γ, and the summation corresponding to $\kappa=0$ is taken to be vacuous.

COROLLARY 2. 1. *If U' in Lemma 2. 1 is discrete harmonic, then*

$$S_R^\circ(U, U')=\sum_{n=\mu+1}^{\nu}U_{(n)}\left(\kappa U'_{(n)}-\sum_{k=1}^{\kappa}U'_{(nl_k)}\right).$$

COROLLARY 2. 2. *If U is a function on R with the boundary property $U(z)=0$ for $z\in A$, and U' is a discrete harmonic function on R, then*

$$S_R(U, U')=S_R^\circ(U, U')=0. \tag{2. 3}$$

Conversely, if a function U' on R satisfies the relation (2. 3) *for every function U on R with the boundary property $U(z)=0$ for $z\in A$, then U' is discrete harmonic on R.*

Proof. The first assertion is obvious by Corollary 2. 1.

If there existed a point $z_m \in R^\circ$ such that $U'_{(m_1)}+U'_{(m_2)}+U'_{(m_3)}+U'_{(m_4)}-4U'_{(m)}\neq 0$, then we would choose the function U so that $U_{(m)}=1$ and $U_{(n)}=0$ for each $z_n\neq z_m$, and by Lemma 2. 1 we would see that $S_R(U, U')=S_R^\circ(U, U')\neq 0$.

COROLLARY 2. 3. *If U is a discrete harmonic function on R, then*

$$\sum_{n=\mu+1}^{\nu}\left(\kappa U_{(n)}-\sum_{k=1}^{\kappa}U_{(nl_k)}\right)=0.$$

3. Boundary value problem, Minimum problem.

LEMMA 2. 2. (Cf. pp. 203–207 of Milne [4].) *Let f be an arbitrarily given function on the boundary Λ of a lattice R. Then there exists one and only one discrete harmonic function U on R which has the boundary property $U(z)=f(z)$ for $z\in\Lambda$.*

Let R be an N-ply connected lattice ($N\geqq 2$), and let Λ_j ($j=0, \cdots, N-1$) be its boundary components. A discrete harmonic function U_j ($j=0, \cdots, N-1$) on R which has the boundary property

$$U_j(z)=\begin{cases}1 & \text{for} \quad z\in\Lambda_j \\ 0 & \text{for} \quad z\in\Lambda-\Lambda_j \qquad (\Lambda=\cup_{j=0}^{N-1}\Lambda_j),\end{cases}$$

is said to be a *discrete harmonic measure of Λ_j on R* respectively.

LEMMA 2. 3. (Cf. p. 206 of Milne [4].) *Let W be a function on a lattice R, and let U be a discrete harmonic function on R with the boundary property $U(z)=W(z)$ for $z\in\Lambda$. Then the inequality*

$$S_R(U)\leqq S_R(W)$$

holds, where the equality appears if and only if $W\equiv U$.

Lemmas 2. 2 and 2. 3 can be also easily proved by making use of Corollary 2. 2.

4. Monotonicity with respect to lattices with common mesh width. Let R_1 and R_2 be two N-ply connected lattices ($N\geqq 2$) which have the properties:

(i) R_1 and R_2 have a common mesh width h;

(ii) $R_1\subset R_2$;

(iii) A boundary component Γ_j^1 ($j=0, \cdots, N-1$) of G_1 is homotopic to a boundary component Γ_j^2 ($j=0, \cdots, N-1$) of G_2 respectively on G_2, where G_1 and G_2 are the lattice domains which define R_1 and R_2 respectively.

LEMMA 2. 4. *Let R_1 and R_2 be the lattices defined as above. Let c_j ($j=1, \cdots, N-1$) be a system of real numbers being not simultaneously zero. Let U^k ($k=1, 2$) be a discrete harmonic function on R_k respectively which has the boundary property*

$$U^k(z)=c_j \qquad \textit{for} \quad z\in\Lambda_j^k=\Gamma_j^k\cap R_k \qquad (j=0, \cdots, N-1;\ c_0=0).$$

Then the inequality

$$S_{R_1}(U^1)\geqq S_{R_2}(U^2)$$

holds.

Proof. We continue U^1 to R_2 by setting $U^1(z)=c_j$ for each point z of R_2 between Γ_j^1 and Γ_j^2 ($j=0, \cdots, N-1$) respectively. Then by Lemma 2. 3

$$S_{R_1}(U^1)=S_{R_2}(U^1)\geqq S_{R_2}(U^2).$$

5. Monotonicity with respect to subdivision of meshes. Let R be an N-ply connected lattice ($N\geqq 2$), and let R' be the lattice which is obtained by dividing each mesh of R to four equal meshes with half width respectively. Let Λ_j $(j=0,\cdots,N-1)$ and Λ'_j $(j=0,\cdots,N-1)$ be boundary components of R and R' respectively with $\Lambda_j\subset\Lambda'_j$.

