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REMARKS ON THE EXISTENCE OF ANALYTIC MAPPINGS

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§1. Introduction. Let R be an ultrahyperelliptic surface defined by $y^2=g(x)$, $g(x)=(e^K-\gamma)(e^K-\delta)$, $\gamma\delta(\gamma-\delta)\neq 0$, $K(0)=0$ with a non-constant entire function K. We already proved that the Picard constant $P(R)$ of R is four and vice versa.

Let S be an ultrahyperelliptic surface defined by $y^2=G(x)$,

$$G(x)\equiv 1-2\beta_1 e^H-2\beta_2 e^L+\beta_1{}^2 e^{2H}-2\beta_1\beta_2 e^{H+L}+\beta_2^2 e^{2L},$$

$$\beta_1\beta_2\neq 0, \qquad H(0)=L(0)=0$$

with two non-constant entire functions H and L. We already proved that the Picard constant $P(S)$ of S is at least three. If K is a polynomial, then R is called to be of finite order. If H and L are polynomials, then S is called to be of finite order.

In our previous paper [5] we proved that if S is of finite order then $P(S)$ is equal to three with four exceptional cases for which $P(S)=4$. As an easy corollary of the above result we proved the following fact:

Let R and S be ultrahyperelliptic surfaces of finite order in the above sense. Assume that $P(S)=3$. Then there is no non-trivial analytic mapping of R into S.

The first purpose of this paper is to prove the following improvement of the above result:

THEOREM 1. *Let R be an ultrahyperelliptic surface of finite order in the above sense with $P(R)=4$. Let S be an ultrahyperelliptic surface defined above without any assumption on its order. Assume that $P(S)=3$. Then there is no non-trivial analytic mapping of R into S.*

Hiromi-Mutō [1] proved the following result: Let R and S be two ultrahyperelliptic surfaces defined by $y^2=g(x)$ and $w^2=G(z)$, respectively, where G and g are entire functions having no zero other than an infinite number of simple zeros. Let g_c and G_c be the canonical products formed by the zeros of g and G, respectively. Assume that the order $\rho_{g_c}<\infty$ and $0<\rho_{G_c}<\infty$ and that there is a non-trivial analytic mapping of R into S. Then ρ_{g_c} is a positive integral multiple of ρ_{G_c}.

We shall prove the following fact, which is the second purpose of this paper:

THEOREM 2. *Under the same assumptions in Hiromi-Mutō's theorem and*

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denoting the lower order of X by μ_X *we have that* $\mu_{N(r;0,g)}$ *is a positive integral multiple of* $\mu_{N(r;0,G)}$.

This theorem 2 gives a powerful criterion for the non-existence of non-trivial analytic mappings.

Niino [3] posed the following problem: Is there any relation between two non-trivial analytic mappings φ_1 and φ_2 which map analytically the same R into the same S?

His formulation of this problem is somewhat restrictive.

The third and final purpose of this paper is to give some informations on this problem and to give an interesting example.

§ **2.**

LEMMA. *Let* $G(x)$ *be*

$$1-2\beta_1e^H-2\beta_2e^L+\beta_1{}^2e^{2H}-2\beta_1\beta_2e^{H+L}+\beta_2{}^2e^{2L},$$

$$\beta_1\beta_2\neq 0, \qquad H(0)=L(0)=0.$$

Then for an arbitrary given $\varepsilon>0$ *and for a sufficiently large* $r\geqq r_0$

$$(2-\varepsilon)\max(m(r,e^H),\ m(r,e^L))\leqq N_2(r;0,G),$$

where $N_2(r;0,G)$ *indicates the N-function of the simple zeros of* G.

