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ON THE MINIMUM MODULUS OF AN ENTIRE ALGEBROID FUNCTION OF LOWER ORDER LESS THAN ONE

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§1. Kjellberg [1] extended the famous Wiman theorem in the following manner:

Let $f(z)$ be an entire function of lower order μ $(0\leqq\mu<1)$. Then

$$\limsup_{r\to\infty}\frac{\log m^*(r)}{\log M(r)}\geqq\cos\pi\mu,$$

where

$$M(r)=\max_{|z|=r}|f(z)|,\qquad m^*(r)=\min_{|z|=r}|f(z)|.$$

In this paper we shall extend this theorem to an n-valued entire algebroid function of lower order less than one. Our theorem is the following:

THEOREM. 1) *Let $y(z)$ be an n-valued entire algebroid function of lower order μ, $0\leqq\mu<1/2$. Then*

$$\limsup_{r\to\infty}\frac{n^2\overset{+}{\log} m^*(r)}{\log M(r)}\geqq\cos\pi\mu,$$

where, denoting the j-th determination of y by y_j,

$$M(r)=\max_{|z|=r}\max_{1\leqq j\leqq n}|y_j|,\qquad m^*(r)=\min_{|z|=r}\max_{1\leqq j\leqq n}|y_j|.$$

2) *Let $1/2\leqq\mu<1$. Then*

$$\limsup_{r\to\infty}\frac{n\log m^*(r)}{\log M(r)}\geqq\cos\pi\mu.$$

§2. **Preliminary considerations.** Let $F(z, y)=y^n+A_1y^{n-1}+\cdots+A_0=0$ be the defining equation of y. Let A, y^* be

$$\max(|A_1|,\cdots,|A_n|),\qquad \max(|y_1|,\cdots,|y_n|),$$

respectively. Then Valiron [2] proved

$$nT(r, y)-m(r, A)=O(1).$$

Evidently

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$$|A_j(z)| \leqq \sum |y_1 \cdots y_j|,$$

where the summation is taken over all products formed by j different y_{β_l}, $l=1, \cdots, j$ among $y_1, \cdots, y_n$. Hence

$$\begin{aligned}\log |A_j(z)| &\leqq \log \sum |y_1 \cdots y_j| \\ &\leqq \log |y_{\alpha_1} \cdots y_{\alpha_j}| + \log \binom{n}{j} \\ &\leqq j \log y^* + \log \binom{n}{j}.\end{aligned}$$

Thus

$$\log A \leqq \max_{1 \leqq j \leqq n} \left(j \log y^* + \log \binom{n}{j} \right),$$

which implies

$$\min_{|z|=r} \log A \leqq \min_{|z|=r} \max_{1 \leqq j \leqq n} \left(j \log y^* + \log \binom{n}{j} \right).$$

If 1) is the case, then

$$(1) \qquad \log m^*(r, A) \leqq n \overset{+}{\log}\, m^*(r) + O(1).$$

Assume that 2) is the case. The following fact is worth while to be remarked. If $m^*(r)$ does not tend to zero as $r \to \infty$, then there is a sequence $\{r_p\}$ for which $m^*(r_p) \geqq c > 0$. Then

$$\limsup_{r \to \infty} \frac{\log m^*(r)}{\log M(r)} \geqq \limsup_{p \to \infty} \frac{\log m^*(r_p)}{\log M(r_p)} \geqq 0.$$

Hence by $\cos \pi\mu \leqq 0$ the desired result holds trivially. Therefore we may assume that $m^*(r) \to 0$ as $r \to \infty$. In this case we have for $r \geqq r_0$

$$(1') \qquad \log m^*(r, A) \leqq \log m^*(r) + O(1)$$

instead of (1).

