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DEFICIENCIES OF AN ENTIRE ALGEBROID FUNCTION, II

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§1. It is well known that there is a big gap between two notions of exceptional values in Picard's sense and in Nevanlinna's. This is still true for an algebroid case in general. The authors [2], however, have obtained some curious results for a two- or three-valued entire algebroid function. A typical one is the following:

Let $f(z)$ be a two-valued entire transcendental algebroid function and a_1, a_2 and a_3 be different finite numbers satisfying

$$\sum_{j=1}^{3} \delta(a_j, f) > 2.$$

Then at least one of $\{a_j\}$ is a Picard exceptional value of f.

Here the curiosity lies in the fact that the condition only on the deficiencies implies the existence of a Picard exceptional value in the two-valued case.

In this paper we shall prove the following results.

THEOREM 1. *Let $f(z)$ be a four-valued entire transcendental algebroid function defined by an irreducible equation*

$$F(z, f) \equiv f^4 + A_3 f^3 + A_2 f^2 + A_1 f + A_0 = 0,$$

where A_j are entire. Let a_j, $j=1, \cdots, 6$ be different finite numbers satisfying $\sum_{j=1}^{6} \delta(a_j, f) > 5$, where $\delta(a_j, f)$ indicates the Nevanlinna-Selberg deficiency of f at a_j. Further assume that any two of $\{F(z, a_j)\}$ are not proportional. Then two of $\{a_j\}$ are Picard exceptional values of f.

In this theorem the non-proportionality condition for every pair of $\{F(z, a_j)\}$ cannot be omitted. We shall give a counter example showing this fact in § 4.

THEOREM 2. *Let $f(z)$ be the same as in the above Theorem 1. Let $\{a_j\}_{j=1}^{7}$ be different finite complex numbers satisfying*

$$\sum_{j=1}^{7} \delta(a_j, f) > 6.$$

Then at least three of $\{a_j\}$, say a_1, a_2 and a_3, are Picard exceptional values of f. Further then $\delta(a_4, f) = \delta(a_5, f) = \delta(a_6, f) = \delta(a_7, f) > 3/4$ and if there is another deficiency of f at a_8, then

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$$\delta(a_8, f) \leqq 1 - \delta(a_7, f) < \frac{1}{4}.$$

§2. Proof of Theorem 1. We put

$$g_j(z) = F(z, a_j), \qquad j = 1, \cdots, 6,$$

and assume that all $g_j(z)$, $j=1, \cdots, 6$ are transcendental.

We firstly have

$$(1) \qquad \sum_{j=1}^{5} \delta(a_j, f) > 4$$

and

$$(2) \qquad \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \alpha_4 g_4 + \alpha_5 g_5 = 1,$$

where

$$\alpha_j = 1 / \prod_{k=1, k \neq j}^{5} (a_j - a_k), \qquad j = 1, \cdots, 5.$$

Applying the method in the proof of Theorem 1 in [2] to our case, we have the linear dependency of $\{g_j\}_{j=1}^5$, that is

$$(3) \qquad \alpha_1' g_1 + \alpha_2' g_2 + \alpha_3' g_3 + \alpha_4' g_4 + \alpha_5' g_5 = 0$$

with constants $\{\alpha_j'\}$ not all zero. Here at least two of $\{\alpha_j'\}$ are not zero. Hence we may assume that $\alpha_4' \alpha_5' \neq 0$ and $\alpha_5' = \alpha_5$. Eliminating g_5 from (2) and (3) we have

$$(\alpha_1 - \alpha_1') g_1 + (\alpha_2 - \alpha_2') g_2 + (\alpha_3 - \alpha_3') g_3 + (\alpha_4 - \alpha_4') g_4 = 1.$$

Since any three of $\{\alpha_j - \alpha_j'\}$ are not zero simultaneously, it is sufficient to study the following subcases:

Case 1). $\alpha_1 \neq \alpha_1'$, $\alpha_2 \neq \alpha_2'$, $\alpha_3 \neq \alpha_3'$, $\alpha_4 \neq \alpha_4'$.