Proof. The last part of this method was suggested by Niino [2]. First of all we shall prove that the equation $y^2=G(x)$ defines an ultrahyperelliptic surface S. Let $f(x)$ be

$$\frac{1}{2}(1+\beta_1e^{2H}-\beta_2e^{2L})+\frac{1}{2}\sqrt{G(x)}.$$

Then f satisfies

$$F(x,f)=f^2-(1+\beta_1e^{2H}-\beta_2e^{2L})f+\beta_1e^{2H}.$$

Now $F(x,0)=\beta_1e^{2H}$ and $F(x,1)=\beta_2e^{2L}$. Thus f is an entire algebroid function, which is at most two-valued and $f\neq 0,1,\infty$ in S. Assume that S is not ultrahyperelliptic. Then either S splits into two punctured discs D_1 and D_2 over $r^*\leqq|x|<\infty$ or S is two-sheeted but one punctured disc over there. If the latter case occurs, then the big Picard theorem implies that the exceptional values are at most two in number when f is transcendental there. When f is not transcendental there, then f can be continued analytically onto $x=\infty$, which shows that f reduces to an algebraic function. Then f takes every value in S at least once excepting ∞. Anyway we arrive at a contradiction. If the former case occurs, we put f_1 and f_2 as two determinations of f in D_1 and D_2, respectively. Assume that both of f_1 and f_2 are transcendental. Then f_1, f_2 have at most two exceptional values ∞, a_1 in D_1 and ∞, a_2 in D_2, respectively. If $a_1\neq a_2$, then a_1

is taken by f_2 in D_2 infinitely often. If $a_1=a_2$, then f has two exceptional values ∞, a_1 in $D_1 \cup D_2$. Anyway f has at most two exceptional values in $D_1 \cup D_2$, which is a contradiction. If one of f_1 and f_2 is not transcendental, we have similarly a contradiction. Thus we have the desired result.

Now we can make use of Selberg's theory on algebroid functions [7]. Since f has two finite exceptional values 0 and 1 and f is regular in S, we have

$$3 \leqq \sum \delta(a) \leqq 2+\xi,$$

where

$$\delta(a)=1-\limsup_{r\to\infty} \frac{N(r; a, f)}{T(r, f)},$$

$$\xi=\liminf_{r\to\infty} \frac{N(r, S)}{T(r, f)},$$

$$2N(r, S)=N_2(r; 0, G)+O(\log r).$$

Hence $\xi \geqq 1$. Further by Valiron's theorem [8] or [7]

$$T(r, f)=\frac{1}{4\pi}\int_0^{2\pi} \log \max (1, |\beta_1 e^{2H}|, \ |1+\beta_1 e^{2H}-\beta_2 e^{2L}|)d\theta+O(1)$$

$$\geqq \frac{1}{2} m(r, \beta_1 e^{2H})+O(1).$$

Therefore $T(r, f) \geqq m(r, e^H)+O(1)$. Thus

$$N(r, S) \geqq (\xi-\varepsilon')T(r, f) \geqq (\xi-\varepsilon')m(r, e^H)+O(1)$$

for $r \geqq r_0$. This implies

$$(2-\varepsilon)m(r, e^H) \leqq N_2(r; 0, G).$$

By symmetry we have

$$(2-\varepsilon)m(r, e^L) \leqq N_2(r; 0, G).$$

Thus we have the desired result.

This Lemma is best possible. Consider the case $2H=L$, $\beta_1^2=16\beta_2$. Then

$$G=(1+\sqrt[4]{\beta_2}e^{H/2})^2(1-\sqrt[4]{\beta_2}e^{H/2})^2(\sqrt{2}-1-\sqrt[4]{\beta_2}e^{H/2})$$

$$\cdot(\sqrt{2}+1+\sqrt[4]{\beta_2}e^{H/2})(\sqrt{2}-1+\sqrt[4]{\beta_2}e^{H/2})(\sqrt{2}+1-\sqrt[4]{\beta_2}e^{H/2}).$$

This implies that

$$N_2(r; 0, G) \sim 4m(r, e^{H/2})=2m(r, e^H).$$

§ 3. Proof of Theorem 1. By our earlier result in [4] we may consider the possibility of the following functional equation

$$f(x)^2 g(x)=G\circ h(x).$$

By the above Lemma we have

$$(2-\varepsilon)\max(m(r, e^{H\circ h}),\ m(r, e^{L\circ h}))\leqq N_2(r; 0, G\circ h).$$

Further we have

$$N_2(r; 0, G\circ h)\leqq N_2(r; 0, g)\sim 2m(r, e^K)$$

$$=\frac{|k_n|}{\pi}r^n\left(1+O\left(\frac{1}{r}\right)\right),$$

where $K(x)=k_n x^n+\cdots+k_1 x,\ k_n\neq 0$. Thus $H\circ h$ and $L\circ h$ must be polynomials. By Pólya's theorem again H, L and h must be polynomials. Now we can make use of our earlier result in [5] and then we have the desired result.