On the other hand as Valiron [2] did by

$$\log |y_1 \cdots y_n| = \log |A_n|,$$

$$(2) \qquad \overset{+}{\log}\, y^* + O(1) \leqq \sum_1^n \overset{+}{\log} |y_j| + O(1) \leqq \overset{+}{\log} |A_n| \leqq \overset{+}{\log}\, A.$$

Further

$$(3) \qquad \log g \leqq \overset{+}{\log}\, A + O(1) \leqq \overset{+}{\log}\, g + O(1),$$

where $g_j=F(z, a_j)$ and $g=\max|g_j|$. Therefore, if 1) is the case, then by (1), (2), (3)

$$\frac{n^2 \overset{+}{\log} m^*(r)}{\log M(r)} \geqq \frac{n \log m^*(r, A)+O(1)}{\overset{+}{\log} M(r, A)+O(1)}$$

$$\geqq \frac{n \min\limits_{|z|=r} \log g+O(1)}{\max\limits_{|z|=r} \overset{+}{\log} g+O(1)} \geqq \frac{\sum\limits_{j=1}^{n} \log m^*(r, g_j)+O(1)}{\sum\limits_{j=1}^{n} \log M(r, g_j)+O(1)}.$$

If 2) is the case, then by the remark already mentioned and by (1′), (2), (3)

$$\frac{n \log m^*(r)}{\log M(r)} \geqq \frac{n \min\limits_{|z|=r} \log A+O(1)}{\max\limits_{|z|=r} \overset{+}{\log} A+O(1)}$$

$$\geqq \frac{\sum\limits_{j=1}^{n} \log m^*(r, g_j)+O(1)}{\sum\limits_{j=1}^{n} \log M(r, g_j)+O(1)}.$$

In both cases we may consider the same expression

$$\frac{\sum\limits_{j=1}^{n} \log m^*(r, g_j)}{\sum\limits_{j=1}^{n} \log M(r, g_j)}.$$

Since

$$\sum_{j=1}^{n} \log M(r, g_j) \leqq n \max_{|z|=r} \log g+O(1)$$

$$\leqq n \max_{|z|=r} \overset{+}{\log} g+O(1) \leqq n \sum_{j=1}^{n} \log M(r, g_j)+O(1),$$

the lower order of $\Pi M(r, g_j)$ is equal to μ. Hence there is a sequence $\{r_n\}$ along which

$$\frac{\log \prod\limits_{j=1}^{n} M(r, g_j)}{r^{\mu+\delta}} \to 0$$

for an arbitrary positive number δ.

In what follows we shall give a proof along Kjellberg's idea, borrowing his several estimates for various quantities, for the two-valued case. The general n-valued case can be handled quite similarly.

§ 3. Case $\mu<1/2$.

Let b_j, $j=1, \cdots, N$ be the zeros of $g_1(z)$ in $|z|<R$. Assume that $g_1(0)=1$. Let

$$g_1{}^1(z) \equiv \prod_{n=1}^{N}\left(1-\frac{z}{b_n}\right),$$

$$g_1{}^2(z) \equiv \prod_{n=1}^{N}\left(1-\frac{z}{|b_n|}\right),$$

and

$$g_1(z) \equiv g_1{}^1(z) g_1{}^3(z).$$

The minimum and the maximum of $|g_1{}^\nu(z)|$, $\nu=1, 2, 3$, on $|z|=r$ are denoted by $m_1{}^\nu(r)$ and $M_1{}^\nu(r)$. Similarly we introduce the corresponding quantities for $g_2(z)$. Now we can make use of several estimations due to Kjellberg [1]. Kjellberg's fundamental inequality is his (23):

$$\int_{R_1}^{R_2} \frac{\log m_j{}^2(r)-\cos \pi\lambda \log M_j{}^2(r)}{r^{1+\lambda}} dr$$

$$>k(\lambda) \frac{\log M_j{}^2(R_1)}{R_1{}^\lambda} - K(\lambda) \frac{\log M_j{}^2(R_2)}{R_2{}^\lambda}.$$

Here λ should be $\mu+\delta$ in our case. Let

$$I_j(R_1, R_2)=\int_{R_1}^{R_2} \frac{\log m_j{}^3(r)-\cos \pi(\mu+\delta) \log M_j{}^3(r)}{r^{1+\mu+\delta}} dr,$$

$$A_j(R_1, R_2)=\int_{R_1}^{R_2} \frac{\log m_j{}^2(r) m_j{}^3(r)-\cos \pi(\mu+\delta) \log M_j{}^2(r) M_j{}^3(r)}{r^{1+\mu+\delta}} dr.$$