Case 2). $\alpha_1 \neq \alpha_1'$, $\alpha_2 \neq \alpha_2'$, $\alpha_3 \neq \alpha_3'$, $\alpha_4 = \alpha_4'$,

(i) $\alpha_1' = \alpha_2' = \alpha_3' = 0$,

(ii) $\alpha_1' = \alpha_2' = 0$, $\alpha_3' \neq 0$,

(iii) $\alpha_1' = 0$, $\alpha_2' \neq 0$, $\alpha_3' \neq 0$, $\alpha_3 \alpha_2' - \alpha_2 \alpha_3' = 0$,

(iv) $\alpha_1' = 0$, $\alpha_2' \neq 0$, $\alpha_3' \neq 0$, $\alpha_3 \alpha_2' - \alpha_2 \alpha_3' \neq 0$,

(v) $\alpha_1' \neq 0$, $\alpha_2' \neq 0$, $\alpha_3' \neq 0$, $\alpha_2 \alpha_1' - \alpha_1 \alpha_2' = \alpha_3 \alpha_1' - \alpha_1 \alpha_3' = 0$,

(vi) $\alpha_1' \neq 0$, $\alpha_2' \neq 0$, $\alpha_3' \neq 0$, $\alpha_2 \alpha_1' - \alpha_1 \alpha_2' = 0$, $\alpha_3 \alpha_1' - \alpha_1 \alpha_3' \neq 0$,

(vii) $\alpha_1' \neq 0$, $\alpha_2' \neq 0$, $\alpha_3' \neq 0$, $\alpha_2 \alpha_1' - \alpha_1 \alpha_2' \neq 0$, $\alpha_3 \alpha_1' - \alpha_1 \alpha_3' \neq 0$.

Case 3). $\alpha_1 \neq \alpha_1'$, $\alpha_2 \neq \alpha_2'$, $\alpha_3 = \alpha_3'$, $\alpha_4 = \alpha_4'$,

(i) $\alpha_1' = \alpha_2' = 0$,

(ii) $\alpha_1' = 0$, $\alpha_2' \neq 0$,

(iii) $\alpha_1' \neq 0$, $\alpha_2' \neq 0$, $\alpha_2\alpha_1' - \alpha_2'\alpha_1 = 0$,

(iv) $\alpha_1' \neq 0$, $\alpha_2' \neq 0$, $\alpha_2\alpha_1' - \alpha_2'\alpha_1 \neq 0$.

The cases 1); 2) (ii), (iv), (vi), (vii); 3) (ii), (iv) lead to an identity of the following type

(A) $$\lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 + \lambda_4 g_4 = 1, \qquad \lambda_1\lambda_2\lambda_3\lambda_4 \neq 0.$$

The cases 2) (v); 3) (iii) lead to the following type

(B) $$\begin{cases} \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 = 1, \\ \lambda_4 g_4 + \lambda_5 g_5 = 1, \qquad \lambda_1 \cdots \lambda_5 \neq 0. \end{cases}$$

The case 3) (i) leads to

(C) $$\begin{cases} \alpha_1 g_1 + \alpha_2 g_2 = 1, \\ \alpha_3 g_3 + \alpha_4 g_4 + \alpha_5 g_5 = 0. \end{cases}$$

The case 2) (iii) leads to

(D) $$\begin{cases} \alpha_1 g_1 + (\alpha_2 - \alpha_2')g_2 + \dfrac{\alpha_3}{\alpha_2}(\alpha_2 - \alpha_2')g_3 = 1, \\ \alpha_1 g_1 + \dfrac{\alpha_4}{\alpha_2'}(\alpha_2' - \alpha_2)g_4 + \dfrac{\alpha_5}{\alpha_2'}(\alpha_2' - \alpha_2)g_5 = 1. \end{cases}$$

The case 2) (i) leads to

(E) $$\begin{cases} \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 = 1, \\ \alpha_4 g_4 + \alpha_5 g_5 = 0. \end{cases}$$

By our assumption the case (E) may be omitted.