LEMMA 2. 5. (Cf. p. 163 of Lelong-Ferrand [3].) *Let R and R' be the lattices defined as above. Let c_j $(j=1,\cdots,N-1)$ be a system of real numbers being not simultaneously zero. Let U and U' be discrete harmonic functions on R and R' respectively which have the boundary properties*

$$U(z)=c_j \qquad \textit{for} \quad z\in\Lambda_j \qquad (j=0,\cdots,N-1;\ c_0=0)$$

and

$$U'(z)=c_j \qquad \textit{for} \quad z\in\Lambda'_j \qquad (j=0,\cdots,N-1;\ c_0=0).$$

Then

$$S_R(U)>S_{R'}(U').$$

Proof. Our proof of which a part is used afterward, is due to Opfer (see Satz 4 of [6]).

The function U is continuously continuable to a function $\tilde{U}$ on the domain G definining R so that for each mesh M of R

$$\tilde{U}=axy+bx+cy+d \qquad (z=x+iy)$$

on the domain $\tilde{M}$ defining M, where a, b, c and d are so determined that $\tilde{U}(z)=U(z)$ for $z\in M$. Especially we can take $\tilde{U}$ as a function on R'. Let z_1, z_2, z_3 and z_4 be four points of M numbered to the positive oriented direction of M. Then an elementary calculation yields

$$S_R(U)-S_{R'}(\tilde{U})=\frac{1}{4}\sum_{M\subset R}(U_{(1)}-U_{(2)}+U_{(3)}-U_{(4)})^2>0, \tag{2. 4}$$

where $U_{(j)}=U(z_j)$ $(j=1,2,3,4)$. Hence by Lemma 2. 3 we see that

$$S_R(U)>S_{R'}(\tilde{U})\geqq S_{R'}(U').$$

6. $\lim_{m,n\to\infty}S_{R_n}(U_n-\tilde{U}_m)=0$. Let R_0 be an N-ply connected lattice on the z-plane ($N\geqq 2$), and R_n $(n=1,2,\cdots)$ be the lattice which is obtained by dividing each mesh of R_{n-1} to four equal meshes with half width respectively. Let Γ_j $(j=0,\cdots,N-1)$ be the boundary components of the domain G defining R_0, and set $\Lambda_j^n=\Gamma_j\cap R_n$ $(j=0,\cdots,N-1;\ n=0,1,\cdots)$. Let c_j $(j=1,\cdots,N-1)$ be a system of real numbers being not simultaneously zero. Let U^n $(n=0,1,\cdots)$ be a discrete harmonic function on R_n which has the boundary property $U^n(z)=c_j$ for $z\in\Lambda_j^n$ $(j=0,\cdots,N-1;\ c_0=0)$

respectively. The function U^n is continuously continuable to a function $\tilde{U}^n$ on G so that for each mesh M of R_n

$$\tilde{U}^n=axy+bx+cy+d \qquad (z=x+iy)$$

on the domain $\tilde{M}$ defining M, where a, b, c and d are so determined that $\tilde{U}^n(z)=U^n(z)$ for $z\in M$. Especially we can take $\tilde{U}^n$ as a function on R_{n+j} ($j\geqq 0$).

By Corollary 2. 2 we see that

$$S_{R_n}(U^n-\tilde{U}^m, U^n)=0 \qquad (n>m). \tag{2. 5}$$

Further by an iteration of the calculation of (2. 4) we see that

$$S_{R_n}(\tilde{U}^m)<S_{R_m}(U^m) \qquad (n>m). \tag{2. 6}$$

(2. 5) and (2. 6) imply that

$$S_{R_n}(U^n-\tilde{U}^m)=S_{R_n}(\tilde{U}^m)-S_{R_n}(U^n)<S_{R_m}(U^m)-S_{R_n}(U^n) \qquad (n>m).$$

Hence we have that there exist the limits

$$\lim_{n\to\infty} S_{R_n}(U^n)=\lim_{m,n\to\infty} S_{R_n}(\tilde{U}^m) \tag{2. 7}$$

and

$$\lim_{m,n\to\infty} S_{R_n}(U^n-\tilde{U}^m)=0. \tag{2. 8}$$

7. Relation between $S_R(U)$ and $D_G(\tilde{U})$. Let R be an N-ply connected lattice on the z-plane ($N\geqq 2$), and Λ_j ($j=0, \cdots, N-1$) be boundary components of R. Let c_j ($j=1, \cdots, N-1$) be a system of real numbers being not simultaneously zero. Let U be a function on R which has the boundary property $U(z)=c_j$ for $z\in\Lambda_j$ ($j=0, \cdots, N-1$; $c_0=0$). The function U is continuously continuable to a function $\tilde{U}$ on the domain G defining R by the same method as $\tilde{U}^n$ in **6**.

