KŌDAI MATH. SEM. REP.
22 (1970), 138—141

ON A CONFORMAL TRANSFORMATION OF A RIEMANNIAN MANIFOLD

BY HITOSI HIRAMATU

Ishihara and Obata [2] proved the following

THEOREM. *If M is a differentiable and connected Riemannian manifold of dimension >2, which is not locally conformally Euclidean and if M admits a conformal transformation φ such that the associated function α_φ satisfies $\alpha_\varphi(x)<1-\varepsilon$ or $\alpha_\varphi(x)>1+\varepsilon$ for each $x\in M$, ε being a positive number, then φ has no fixed point.*

On the other hand, the author [1] studied that a differentiable and connected Riemannian manifold admitting a conformal transformation, group of sufficiently high dimension is locally conformally Euclidean. In connection with the above theorem, in this note, the author will obtain results concerning the fixed point of a conformal transformation of a Riemannian manifold and concerning the locally conformally flatness of the Riemannian manifold.

Let $\mathfrak{M}$ be a differentiable[1] and connected Riemannian manifold with the fundamental metric tensor field g. A diffeomorphism φ on $\mathfrak{M}$ is called a conformal transformation on $\mathfrak{M}$ if there exists a positive valued function α_φ on $\mathfrak{M}$ such that $\varphi g=\alpha_\varphi g$[2] holds, and a homothetic transformation on $\mathfrak{M}$ if α_φ is constant on $\mathfrak{M}$. The function α_φ connected with φ is called the associated function of φ. The α_φ is necessarily differentiable. If α_φ is identically equal to unity, then φ is nothing else than an isometry on $\mathfrak{M}$.

Let φ be a conformal transformation on $\mathfrak{M}$ and a_φ and A_φ denote $\inf\{\alpha_\varphi(x);x\in\mathfrak{M}\}$ $(\geqq 0)$ and $\sup\{\alpha_\varphi(x);x\in\mathfrak{M}\}$ $(\leqq\infty)$ respectively. Then $a<A_\varphi$ if and only if φ is not a homothetic transformation, $a_\varphi=A_\varphi$ if and only if φ is a homothetic transformation and $a_\varphi=A_\varphi=1$ if and only if φ is an isometry.

We shall denote by (A) the following property: *there exists a real number ε such that $0<\varepsilon<1$ and such that for each point $x\in\mathfrak{M}$ either $\alpha_\varphi(x)<1-\varepsilon$ or $\alpha_\varphi(x)>1+\varepsilon$ holds.* Since $\mathfrak{M}$ is assumed to be connected and α_φ is continuous on $\mathfrak{M}$, $\{\alpha_\varphi(x);x\in\mathfrak{M}\}$ is a connected subset in real number space. Therefore the property (A) is equivalent to a property that only one of the following (1) and (2) occurs: (1) $\alpha_\varphi(x)<1-\varepsilon$ for all $x\in\mathfrak{M}$ and (2) $\alpha_\varphi(x)>1+\varepsilon$ for all $x\in\mathfrak{M}$. We remark that if (A)

Received November 13, 1969.

1) Here and hereafter, by differentiability we understand that of class C^∞.

2) Definition of φ is as follows. If f is a function on $\mathfrak{M}$, $\varphi f=f\circ\varphi^{-1}$; if X is a contravariant vector field on $\mathfrak{M}$, $(\omega X)f=\varphi(X(\varphi^{-1}f))$ for all functions f on $\mathfrak{M}$; if ω is a covariant vector field on $\mathfrak{M}$, $(\varphi\omega)X=\varphi(\omega(\varphi^{-1}X))$ for all contravariant vector fields X on $\mathfrak{M}$; and so on.

is assumed the φ is not an isometry.

LEMMA 1. *If* (A) *is assumed, then* $a_\varphi>1$ *or* $a_{\varphi^{-1}}>1$, φ^{-1} *being the inverse of* φ *and* $a_{\varphi^{-1}}=\inf\{\alpha_{\varphi^{-1}}(x); x\in\mathfrak{M}\}$.

Proof. If the case (2) occurs, the result is clear. To prove our result, it suffices to consider the case in which (1) occurs. Considering the inverse φ^{-1} of φ, we have

$$(\varphi^{-1}\varphi)g=\varphi^{-1}(\alpha_\varphi g)=(\varphi^{-1}\alpha_\varphi\cdot\alpha_{\varphi^{-1}})g$$

from which $1/\alpha_\varphi\circ\varphi=\alpha_{\varphi^{-1}}$ because $\alpha_{\varphi^{-1}\varphi}=1$. It follows that $a_{\varphi^{-1}}>1$.

Under the condition (A), by considering the inverse φ^{-1} of φ if necessary, we can assume without loss of generality that $a_\varphi>1$. Hereafter we shall use this fact.

LEMMA 2. *If* (A) *is assumed, then for any given points* p *and* q *of* $\mathfrak{M}$ *and for any given positive integer* m *the relation*

$$d(\varphi^m p, \varphi^m q)\leqq(a_\varphi)^{-m/2}d(p, q)$$

holds, where d *denotes the metric function on* $\mathfrak{M}$ *connected with* g.