In the first place we remark that Valiron [3] proved

$$T(r, f) = \mu(r, A) + O(1),$$

where $A = \max_{0 \leqq j \leqq 3}(1, |A_j|)$ and

$$4\mu(r, A) = \frac{1}{2\pi}\int_0^{2\pi} \log A \, d\theta.$$

Further we have

$$4\mu(r, A)=m(r, g)+O(1),$$

$$g=\max_{1\leqq j\leqq 4}(1, |g_j|).$$

The case (A). In this case we have

$$\sum_{j=1}^{4}\delta(a_j, f)>3$$

and

$$4T(r, f)=m(r, g)+O(1)=m(r, g_1^*)+O(1),$$

where $g_1^*=\max_{1\leqq j\leqq 3}(1, |g_j|)$. Therefore the reasoning in the proof of Theorem 2 in [2] leads to the following type

$$\begin{cases}\lambda_1 g_1+\lambda_2 g_2=1,\\ \lambda_3 g_3+\lambda_4 g_4=0.\end{cases}\tag{4}$$

Further we have

$$\beta_1 g_1+\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_6 g_6=1,\tag{5}$$

where $\beta_j=1/\Pi_{k=1, k\neq j, 5}^{6}(a_j-a_k)$, $j=1, 2, 3, 4, 6$. Eliminating g_1 and g_3 from (4), (2) and (5), we have

$$\begin{cases}\left(\alpha_2-\dfrac{\lambda_2}{\lambda_1}\alpha_1\right)g_2+\left(\alpha_4-\dfrac{\lambda_4}{\lambda_3}\alpha_3\right)g_4+\alpha_5 g_5=1-\dfrac{\alpha_1}{\lambda_1},\\ \left(\beta_2-\dfrac{\lambda_2}{\lambda_1}\beta_1\right)g_2+\left(\beta_4-\dfrac{\lambda_4}{\lambda_3}\beta_3\right)g_4+\beta_6 g_6=1-\dfrac{\beta_1}{\lambda_1}.\end{cases}$$

Since $1-\alpha_1/\lambda_1$ and $1-\beta_1/\lambda_1$ are not zero simultaneously, we may assume $1-\alpha_1/\lambda_1\neq 0$. We consider the following subcases:

$$\alpha_2-\frac{\lambda_2}{\lambda_1}\alpha_1=\alpha_4-\frac{\lambda_4}{\lambda_3}\alpha_3=0,\tag{i}$$

$$\alpha_2-\frac{\lambda_2}{\lambda_1}\alpha_1=0,\qquad \alpha_4-\frac{\lambda_4}{\lambda_3}\alpha_3\neq 0,\tag{ii}$$

$$\alpha_2-\frac{\lambda_2}{\lambda_1}\alpha_1\neq 0,\qquad \alpha_4-\frac{\lambda_4}{\lambda_3}\alpha_3=0,\tag{iii}$$

$$\alpha_2-\frac{\lambda_2}{\lambda_1}\alpha_1\neq 0,\qquad \alpha_4-\frac{\lambda_4}{\lambda_3}\alpha_3\neq 0.\tag{iv}$$

The case (i) gives trivially a contradiction.

The case (ii) leads to

$$\begin{cases}\lambda_1 g_1+\lambda_2 g_2=1.\\ \left(\alpha_4-\dfrac{\lambda_4}{\lambda_3}\alpha_3\right)g_4+\alpha_5 g_5=1-\dfrac{\alpha_1}{\lambda_1}.\end{cases}$$

In this case we have

$$\delta(a_1,f)+\delta(a_2,f)+\delta(a_4,f)+\delta(a_5,f)>3$$

and

$$4T(r,f)=m(r,g_2{}^*)+O(1),$$

where $g_2{}^*=\max(1, |g_1|, |g_4|)$. By the reasoning of the case (B) in the proof of Theorem 2 in [2] we arrive at a contradiction.

The case (iii) leads to

$$\begin{cases}\alpha_1 g_1+\alpha_2 g_2+\alpha_5 g_5=1,\\ \alpha_3 g_3+\alpha_4 g_4=0,\end{cases}$$

which is the type of case (E). Hence this case may be omitted by our assumption.

Consider the case (iv). In this case we have

$$\delta(a_2,f)+\delta(a_4,f)+\delta(a_5,f)>2$$

and

$$4T(r,f)=m(r,g_3{}^*)+O(1),$$

where $g_3{}^*=\max(1, |g_2|, |g_4|)$. Hence by virtue of the argument in the case (A) in the proof of Theorem 2 in [2] we arrive at a contradiction.

