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# ON THE GROWTH OF ALGEBROID FUNCTIONS WITH SEVERAL DEFICIENCIES, II

BY MITSURU OZAWA

In our previous paper [6] we proved the following result:

Let $y(z)$ be an $n$-valued transcendental entire algebroid function with $n$ finite deficient values $a_j$, $j=1, \cdots, n$. Then the lower order of $y(z)$ is positive.

A corresponding result for a general algebroid function was established with an additional condition. In this paper we shall prove the following theorem:

THEOREM 1. *Let $y(z)$ be an $n$-valued transcendental algebroid function. Assume that $y$ has $n+1$ deficient values $a_j$, $j=1, \cdots, n+1$. Then the lower order of $y$ is positive.*

Toda [7] generalized the following Nevanlinna theorem [4] to algebroid functions: Let $f(z)$ be a meromorphic function of order $\lambda<\infty$. Then there is a positive constant $k(\lambda)$ for which

$$K(f)=\overline{\lim_{r\to\infty}}\frac{N(r; 0, f)+N(r; \infty, f)}{T(r, f)}\geqq k(\lambda),$$

unless $\lambda$ is a positive integer.

Toda's definition of $k(\lambda)$ is

$$\inf K(f)=\inf \overline{\lim_{r\to\infty}}\frac{\sum_{j=1}^{n+1} N(r; a_j, y)}{T(r, y)},$$

where infimum is taken over all the $n$-valued algebroid functions of order $\lambda$.

Again it is an important problem to determine the exact value of $k(\lambda)$. We shall determine it for $0\leqq\lambda\leqq 1$.

THEOREM 2.

$$k(\lambda)=\begin{cases}1 & \text{for } 0\leqq\lambda<1/2,\\ \sin\pi\lambda & \text{for } 1/2\leqq\lambda\leqq 1.\end{cases}$$

As an obvious corollary we have

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COROLLARY 1.

$$\sum_{j=1}^{n+1} \delta(a_j, y) \leqq \begin{cases} n & for \quad 0 \leqq \lambda < 1/2, \\ n+1-\sin \pi\lambda & for \quad 1/2 \leqq \lambda \leqq 1. \end{cases}$$

§1. **Proof of Theorem 1.** Edrei-Fuchs [2] proved the following inequality: For a meromorphic function $f(z)$

$$T(r, f) \leqq \frac{4}{\sigma-1} T(\sigma r, f) + \max\{N(\sigma r; 0, f),\ N(\sigma r; \infty, f)\} + O(\log r),$$

where $\sigma > 1$, $r > 2$. Let $F(z, y) \equiv A_n y^n + \cdots + A_0 = 0$ be the defining equation of $y$. Let $g_j$ be $F(z, a_j)$. Put $f_j = g_j / g_{n+1}$. Applying the above inequality to $f_j$, we have

$$T(r, f_j) \leqq \frac{4}{\sigma-1} T(\sigma r, f_j) + \max\{N(\sigma r; 0, g_j),\ N(\sigma r; 0, g_{n+1})\} + O(\log r).$$

Summing up these inequalities, we have

$$\sum_{j=1}^{n} T(r, f_j) \leqq \frac{4}{\sigma-1} \sum_{j=1}^{n} T(\sigma r, f_j)$$

$$+ \sum_{j=1}^{n} \max\{N(\sigma r; 0, g_j),\ N(\sigma r; 0, g_{n+1})\} + O(\log r).$$

By Cartan's [1] and Toda's inequalities [7] we have

$$nT(r, y) \leqq \frac{4n^2}{\sigma-1} T(\sigma r, y) + n^2 c' T(\sigma r, y) + O(\log r),$$

where $\gamma = \max(1-\delta(a_j, y)) < c' < c < 1$ and $r \geqq r_0 > 2$. Now we have

$$\frac{T(\sigma r, y)}{T(r, y)} \geqq \frac{1}{n} \frac{1}{\dfrac{4}{\sigma-1} + c' + \dfrac{A}{n^2} \dfrac{\log r}{T(\sigma r, y)}}$$

$$\geqq \frac{1}{n} \frac{1}{\dfrac{4}{\sigma-1} + c}$$

for $r \geqq r_1 \geqq r_0$. Taking $\sigma = 1 + 4/c(1-c)$, we have

$$\frac{T(\sigma r, y)}{T(r, y)} \geqq \frac{1}{n} \frac{1}{c(2-c)}.$$

The same reasoning remains valid as in [2] and then we have the desired result.