Let M be a mesh of R, let z_1, z_2, z_3 and z_4 be four points of M numbered to the positive oriented direction of M, and let us denote $U(z_j)=U_{(j)}$ ($j=1, 2, 3, 4$). An elementary calculation yields

$$D_{\tilde{M}}(\tilde{U})=\frac{1}{3}((U_{(1)}-U_{(2)})^2+(U_{(2)}-U_{(3)})^2+(U_{(3)}-U_{(4)})^2+(U_{(4)}-U_{(1)})^2$$
$$-(U_{(1)}-U_{(2)})(U_{(3)}-U_{(4)})-(U_{(2)}-U_{(3)})(U_{(4)}-U_{(1)})),$$

where $\tilde{M}$ is the domain defining M. We set

$$T_M(U)=(U_{(1)}-U_{(2)})^2+(U_{(2)}-U_{(3)})^2+(U_{(3)}-U_{(4)})^2+(U_{(4)}-U_{(1)})^2.$$

Then

$$\frac{1}{2}T_M(U)-D_{\tilde{M}}(\tilde{U})=\frac{1}{3}(U_{(1)}-U_{(2)}+U_{(3)}-U_{(4)})^2.$$

Hence we have that

$$
\begin{aligned}
S_R(U)-D_G(\tilde{U}) &= \frac{1}{2}\sum_{M\subset R} T_M(U)-\sum_{M\subset R} D_{\tilde{M}}(\tilde{U}) \\
&= \frac{1}{3}\sum_{M\subset R}(U_{(1)}-U_{(2)}+U_{(3)}-U_{(4)})^2>0.
\end{aligned} \tag{2. 9}
$$

8. $\lim_{n\to\infty} S_{R_n}(U^n)=D_G(u)$. With the notations in **6**, let u be a harmonic function on G which has the boundary property $u=c_j$ on Γ_j $(j=0, \cdots, N-1)$. (2. 8) and (2. 9) imply that

$$
\lim_{m,n\to\infty} D_G(\tilde{U}^n-\tilde{U}^m)=0. \tag{2. 10}
$$

On the other hand, by a consequence of Courant, Friedrichs and Lewy (see pp. 47–54 of [2]) we see that $\{\partial\tilde{U}^n/\partial x\}$ and $\{\partial\tilde{U}^n/\partial y\}$ uniformly converge to the functions $\partial u/\partial x$ and $\partial u/\partial y$ respectively almost everywhere on every compact subregion of G. Hence we obtain that

$$
\lim_{n\to\infty} D_G(u-\tilde{U}^n)=0,
$$

which implies that

$$
\lim_{n\to\infty} D_G(\tilde{U}^n)=D_G(u). \tag{2. 11}
$$

By (2. 4), (2. 7) and (2. 9) we see that

$$
\lim_{n\to\infty} S_{R_n}(U^n)=\lim_{n\to\infty} D_G(\tilde{U}^n). \tag{2. 12}
$$

(2. 11) and (2. 12) yield that

$$
S_{R_n}(U^n)\searrow D_G(u) \qquad (n\to\infty).
$$

LEMMA 2. 6. *Let R_0 be an N-ply connected lattice, let R_n $(n=1, 2, \cdots)$ be the lattice which is obtained by dividing each mesh of R_{n-1} to four equal meshes with half width respectively, let G be the lattice domain which defines R_0, and let Γ_j $(j=0, \cdots, N-1)$ be boundary components of G. Let c_j $(j=1, \cdots, N-1)$ be a system of real numbers being not simultaneously zero. Let U^n $(n=0, 1, \cdots)$ be a discrete harmonic function on R^n respectively which has the boundary property $U^n(z)=c_j$ for $z\in\Lambda_j^n=\Gamma_j\cap R_n$ $(j=0, \cdots, N-1; c_0=0)$, and let u be a harmonic function on G which has the boundary property $u=c_j$ on Γ_j $(j=0, \cdots, N-1)$. Then*

$$
S_{R_n}(U^n)\searrow D_G(u) \qquad (n\to\infty).
$$

9. Monotone convergence theorem of $S_{R_n}(U^n)$. Let G be a non-degenerated N-ply connected bounded domain $(N\geqq 2)$. For each sufficiently small $h>0$ there exists a maximal N-ply connected lattice domain $G_0\subset G$ with mesh width h which has the properties:

(i) A boundary component Γ_j^0 $(j=0, \cdots, N-1)$ of G_0 is homotopic to a boundary component Γ_j $(j=0, \cdots, N-1)$ of G respectively on G;

(ii) G_0 defines a lattice R_0.