The case (B). In this case we have

$$4T(r,f)=m(r,g_4{}^*)+O(1),$$

where $g_4{}^*=\max_{2\leq j\leq 4}(1, |g_j|)$. By virtue of the argument in the case (B) in the proof of Theorem 2 in [2], we similarly have a contradiction.

The case (C). Eliminating g_1 from (C) and (5) we have

$$\frac{1}{\alpha_1}(\alpha_1\beta_2-\alpha_2\beta_1)g_2+\beta_3 g_3+\beta_4 g_4+\beta_6 g_6=1-\frac{\beta_1}{\alpha_1}\neq 0$$

Since $(\alpha_1\beta_2-\alpha_2\beta_1)\neq 0$ and $4T(r,f)=m(r,g_4{}^*)+O(1)$, this case reduces to the case (A), which is a contradiction.

The case (D). Eliminating g_1 from (D) and (5), we have

$$\{\beta_1(\alpha_2-\alpha_2')-\alpha_1\beta_2\}g_2+\left\{\beta_1\frac{\alpha_3}{\alpha_2}(\alpha_2-\alpha_2')-\alpha_1\beta_3\right\}g_3-\alpha_1\beta_4 g_4-\alpha_1\beta_6 g_6=\beta_1-\alpha_1\neq 0.$$

Since the coefficients of g_2 and g_3 are not zero simultaneously, we consider the following subcases:

(i) $\beta_1(\alpha_2-\alpha_2')-\alpha_1\beta_2\neq 0,\quad \beta_1\alpha_3(\alpha_2-\alpha_2')-\alpha_1\alpha_2\beta_3\neq 0,$

(ii) $\beta_1(\alpha_2-\alpha_2')-\alpha_1\beta_2=0,\quad \alpha_1\alpha_3\beta_2\neq\alpha_2\beta_1\beta_3,$

(iii) $\beta_1(\alpha_2-\alpha_2')-\alpha_1\beta_2=0,\quad \alpha_1\alpha_3\beta_2=\alpha_2\beta_1\beta_3,\quad \beta_4(\alpha_2\beta_1-\alpha_1\beta_2)\neq\alpha_4\beta_2(\beta_1-\alpha_1),$

(iv) $\beta_1(\alpha_2-\alpha_2')-\alpha_1\beta_2=0,\quad \alpha_1\alpha_3\beta_2=\alpha_2\beta_1\beta_3,\quad \beta_4(\alpha_2\beta_1-\alpha_1\beta_2)=\alpha_4\beta_2(\beta_1-\alpha_1),$

(v) $\beta_1\alpha_3(\alpha_2-\alpha_2')-\alpha_1\alpha_2\beta_3=0,\quad \alpha_1\alpha_2\beta_3\neq\alpha_3\beta_1\beta_2,$

(vi) $\beta_1\alpha_3(\alpha_2-\alpha_2')-\alpha_1\alpha_2\beta_3=0,\quad \alpha_1\alpha_2\beta_3=\alpha_3\beta_1\beta_2,\quad \beta_4(\alpha_3\beta_1-\alpha_1\beta_3)\neq\alpha_4\beta_3(\beta_1-\alpha_1),$

(vii) $\beta_1\alpha_3(\alpha_2-\alpha_2')-\alpha_1\alpha_2\beta_3=0,\quad \alpha_1\alpha_2\beta_3=\alpha_3\beta_1\beta_2,\quad \beta_4(\alpha_3\beta_1-\alpha_1\beta_3)=\alpha_4\beta_3(\beta_1-\alpha_1).$

All of these cases reduce to the case (A), which is a contradiction.

Thus we obtain a desired contradiction in every case. Therefore at least one of $\{g_j\}_{j=1}^6$ must be a polynomial.