Let $I(R_1, R_2)=I_1(R_1, R_2)+I_2(R_1, R_2)$, $A(R_1, R_2)=A_1(R_1, R_2)+A_2(R_1, R_2)$. Then

$$\text{(A)} \quad \begin{aligned} A(R_1, R_2) > & k(\mu+\delta) \frac{\log M_1{}^2(R_1) M_2{}^2(R_1)}{R_1{}^{\mu+\delta}} - K(\mu+\delta) \frac{\log M_1{}^2(R_2) M_2{}^2(R_2)}{R_2{}^{\mu+\delta}} \\ & + I(R_1, R_2). \end{aligned}$$

Let R be a sufficiently large value belonging to $\{r_n\}$ for which

$$\log M(2R, g_1) M(2R, g_2) < \varepsilon(2R)^{\mu+\delta}$$

for an arbitrary given $\varepsilon>0$. Let $R_2=R/2$. By the same method as in Kjellberg's paper we finally have

$$A\left(R_1, \frac{1}{2}R\right) > k(\mu+\delta) \frac{\log M(R_1, g_2) M(R_1, g_2)}{R_1{}^{\mu+\delta}} - k(\mu+\delta) \varepsilon 2^{\mu+\delta+2} \left(\frac{R_1}{R}\right)^{1-\mu-\delta}$$

$$-K(\mu+\delta)\varepsilon 2^{1+2\mu+2\delta}-\frac{\varepsilon}{1-\mu-\delta}2^{2\mu+2\delta+2}.$$

Now we chose ε sufficiently small for which

$$K(\mu+\delta)\varepsilon 2^{1+2\mu+2\delta}+\frac{\varepsilon}{1-\mu-\delta}2^{2\mu+2\delta+2}$$

$$<\frac{1}{4}k(\mu+\delta)\frac{\log M(R_1, g_1)M(R_1, g_2)}{R_1^{\mu+\delta}}.$$

Next we choose R satisfying

$$\log M(2R, g_1)M(2R, g_2)<\varepsilon(2R)^{\mu+\delta}$$

and

$$k(\mu+\delta)\varepsilon 2^{2+\mu+\delta}\left(\frac{R_1}{R}\right)^{1-\mu-\delta}$$

$$<\frac{1}{4}k(\mu+\delta)\frac{\log M(R_1, g_1)M(R_1, g_2)}{R_1^{\mu+\delta}}.$$

Thus we have

$$A\left(R_1, \frac{1}{2}R\right)>\frac{1}{2}k(\mu+\delta)\frac{\log M(R_1, g_1)M(R_1, g_2)}{R_1^{\mu+\delta}}>0.$$

Hence there is a sequence $\{r_n^*\}$ such that

$$\log m_1^2(r)m_1^3(r)m_2^2(r)m_2^3(r)-\cos\pi(\mu+\delta)\log M_1^2(r)M_1^3(r)M_2^2(r)M_2^3(r)>0$$

along $\{r_n^*\}$, $r_n^*\to\infty$ as $n\to\infty$. By

$$m^*(r, g_j)\geqq m_j^1(r)m_j^3(r)\geqq m_j^2(r)m_j^3(r),$$

$$M(r, g_j)\leqq M_j^1(r)M_j^3(r)\leqq M_j^2(r)M_j^3(r)$$

and by $\cos\pi(\mu+\delta)>0$,

$$\log m^*(r, g_1)m^*(r, g_2)-\cos\pi(\mu+\delta)\log M(r, g_1)M(r, g_2)>0$$

along $\{r_n^*\}$. Thus

$$\limsup_{r\to\infty}\frac{\log m^*(r, g_1)m^*(r, g_2)}{\log M(r, g_1)M(r, g_2)}\geqq\cos\pi(\mu+\delta).$$

Letting $\delta\to 0$, theorem follows for $0\leqq\mu<1/2$.

§4. Case $1/2\leqq\mu<1$.

As Kjellberg did we replace $I(R_1, R_2)$ by

$$J(R_1, R_2)=\int_{R_1}^{R_2} \frac{(1-\cos \pi(\mu+\delta)) \log m_1^3(r) m_2^3(r)}{r^{1+\mu+\delta}} dr$$

in (A). Then we arrive at the final result similarly.

Further we can remove the assumptions $g_1(0)=1$, $g_2(0)=1$ as in Kjellberg's.

REFERENCES

[1] KJELLBERG, B., On the minimum modulus of entire functions of lower order less than one. Math. Scand. **8** (1960), 189–197.

[2] VALIRON, G., Sur la dérivée des fonctions algébroïdes. Bull. Soc. Math. **59** (1931), 17–39.

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