The lattice R_0 is said to be an *inner maximal lattice of G with mesh width h.*

Let $\{R_n\}_{n=0}^{\infty}$ be a sequence of inner maximal lattices of G with mesh width $h/2^n$ $(n=0, 1, \cdots)$ respectively, and let G_n $(n=0, 1, \cdots)$ be the domain defining R_n respectively. If the sequence $\{G_n\}_{n=0}^{\infty}$ is an exhaustion of G, then $\{R_n\}_{n=0}^{\infty}$ is said to *converge to* G, denoted by $R_n \nearrow G$ $(n\to\infty)$ (Cf. Opfer [6]).

By Lemmas 1. 1, 2. 4 and 2. 6 we can easily conclude the theorem.

THEOREM 2. 1. *Let G be a non-degenerated N-ply connected bounded domain $(N\geqq 2)$, and $\{R_n\}_{n=0}^{\infty}$ be a sequence of inner maximal lattices of G with mesh width $h/2^n$ $(n=0, 1, \cdots)$ respectively. Let Γ_j $(j=0, \cdots, N-1)$ be boundary components of G, and let Γ_j^n $(j=0, \cdots, N-1)$ be boundary components of the domain G_n defining R_n $(n=0, 1, \cdots)$ respectively so determined that Γ_j^n is homotopic to Γ_j on G respectively. Let c_j $(j=1, \cdots, N-1)$ be a system of real numbers being not simultaneously zero. Let U^n $(n=0, 1, \cdots)$ be a discrete harmonic function on R_n respectively which has the boundary property $U^n(z)=c_j$ for $z\in \Lambda_j^n=\Gamma_j^n\cap R_n$ $(j=0, \cdots, N-1; c_0=0)$, and let u be a harmonic function on G which has the boundary property $u=c_j$ on Γ_j $(j=0, \cdots, N-1)$. Then*

$$S_{R_n}(U^n)>D_G(u) \qquad (n=0, 1, \cdots),$$

and if $R_n \nearrow G$ $(n\to\infty)$,

$$S_{R_n}(U^n)\searrow D_G(u) \qquad (n\to\infty).$$

COROLLARY 2. 4. *With the notations of Theorem 2. 1, let U_j^n $(j=1, \cdots, N-1)$ be a discrete harmonic measure of Λ_j^n on R_n $(n=0, 1, \cdots)$ respectively, and σ_{jk} $(j, k=1, \cdots, N-1)$ be the system of modified periodicity moduli of G. Then*

$$S_{R_n}(U_j^n+U_k^n)>\sigma_{jk} \qquad (j, k=1, \cdots, N-1;\ n=0, 1, \cdots),$$

and if $R_n \nearrow G$ $(n\to\infty)$,

$$S_{R_n}(U_j^n+U_k^n)\searrow \sigma_{jk} \qquad (n\to\infty;\ j, k=1, \cdots, N-1).$$

§ 3. Monotone convergence of periodicity moduli.

1. Period of conjugate discrete harmonic function. Let M be a mesh $\{z, z+h, z+ih, z+h(1+i)\}$ in L_h. The point $z+h(1+i)/2$ is said to be a *middle point* of M. A middle point z_1 is said to be *neighboring* to a middle point z_2, if $|z_1-z_2|=h$. Let R be a lattice with mesh width h, and let U be a discrete harmonic function on R. Let γ be a Jordan curve which consists of the segments each of which joins two neighboring middle points of meshes of R, and let $z_0, z_1, \cdots, z_\iota=z_0$ be the middle points through which γ runs and which are numbered successively in positive direction of γ. Then the points

$$z_{j_r}=\frac{z_{j-1}+z_j}{2}+i\frac{z_{j-1}-z_j}{2} \quad \text{and} \quad z_{j_l}=\frac{z_{j-1}+z_j}{2}+i\frac{z_j-z_{j-1}}{2} \qquad (j=1,\cdots,\iota)$$

belong to R. We set

$$\delta t_{(j)}=U(z_{j_r})-U(z_{j_l}) \qquad (j=1,\cdots,\iota)$$

and

$$t_\gamma=\sum_{j=1}^{\iota}\delta t_{(j)}.$$

LEMMA 3.1. (Cf. Satz 1 of Opfer [6].) *If γ and γ' are two Jordan curves defined as above and which are homotopic each other on the domain G defining R, then $t_\gamma=t_{\gamma'}$.*