§ 4. **Proof of Theorem 2.** We may change the last past of Hiromi-Mutō's proof of their theorem. Let $h(x)$ be a polynomial of the form $a_0x^\nu+a_1x^{\nu-1}+\cdots+a_\nu$. Then we have for an arbitrary positive number ε (<1)

$$\nu n(|a_0|r^\nu(1+\varepsilon); 0, G_c)+O(1)$$

$$\geqq n(r; 0, G_c\circ h)\geqq \nu n(|a_0|r^\nu(1-\varepsilon); 0, G_c)-O(1)$$

for $r\geqq r_0$. Hence

$$N(|a_0|r^\nu(1+\varepsilon); 0, G_c)+O(\log r)$$

$$\geqq N(r; 0, G_c\circ h)\geqq N(|a_0|r^\nu(1-\varepsilon); 0, G_c)-O(\log r).$$

Since G_c is transcendental and is a canonical product, we have

$$(1+\delta)N(|a_0|r^\nu(1+\varepsilon); 0, G)\geqq N(r; 0, G_c\circ h)\geqq(1-\delta)N(|a_0|r^\nu(1-\varepsilon); 0, G).$$

Hence

$$\nu\mu_{N(r; 0, G)}\geqq \mu_{N(r; 0, G_c\circ h)}\geqq \nu\mu_{N(r; 0, G)}.$$

Further

$$N_2(r; 0, G_c\circ h)\leqq N(r; 0, g_c)=N(r; 0, g)$$

and

$$N(r; 0, g_c)=N(r; 0, G_c\circ h)-2N(r; 0, f),$$

$$N(r; 0, f)\leqq 2T(r, h)=O(\log r),$$

$$N(r; 0, G_c\circ h)=N_2(r; 0, G_c\circ h)+O(\log r).$$

Hence we have

$$\mu_{N(r;0,g)}=\mu_{N(r;0,G_c\circ h)}.$$

Thus we have the desired result:

$$\mu_{N(r;0,g)}=\nu\mu_{N(r;0,G)}.$$

By Theorem 2 together with Hiromi-Mutō's theorem the regularity of growth is preserved by non-trivial analytic mappings under our assumptions.

§ 5. Let φ_1 and φ_2 be non-trivial analytic mappings of R into S. Let h_1 and h_2 be their projections. Assume that there is an algebraic relation between h_1 and h_2, that is, there is an irreducible algebraic equation $F(x, y)=0$ satisfying $F(h_1, h_2)\equiv 0$. Then if one of h_1, h_2 is transcendental the Riemann surface W defined by $F(x, y)=0$ must be of genus at most one by Picard's uniformization theorem, since h_1, h_2 are defined in $|z|<\infty$. Assume that W is of genus one. Then h_j must be doubly periodic. Hence h_j must have poles, which contradicts the regularity. Thus W must be of genus zero.

THEOREM 3. *Suppose that two non-trivial analytic mappings of an ultrahyperelliptic surface R into another such surface S satisfies an algebraic relation. Then the surface defined by the algebraic relation is of genus zero, if at least one of the two projections is transcendental.*

Next we shall give an example. Let $G(z)$ be an entire function whose zeros are $\pm p_n i$, $\pm\sqrt{1+p_n^2}$, where p_n is real positive $\geqq 1$ and $p_n<p_{n+1}$, $p_n\to\infty$ as $n\to\infty$. Let $g(z)$ be $G\circ\sin z$. Then $g(z)$ has no zeros other than an infinite number of simple zeros. On the other hand $G\circ\cos z\equiv g^*(z)$ has the same zeros as $g(z)$. Hence $g^*(z)=e^{L(z)}g(z)$. Let R and S be two surfaces defined by $y^2=g(z)$, $w^2=G(z)$, respectively. Then there are two analytic mappings whose projections are $\cos z$, $\sin z$, respectively.

Hence

$$x^2+y^2=1$$

is satisfied by $x=\cos z$, $y=\sin z$. This is really a circle.

Niino has given an example of parabola recently. Following is an open problem. Is there any example of $y=ax^n$, $n=3$?

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