Proof. Let σ: $[t_0, t_1]\to\mathfrak{M}$ be a piecewisely C'-differentiable curve joining p to q. Then the length $L(\varphi\circ\sigma)$ of the transformed curve $\varphi\circ\sigma$ joining φp to φq is given by the integral

$$L(\varphi\circ\sigma)=\int_{t_0}^{t_1}\left[g_{(\varphi\circ\sigma)(t)}\left(\varphi\frac{d\sigma}{dt}, \varphi\frac{d\sigma}{dt}\right)\right]^{1/2}dt$$

$$=\int_{t_0}^{t_1}\left[\frac{1}{\alpha_\varphi((\varphi\circ\sigma)(t))}g_{\sigma(t)}\left(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\right)\right]^{1/2}dt.$$

Since $a_\varphi\leqq\alpha_\varphi((\varphi\circ\sigma)(t))$ for all $t\in[t_0, t_1]$, we have

$$L(\varphi\circ\sigma)\leqq(a_\varphi)^{-1/2}\int_{t_0}^{t_1}\left[g_{\sigma(t)}\left(\frac{d\sigma}{dt}, \frac{d\sigma}{dt}\right)\right]^{1/2}dt$$

$$=(a_\varphi)^{-1/2}L(\sigma),$$

where $L(\sigma)$ denotes the length of σ. It follows from the above relation that

$$d(\varphi p, \varphi q)\leqq(a_\varphi)^{-1/2}d(p, q)$$

and consequently for any given positive integer m

$$d(\varphi^m p, \varphi^m q)\leqq(a_\varphi)^{-m/2}d(p, q).$$

Now we shall prove the following

THEOREM 1. *Let $\mathfrak{M}$ be a differentiable, connected and complete Riemannian manifold and let φ be a conformal transformation on $\mathfrak{M}$. If* (A) *is assumed, then φ has only one fixed point.*

Proof. From the assumption (A), by using Lemma 1, we can assume without loss of generality that $a_\varphi > 1$. Take any point p of $\mathfrak{M}$. Then, for any given positive integers m and l, we have by using Lemma 2

$$\begin{aligned} d(\varphi^m p, \varphi^{m+l} p) &\leqq d(\varphi^m p, \varphi^{m+1} p) + d(\varphi^{m+1} p, \varphi^{m+2} p) + \cdots + d(\varphi^{m+l-1} p, \varphi^{m+l} p) \\ &\leqq (a_\varphi)^{-m/2} d(p, \varphi p) + \cdots + (a_\varphi)^{-(m+l-1)/2} d(p, \varphi p) \\ &< (a_\varphi)^{-m/2} d(p, \varphi p) \sum_{s=0}^{\infty} (a_\varphi)^{-s/2}. \end{aligned}$$

It follows from the above relation that a sequence of points $\{\varphi^m p\}_{m=1}^{\infty}$ is a Cauchy sequence because the series in the right hand side of the above relation converges. Since $\mathfrak{M}$ is assumed to be complete, the sequence of points has the limit point p_0. It is easily proved that φ leaves p_0 invariant. Next, if x_0 and y_0 are two fixed points of φ, then from Lemma 2, we have

$$d(x_0, y_0) = d(\varphi^m x_0, \varphi^m y_0) \leqq (a_\varphi)^{-m/2} d(x_0, y_0)$$

for any positive integer m from which $d(x_0, y_0) = 0$ and hence $x_0 = y_0$.

From Theorem 1 and the already expressed theorem of Ishihara and Obata, we have

THEOREM 2. *Let $\mathfrak{M}$ be a differentiable, connected and complete Riemannian manifold of dimension >2 and let φ be a conformal transformation on $\mathfrak{M}$. If* (A) *is assumed, then $\mathfrak{M}$ is locally conformally Euclidean.*

As a corollary to Theorem 2, we have the following fact due to Ishihara and Obata [2].

COROLLARY 1. *Let $\mathfrak{M}$ be a differentiable, connected and complete Riemannian manifold of dimension >2, which is not locally conformally Euclidean. If $\mathfrak{M}$ admits a conformal transformation φ, then α_φ can take value unity or an arbitrary value closed to unity.*

Since $\mathfrak{M}$ is assumed to be connected and α_φ is continuous, if $\mathfrak{M}$ is compact, then the set $\{\alpha_\varphi(x); x \in \mathfrak{M}\}$ is compact and connected subset in real number space and hence is a closed interval. Therefore, we have

COROLLARY 2. *Let $\mathfrak{M}$ be a differentiable, connected and compact Riemannian manifold of dimension >2, which is not locally conformally Euclidean. If $\mathfrak{M}$ admits a conformal transformation φ, then α_φ takes value unity.*

Bibliography

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FACULTY OF ENGINEERING,
KUMAMOTO UNIVERSITY.