§2. **Proof of Theorem 2.** Firstly assume that $\lambda = 0$. Then by Theorem 1 there are at most $n$ deficient values. Hence

$$K(y)=\varlimsup_{r\to\infty}\frac{\sum_{j=1}^{n+1}N(r;a_j,y)}{T(r,y)}\geqq 1.$$

Let $y$ be

$$g(z)y^n-g(z)+1=0,$$

where $g(z)$ is an arbitrary transcendental entire function of order zero. Evidently

$$nT(r,y)\sim T(r,g).$$

By the well-known result there is no deficient value of $g(z)$ other than $\infty$. Hence $\delta(\infty,y)=0$, which shows

$$\varlimsup_{r\to\infty}\frac{N(r;\infty,y)}{T(r,y)}=1.$$

However $y$ has $n$ Picard exceptional values $\exp(2\pi ji/n)$, $j=1,\cdots,n$. Hence $K(y)=1$. Thus $k(0)=1$.

Secondly assume that $\lambda=1$. We may consider

$$y^n+e^z-1=0.$$

Evidently $K(y)=0$. Thus $k(1)=0$.

In the third place assume that $0<\lambda<1$. Let $g_j$ be $F(z,a_j)$. Denote its zeros by $b_\nu$. Then

$$g_j(z)=c\prod_{\nu=1}^{\infty}\left(1-\frac{z}{b_\nu}\right).$$

Here we may assume that $g_j(0)\neq 0$. This assumption does not make any trouble in our problem. Let $\hat{g}_j(z)$ be

$$|c|\prod_{\nu=1}^{\infty}\left(1+\frac{z}{|b_\nu|}\right).$$

Then

$$\begin{aligned} m(r,g_j)&\leqq m(r,\hat{g}_j)\\ &=\frac{1}{\pi}\int_0^\infty N(t;0,\hat{g}_j)\frac{r\sin\beta_j}{t^2+2tr\cos\beta_j+r^2}dt+O(\log r)\\ &=\frac{1}{\pi}\int_0^\infty N(t;0,g_j)\frac{r\sin\beta_j}{t^2+2tr\cos\beta_j+r^2}dt+O(\log r)\end{aligned}$$

where $\beta_j$ depends on $r$. Since

$$nT(r, y) \leqq \sum_{j=1}^{n+1} m(r, g_j),$$

$$nT(r, y) \leqq \sum_{j=1}^{n+1} \int_0^\infty N(t; 0, g_j)\ P(t, r, \beta_j)dt + O(\log r),$$

where

$$P(t, r, \beta_j) = \frac{1}{\pi} \frac{r \sin \beta_j}{t^2 + 2tr \cos \beta_j + r^2}.$$

Let $P(t, r, \tau) = \max P(t, r, \beta_j)$. Then

$$nT(r, y) \leqq \int_0^\infty \sum_{j=1}^{n+1} N(t; 0, g_j)\ P(t, r, \tau)dt + O(\log r).$$

Hence

$$nT(r, y) \leqq nK(y) \int_0^\infty T(t, y) P(t, r, \tau)dt + O(\log r).$$

Now we make use of the same process as in [3]. Then we have

$$1 \leqq \sup_{0 \leqq \tau \leqq \pi} K(y) \frac{\sin \tau\lambda}{\sin \pi\lambda}.$$

If $0 < \lambda < 1/2$, then $\sin \tau\lambda \leqq \sin \pi\lambda$. Hence

$$K(y) \geqq 1.$$

If $1/2 \leqq \lambda < 1$, then $\sin \tau\lambda \leqq 1$. Hence

$$K(y) \geqq \sin \pi\lambda.$$

Now we consider equality parts. Let $f(z; \lambda)$ be the Lindelöf function

$$f(z; \lambda) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{b_\nu}\right),$$

$$b_\nu = \nu^{1/\lambda}, \qquad \nu = 1, 2, 3, \cdots.$$

Let $h_\alpha(z) = f(\alpha^{1/\lambda}(z+c); \lambda)$. The asymptotic behavior of $f(z; \lambda)$ is well known [4]. Now we consider

$$h_\alpha(z) y^n - h_\alpha(z) + 1 = 0.$$

Then we have

$$K(y) = \overline{\lim_{r\to\infty}} \frac{\sum_{j=1}^{n+1} N(r; a_j, y)}{T(r, y)} = \overline{\lim_{r\to\infty}} \frac{N(r; \infty, y)}{T(r, y)}$$

for $a_j=\exp(2\pi ji/n)$, $j=1, \cdots, n$; $a_{n+1}=\infty$ and further

$$K(y)=\begin{cases} 1 & for \quad 0<\lambda<1/2, \\ \sin \pi\lambda & for \quad 1/2\leqq\lambda<1. \end{cases}$$

Hence Theorem 2 follows.

§ **3.** By the way we state the following theorem.