Next we assume that one of $\{g_j\}_{j=1}^6$, say g_1, is a polynomial. Further assume that the others g_j are transcendental. If $\alpha_1 g_1\equiv 1$, then the identity (5) implies

$$\beta_2 g_2+\beta_3 g_3+\beta_4 g_4+\beta_6 g_6=1-\frac{\beta_1}{\alpha_1}\neq 0,$$

which is the type of our case (A). This is a contradiction. If $\alpha g_1\not\equiv 1$, then the identity (2) implies

$$\alpha_2 g_2+\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=1-\alpha_1 g_1.$$

By the reasoning in the proof of Theorem 1 in [2], this case can be handled in the same method as our case (A). Hence we have a contradiction. Therefore at least one of $\{g_j\}_{j=2}^6$ must be a polynomial and the proof of our Theorem 1 is complete.

§ 3. Proof of Theorem 2.

We set

$$g_j(z)=F(z,a_j),\qquad j=1,\cdots,7,$$

and assume that all $g_j(z)$, $j=1,\cdots,7$ are transcendental. Then by the proof of Theorem 1 $\sum_{j=1}^6\delta(a_j,f)>5$ leads to the following type

$$\text{(E)}\qquad \begin{cases}\alpha_1 g_1+\alpha_2 g_2+\alpha_3 g_3=1,\\ \alpha_4 g_4+\alpha_5 g_5=0.\end{cases}$$

Further we have

$$(6)\qquad \gamma_1 g_1+\gamma_2 g_2+\gamma_3 g_3+\gamma_4 g_4+\gamma_7 g_7=1,$$

where $\gamma_j=1/\prod_{k=1,k\neq j,5,6}^7(a_j-a_k)$, $j=1,2,3,4,7$. Firstly eliminating g_1 from (E), (5) and (6) we have

$$(7) \qquad (\alpha_1\beta_2-\alpha_2\beta_1)g_2+(\alpha_1\beta_3-\alpha_3\beta_1)g_3+\alpha_1\beta_4g_4+\alpha_1\beta_6g_6=\alpha_1-\beta_1,$$

$$(8) \qquad (\alpha_1\gamma_2-\alpha_2\gamma_1)g_2+(\alpha_1\gamma_3-\alpha_3\gamma_1)g_3+\alpha_1\gamma_4g_4+\alpha_1\gamma_7g_7=\alpha_1-\gamma_1.$$

All the coefficients of these terms are not zero. It is sufficient from (7) and the argument of our case (A) to consider the following cases:

$$(\mathrm{i}) \qquad \begin{cases}(\alpha_1\beta_2-\alpha_2\beta_1)g_2+(\alpha_1\beta_3-\alpha_3\beta_1)g_3=\alpha_1-\beta_1,\\ \beta_4g_4+\beta_6g_6=0,\end{cases}$$

$$(\mathrm{ii}) \qquad \begin{cases}(\alpha_1\beta_2-\alpha_2\beta_1)g_2+\alpha_1\beta_4g_4=\alpha_1-\beta_1,\\ (\alpha_1\beta_3-\alpha_3\beta_1)g_3+\alpha_1\beta_6g_6=0,\end{cases}$$

$$(\mathrm{iii}) \qquad \begin{cases}(\alpha_1\beta_2-\alpha_2\beta_1)g_2+\alpha_1\beta_6g_6=\alpha_1-\beta_1,\\ (\alpha_1\beta_3-\alpha_3\beta_1)g_3+\alpha_1\beta_4g_4=0,\end{cases}$$

$$(\mathrm{iv}) \qquad \begin{cases}\alpha_1\beta_4g_4+\alpha_1\beta_6g_6=\alpha_1-\beta_1,\\ (\alpha_1\beta_2-\alpha_2\beta_1)g_2+(\alpha_1\beta_3-\alpha_3\beta_1)g_3=0.\end{cases}$$

Assume that the case (i) occurs. Then eliminating g_2 from (8) and (i) we have

$$\begin{aligned}&\{\alpha_1(\beta_2\gamma_3-\beta_3\gamma_2)-\alpha_2(\beta_1\gamma_3-\beta_3\gamma_1)+\alpha_3(\beta_1\gamma_2-\beta_2\gamma_1)\}g_3\\ &\quad+(\alpha_1\beta_2-\alpha_2\beta_1)\gamma_4g_4+(\alpha_1\beta_2-\alpha_2\beta_1)\gamma_7g_7=(\alpha_1\beta_2-\alpha_2\beta_1)-(\alpha_1\gamma_2-\alpha_2\gamma_1)+(\beta_1\gamma_2-\beta_2\gamma_1).\end{aligned}$$

All the coefficients of these terms are not zero. Hence we have a contradiction.