Proof. It is immediately shown by making use of Corollary 2.3.

t_γ is said to be a *period of the conjugate discrete harmonic function of U along γ.*

2. Periodicity moduli of N-ply connected lattice. Let R be an N-ply connected lattice ($N\geqq 2$), and let A_j ($j=0,\cdots,N-1$) be its boundary components. Let U_j ($j=0,\cdots,N-1$) be the discrete harmonic measure of A_j on R respectively. Let γ_j ($j=0,\cdots,N-1$) be a Jordan curve which consists of the segments each of which joins two neighboring middle points of meshes of R, and which is homotopic to Γ_j respectively on the domain G defining R, where Γ_j is a boundary component of G such that $\Gamma_j\cap R=A_j$. By t_{jk} ($j,k=0,\cdots,N-1$) we denote the period of the conjugate discrete harmonic function of U_j along γ_k respectively. By Lemma 3.1, t_{jk} is independent of a particular choice of γ_k. It is immediately seen that

$$\sum_{j=0}^{N-1}t_{jk}=0 \qquad (k=0,\cdots,N-1).$$

Furthermore by Corollary 2.1 we see that

$$S_R(U_j,U_k)=S_R^\circ(U_j,U_k)=\sum_{z_n\in A_j}\left(\kappa U_{k(n)}-\sum_{p=1}^{\kappa}U_{k(nl_p)}\right)=t_{kj}, \tag{3.1}$$

which implies

$$t_{jk}=t_{kj} \qquad (j,k=0,\cdots,N-1),$$

where $U_{k(n)}=U_k(z_n)$. The collection of t_{jk} ($j,k=1,\cdots,N-1$) is said to be a *system of periodicity moduli of R.* Furthermore a *system of modified periodicity moduli of R* is defined by a collection of quantities

$$s_{jk}\equiv S_R(U_j+U_k)=t_{jj}+2t_{jk}+t_{kk} \qquad (j,k=1,\cdots,N-1).$$

By Corollary 2.1 we see that s_{jk} is a period of the conjugate discrete harmonic function of U_j+U_k along $\gamma_j+\gamma_k$ respectively.

3. Monotone convergence theorem of periodicity moduli.

THEOREM 3.1. *Under the same condition as Theorem* 2.1, *the following hold*:

(i)
$$\sum_{j,k=1}^{N-1} c_j c_k t_{jk}^n > \sum_{j,k=1}^{N-1} c_j c_k \tau_{jk} \qquad (n=0,1,\cdots);$$

(ii) *If* $R_n \nearrow G$ $(n\to\infty)$, *then*

$$\sum_{j,k=1}^{N-1} c_j c_k t_{jk}^n \searrow \sum_{j,k=1}^{N-1} c_j c_k \tau_{jk} \qquad (n\to\infty),$$

where by t_{jk}^n $(j,k=1,\cdots,N-1)$ *and* τ_{jk} $(j,k=1,\cdots,N-1)$ *we denote the systems of periodicity moduli of* R_n *and* G *respectively.*

Proof. When we note that in Theorem 2.1

$$S_{R_n}(U^n)=\sum_{j,k=1}^{N-1} c_j c_k t_{jk}^n \qquad \text{and} \qquad D_G(u)=\sum_{j,k=1}^{N-1} c_j c_k \tau_{jk}$$

because of (3.1), Theorem 2.1 implies the present theorem.

When we set $c_j=c_k=1$ and $c_l=0$ $(l\neq j,k)$ in Theorem 3.1, we obtain the corollary.

COROLLARY 3.1. *With the notations of Theorem* 3.1, *let* s_{jk}^n $(j,k=1,\cdots,N-1)$ *and* σ_{jk} $(j,k=1,\cdots,N-1)$ *be the systems of modified periodicity moduli of* R_n *and* G *respectively. Then the following hold*:

(i)
$$s_{jk}^n > \sigma_{jk} \qquad (j,k=1,\cdots,N-1;\ n=0,1,\cdots);$$

(ii) *If* $R_n \nearrow G$ $(n\to\infty)$, *then*

$$s_{jk}^n \searrow \sigma_{jk} \qquad (n\to\infty;\ j,k=1,\cdots,N-1),$$

and thus

$$t_{jk}^n \to \tau_{jk} \qquad (n\to\infty;\ j,k=1,\cdots,N-1).$$

If $N=2$, then Theorem 3.1 and Corollary 3.1 coincide to Satz 7 of Opfer [6].

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