THEOREM 3. *Let $y(z)$ be an n-valued transcendental entire algebroid function of order $\lambda$, $0<\lambda<1$. Let $M(r, y)$ be the maximum modulus of $y$ on $|z|=r$. Then there is at least one $a_j$ among $n$ different finite numbers $a_\nu, \nu=1, \cdots, n$, satisfying*

$$\varlimsup_{r\to\infty} \frac{nN(r; a_j, y)}{\log M(r, y)} \geqq \frac{\sin \pi\lambda}{\pi\lambda}.$$

*Proof.* Evidently we have

$$\begin{aligned}\log M(r, y)&=\max_{|z|=r} \max_{1\leqq\nu\leqq n} \log|y_\nu(z)| \\ &\leqq \max_{|z|=r} \max_{1\leqq\nu\leqq n} \overset{+}{\log}|y_\nu(z)| \\ &\leqq \max_{|z|=r} \sum_1^n \overset{+}{\log}|y_\nu(z)|.\end{aligned}$$

By Valiron's argument [8]

$$\begin{aligned}\sum_1^n \overset{+}{\log}|y_\nu(z)| &\leqq \log A(z)+O(1) \\ &\leqq \log g(z)+O(1),\end{aligned}$$

where

$$A(z)=\max(1, |A_{n-1}|, \cdots, |A_0|),$$

$$g(z)=\max(|g_1|, \cdots, |g_n|),$$

$$g_\nu(z)=F(z, a_\nu).$$

Here $F(z, y)=0$ is the defining equation of $y$ and $A_\nu$ is the coefficient of $y^\nu$, $A_n\equiv1$. Further we have

$$\begin{aligned}\max_{|z|=r} \log g(z)&=\log \max_{|z|=r} g(z) \\ &=\log \max_{1\leqq\nu\leqq n} \max_{|z|=r} |g_\nu(z)| \\ &=\max_{1\leqq\nu\leqq n} \log M(r, g_\nu).\end{aligned}$$

Let $g_\nu(z)$ be

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{b_k}\right)$$

and $\hat{g}$ be

$$\sum_{k=1}^{\infty}\left(1+\frac{z}{|b_k|}\right).$$

Then

$$M(r, g_\nu)\leqq M(r, \hat{g}_\nu).$$

Further

$$\log M(r, \hat{g}_\nu)=r\int_0^{\infty} N(t; 0, g_\nu)\frac{dt}{(t+r)^2}$$

Hence

$$\log M(r, y)\leqq r\max_{1\leqq\nu\leqq n}\int_0^{\infty} N(t; 0, g_\nu)\frac{dt}{(t+r)^2}+O(1)$$

$$=r\max_{1\leqq\nu\leqq n} n\int_0^{\infty} N(r; a_\nu, y)\frac{dt}{(t+r)^2}+O(1).$$

Assume that for all $\nu$

$$\varlimsup_{r\to\infty}\frac{nN(r; a_\nu, y)}{\log M(r, y)}<\frac{\sin\pi\lambda}{\pi\lambda}.$$

Then

$$\frac{nN(r; a_\nu, y)}{\log M(r, y)}<\frac{\sin\pi\lambda}{\pi\lambda}-\varepsilon\equiv U, \qquad \varepsilon>0$$

for $r\geqq r_0$. Thus

$$\log M(r, y)<rU\int_{r_0}^{\infty}\log M(t, y)\frac{dt}{(t+r)^2}+O(1).$$

Now we make use of the notion of Pólya peaks. Let $\lambda>\delta>0$, $\lambda+\delta<1$. Then there is a sequence $\{r_n\}$ such that

$$\frac{\log M(t, y)}{t^{\lambda-\delta}}\leqq\frac{\log M(r_n, y)}{r_n^{\lambda-\delta}}, \qquad r_0\leqq t\leqq r_n,$$

$$\frac{\log M(t, y)}{t^{\lambda+\delta}}\leqq\frac{\log M(r_n, y)}{r_n^{\lambda+\delta}} \qquad r_n\leqq t.$$

Thus, using $r$ instead of $r_n$

$$\log M(r, y) < Ur\int_{t_0}^{r} \log M(r, y)\left(\frac{t}{r}\right)^{\lambda-\delta} \frac{dt}{(t+r)^2}$$

$$+Ur\int_{t_0}^{\infty} \log M(r, y)\left(\frac{t}{r}\right)^{\lambda+\delta} \frac{dt}{(t+r)^2} + O(1)$$

$$= Ur \log M(r, y)\left[\int_{t_0}^{r}\left(\frac{t}{r}\right)^{\lambda-\delta} \frac{dt}{(t+r)^2} + \int_{r}^{\infty}\left(\frac{t}{r}\right)^{\lambda+\delta} \frac{dt}{(t+r)^2}\right] + O(1).$$