In the cases (ii) and (iii) we have

$$4T(r,f)=m(r,g_5^*)+O(1), \qquad g_5^*=\max(1, |g_1|, |g_3|),$$

$$\alpha_1g_1+\alpha_2g_2+\alpha_3g_3=1$$

and

$$\delta(a_1,f)+\delta(a_2,f)+\delta(a_3,f)>2,$$

which gives similarly a contradiction.

The case (iv) leads to our case (B). Hence we have a contradiction.

Thus we obtain a desired contradiction in every case. Therefore at least one of $\{g_j\}_{j=1}^7$ must be a polynomial. We may suppose without loss in generality that g_1 is a polynomial. Further suppose that the others g_j are transcendental. Then we have

$$\begin{cases}\alpha_2g_2+\alpha_3g_3+\alpha_4g_4+\alpha_5g_5=1-\alpha_1g_1,\\ \beta_2g_2+\beta_3g_3+\beta_4g_4+\beta_6g_6=1-\beta_1g_1,\\ \gamma_2g_2+\gamma_3g_3+\gamma_4g_4+\gamma_7g_7=1-\gamma_1g_1.\end{cases}$$

Here we may assume that $(1-\alpha_1 g_1)(1-\beta_1 g_1)\not\equiv 0$. It is sufficient from the argument of our case (A) to consider the following two cases:

$$\text{(i)}\quad \begin{cases}\alpha_4 g_4+\alpha_5 g_5=1-\alpha_1 g_1,\\ \alpha_2 g_2+\alpha_3 g_3=0,\end{cases}\qquad\qquad \text{(ii)}\quad \begin{cases}\alpha_2 g_2+\alpha_3 g_3=1-\alpha_1 g_1,\\ \alpha_4 g_4+\alpha_5 g_5=0.\end{cases}$$

In the case (i) we have

$$\frac{1}{\alpha_2}(\alpha_2\beta_3-\alpha_3\beta_2)g_3+\beta_4 g_4+\beta_6 g_6=1-\beta_1 g_1\not\equiv 0,$$

which is a contradiction by our standard method.

In the case (ii) we have

$$(\alpha_2\beta_3-\alpha_3\beta_2)g_3+\alpha_2\beta_4 g_4+\alpha_2\beta_6 g_6=\alpha_2-\beta_2-(\alpha_2\beta_1-\alpha_1\beta_2)g_1,$$

$$(\alpha_2\gamma_3-\alpha_3\gamma_2)g_3+\alpha_2\gamma_4 g_4+\alpha_2\gamma_7 g_7=\alpha_2-\gamma_2-(\alpha_2\gamma_1-\alpha_1\gamma_2)g_1.$$

Since the right hand side terms of the above identities are not zero simultaneously, we similarly have a contradiction. Hence at least one of $\{g_j\}_{j=2}^7$ must be a polynomial.

We may suppose that g_2 is a polynomial. Further suppose that g_j, $j=3,\cdots,7$ are transcendental. Then we have

$$\begin{cases}\alpha_3 g_3+\alpha_4 g_4+\alpha_5 g_5=1-\alpha_1 g_1-\alpha_2 g_2,\\ \beta_3 g_3+\beta_4 g_4+\beta_6 g_6=1-\beta_1 g_1-\beta_2 g_2,\\ \gamma_3 g_3+\gamma_4 g_4+\gamma_7 g_7=1-\gamma_1 g_1-\gamma_2 g_2.\end{cases}$$

Since the right hand side terms of the above identities are not zero simultaneously, we similarly have a contradiction.

Therefore at least one of $\{g_j\}_{j=3}^7$, say g_3, must be a polynomial. Since f is transcendental, it clearly follows that all $g_j, j=4,5,6,7$ are transcendental. And we have

$$4T(r,f)=m(r,g_4)+O(1),$$

and

$$\delta(a_4,f)+\delta(a_j,f)>1,\qquad j=5,6,7.$$

Hence by virtue of our standard method we obtain

$$(9)\qquad \begin{cases}\alpha_4 g_4+\alpha_5 g_5=1-\alpha_1 g_1-\alpha_2 g_2-\alpha_3 g_3=0,\\ \beta_4 g_4+\beta_6 g_6=1-\beta_1 g_1-\beta_2 g_2-\beta_3 g_3=0,\\ \gamma_4 g_4+\gamma_7 g_7=1-\gamma_1 g_1-\gamma_2 g_2-\gamma_3 g_3=0.\end{cases}$$

Therefore we obtain a part of the desired result:

$$\delta(a_4, f)=\delta(a_5, f)=\delta(a_6, f)=\delta(a_7, f)>\frac{3}{4}.$$

Suppose that there is another deficiency $\delta(a_8, f)$ satisfying

$$\delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)+\delta(a_4, f)+\delta(a_8, f)>4.$$

Then we have

$$\mu_1 g_1+\mu_2 g_2+\mu_3 g_3+\mu_4 g_4+\mu_8 g_8=1, \tag{10}$$

where $\mu_j=1/\Pi_{k=1, k\neq j, 5, 6, 7}^{8}(a_j-a_k)$. Eliminating g_1, g_2 and g_3 from (9) and (10) we have

$$\mu_4 g_4+\mu_8 g_8=-\begin{vmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & \beta_1 & \beta_2 & \beta_3 \\ 1 & \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & \mu_1 & \mu_2 & \mu_3 \end{vmatrix}\neq 0,$$

which is a contradiction. Hence we have

$$\delta(a_1, f)+\delta(a_2, f)+\delta(a_3, f)+\delta(a_4, f)+\delta(a_8, f)\leqq 4,$$

that is

$$\delta(a_8, f)\leqq 1-\delta(a_4, f)=1-\delta(a_7, f)<\frac{1}{4}.$$

Thus the proof of Theorem 2 is complete.

§4. A counter-example. We shall give here a counter-example showing that the non-proportionality condition for every pair of $\{F(z, a_j)\}$ in Theorem 1 cannot be omitted.

Let g_1 be a transcendental entire function, whose modulus satisfies

$$|g_1(re^{i\theta})|=o(e^{r^2}).$$

Let g_4 be the famous Lindelöf function $f(z; 2, \alpha)$ with $0<\alpha<1$ (cf. [1]). We set

$$g_2=\frac{1}{2}g_1+6, \qquad g_3=g_1-12,$$

$$g_5=-g_4, \qquad g_6=4g_4.$$

Now we consider a four-valued entire algebroid function y defined by

$$F(z, y)=y^4+A_3y^3+A_2y^2+A_1y+A_0=0,$$

where $A_0=g_1$, $A_1=(1/6)(12-3g_1+2g_2-g_3-g_4)$, $A_2=-(1/2)(2+2g_1-g_2-g_3)$ and A_3

$=-(1/6)(12-3g_1+3g_2+g_3-g_4)$. Then by virtue of the same argument as § 6 in [2] we have

$$4T(r, y)=m(r, g_4)(1+\varepsilon(r)), \qquad \lim_{r\to\infty} \varepsilon(r)=0.$$

Since $F(z, 0)=g_1$, $F(z, 1)=g_2$, $F(z, -1)=g_3$, $F(z, 2)=g_4$, $F(z, -2)=g_5$ and $F(z, 3)=g_3$, we obtain

$$\delta(0, y)=\delta(1, y)=\delta(-1, y)=\delta(2, y)=\delta(-2, y)=\delta(3, y)=1.$$

However there is no Picard exceptional value among $\{0, 1, -1, 2, -2, 3\}$.

Further we know that there is no other deficiency of y. In fact, suppose, to the contrary, that there is another deficiency of y at a_7. Then

$$\delta(0, y)+\delta(1, y)+\delta(-1, y)+\delta(2, y)+\delta(-2, y)+\delta(3, y)+\delta(a_7, y)>6.$$

Hence by Theorem 2 there are at least three Picard exceptional values among $\{0, 1, -1, 2, -2, 3, a_7\}$, which is a contradiction.

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