Hence

$$1 < U\cdot V + O\left(\frac{1}{\log M(r, y)}\right),$$

$$V = r\int_{t_0}^{r}\left(\frac{t}{r}\right)^{\lambda-\delta} \frac{dt}{(t+r)^2} + r\int_{r}^{\infty}\left(\frac{t}{r}\right)^{\lambda+\delta} \frac{dt}{(t+r)^2}.$$

$V$ can be obtained explicitely.

$$V = \frac{\pi(\lambda+\delta)}{\sin \pi(\lambda+\delta)} + O(\delta) + O\left(\frac{1}{r}\right).$$

Thus $r\to\infty$ along $\{r_n\}$ implies

$$1 \leqq U\left\{\frac{\pi(\lambda+\delta)}{\sin \pi(\lambda+\delta)} + O(\delta)\right\}$$

and then letting $\delta\to 0$ we have

$$1 \leqq U\frac{\pi\lambda}{\sin \pi\lambda} = \left(\frac{\sin \pi\lambda}{\pi\lambda} - \varepsilon\right)\frac{\pi\lambda}{\sin \pi\lambda}$$

$$= 1 - \varepsilon\frac{\pi\lambda}{\sin \pi\lambda} < 1,$$

which is a contradiction. Hence Theorem 3 follows.

§ **4.** It is very easy to prove

$$k(\lambda) \leqq \begin{cases} \dfrac{|\sin \pi\lambda|}{q+|\sin \pi\lambda|}, & q < \lambda \leqq q+\dfrac{1}{2}, \quad q\text{: integer}, \\[2ex] \dfrac{|\sin \pi\lambda|}{q+1}, & q+\dfrac{1}{2} < \lambda \leqq q+1, \quad q\text{: integer}. \end{cases}$$

Consider the Lindelöf function $f(z; \lambda)$ already defined. In this case $\lambda \geqq 1$. Consider $f(z; \lambda)y^n - f(z; \lambda) + 1 = 0$. Evidently we have

$$K(y)=K(f(z;\lambda))=1-\delta(0,f(z;\lambda))$$

$$=\begin{cases}\dfrac{|\sin\pi\lambda|}{q+|\sin\pi\lambda|}, & q\leqq\lambda\leqq q+\dfrac{1}{2},\quad q=[\lambda],\\[2ex] \dfrac{|\sin\pi\lambda|}{q+1}, & q+\dfrac{1}{2}<\lambda<q+1,\quad q=[\lambda].\end{cases}$$

Thus we have

$$k(\lambda)\leqq K(y),$$

which is the desired result.

§ **5.** It should be remarked that theorems 2 and 3 can be formulated by making use of the lower order $\mu$ instead of the order $\lambda$. We shall not give any proof of them here.

§ **6.** By the way we shall give a supplementary fact to our previous result [5].

THEOREM 4. *Let $y$ be a two-valued entire algebroid function of order $\lambda$ (or of lower order $\mu$)* $0\leqq\lambda\leqq 1$ *(or* $0\leqq\mu\leqq 1$*). Suppose that there are three finite different values $a_1, a_2, a_3$ satisfying*

$$\delta(a_1,y)+\delta(a_2,y)+\delta(a_3,y)>2.$$

*Then* $\lambda>5/6$ *(or* $\mu>5/6$*).*

*Proof.* By the previous result in [8] we have

$$\delta(a_1,y)=1,\qquad \delta(a_2,y)=\delta(a_3,y)>\frac{1}{2}$$

for example. Hence by corollary 1 for $1/2\leqq\lambda\leqq 1$

$$\frac{5}{2}<\delta(\infty,y)+\delta(a_1,y)+\delta(a_2,y)\leqq 3-\sin\pi\lambda.$$

Thus

$$\sin\pi\lambda<\frac{1}{2}.$$

This implies $\lambda>5/6$. For $0\leqq\lambda<1/2$

$$\delta(\infty,y)+\delta(a_1,y)+\delta(a_2,y)\leqq 2$$

by corollary 1, which is untenable.

This is best possible. Consider again $f(z;\lambda)$. Then the two-valued entire algebroid function $y$ defined by

$$y^2+f(z;\lambda)y-1=0$$

satisfies $\delta(0, y)=1$, $\delta(1, y)=\delta(-1, y)=1-\sin \pi\lambda$ for $\lambda>1/2$. Then

$$\delta(0, y)+\delta(1, y)+\delta(-1, y)=3-2 \sin \pi\lambda>2$$

if and only if $5/6<\lambda\leqq 1$.

We can prove a similar result for the three-valued case.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.