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MINIMAL IMMERSIONS OF COMPACT RIEMANNIAN MANIFOLDS IN COMPLETE AND NON-COMPACT RIEMANNIAN MANIFOLDS

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§ 0. Introduction.

As is well known a closed surface in a Euclidian 3-space has at least one point where Gaussian curvature is positive, and hence no closed minimal surface exists in a Euclidean 3-space. This result was generalized by Myers [3] to higher dimensions and certain Riemannian manifolds. One of his results in [3] states that there exists no closed minimal hypersurface in a complete and simply connected Riemannian manifold of non-positive curvature. One of the essential ideals for proof of the theorem is "local concavity". This idea was used by Tompkins [4] in order to investigate the lower bound of the dimension of the Euclidean space in which a compact and flat Riemannian manifold can be immersed isometrically.

In this paper, we shall observe the idea of Tompkins from another point of view. We shall find certain property of second fundamental form of a compact Riemannian manifold immersed in a complete and non-compact Riemannian manifold of non-negative curvature. We shall prove the existence of a point and a unit normal vector e at the point on a compact Riemannian manifold immersed in a complete and non-compact Riemannian manifold of non-negative (positive) curvature at which the eigenvalues of the second fundamental form with respect to e are all non-negative (positive). Our main results obtained in the present paper will state as follows.

THEOREM A. *A compact Riemannian manifold N of dimension n cannot be immersed minimally in an $(n+1)$-dimensional complete and non-compact Riemannian manifold of positive Ricci curvature.*

THEOREM B. *A compact Riemannian manifold N of dimension n cannot be immersed minimally in an $(n+m)$-dimensional complete and non-compact Riemannian manifold of positive curvature.*

Our essential tool for the proofs of the results is quite analogous as the local concavity except the point of view. The compactness of an immersed Riemannian manifold N in a complete and non-compact Riemannian manifold M ensures the existence of a point on N which is the nearest to the point at infinity. If the

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ambient manifold has non-negative curvature, the point of N nearest to the point at infinity will have the semi-definite second fundamental form with respect to the unit normal vector to N which is the starting direction to a ray from N to the point at infinity.

In § 1, we state definitions and notations used in the present paper. In § 2, we shall prove the basic lemma which plays an important role for the proofs of our results. In § 3, we shall prove some consequences of the lemmas stated in § 2. Theorems A and B will be proved in this section.

§ 1. Preliminaries.

Throughout this paper let M be an $(n+m)$-dimensional complete connected and non-compact Riemannian manifold of class C^∞ and N be an n-dimensional compact Riemannian manifold of class C^∞ immersed isometrically in M. For a point $x \in M$, we denote by M_x the tangent space of M at x, and for two tangent vectors $u, v \in M_x$, we also denote by $\langle u, v\rangle$ the inner product of u and v with respect to the Riemannian metric of M. Geodesics of M are all parametrized by arc length. For any two disjoint compact sets A and B in M, let $G(A, B)$ be the set of all shortest geodesics starting from a point $x \in A$ and ending at a point $y \in B$ such that $d(x, y) = d(A, B)$, where $d(x, y)$ means the distance between x and y with respect to the Riemannian metric of M. For a compact subset A in M, a geodesic $L=\{\gamma(t)\}$ $(0 \leqq t < \infty)$ is by definition a ray from A to the point at infinity ∞ if both $\gamma(0) \in A$ and $\Gamma \mid [0, t] \in G(A, \gamma(t))$ hold for any $t>0$. We denote by $G(A, \infty)$ the set of all rays from A to ∞.

Now let ι be an isometric immersion $\iota\colon N \to M$. $\mathcal{F}(M)$ and $\mathcal{F}(N)$ be the orthonormal frame bundles of M and N respectively, and B the set of all frames $b=(p, e_1, \cdots, e_n, \cdots, e_{n+m})$ such that $(p, e_1, \cdots, e_n) \in \mathcal{F}(N)$ and $(\iota(p), \iota_*(e_1), \cdots, \iota_*(e_n), e_{n+1}, \cdots, e_{n+m}) \in \mathcal{F}(M)$. Then the map $\tilde{\iota}\colon B \to \mathcal{F}(M)$ is naturally defined by $\tilde{\iota}(b)=(\iota(p), \iota_*(e_1), \cdots, \iota_*(e_n), e_{n+1}, \cdots, e_{n+m}) \in \mathcal{F}(M)$, where $b=(p, e_1, \cdots, e_{n+m})$. The structure equations of $\mathcal{F}(M)$ are given by

$$
\begin{aligned}
&d\bar{\omega}_A = \sum \bar{\omega}_{AB} \wedge \bar{\omega}_B,\\
&d\bar{\omega}_{AB} = \sum \bar{\omega}_{AC} \wedge \bar{\omega}_{CB} + \bar{\Omega}_{AB}, \qquad \bar{\Omega}_{AB} = \frac{1}{2} \sum \bar{R}_{ABCD} \bar{\omega}_C \wedge \bar{\omega}_D,\\
&\bar{\omega}_{AB} + \bar{\omega}_{BA} = 0,\\
&A, B, C, D = 1, \cdots, n+m,
\end{aligned}
$$

where $\bar{\omega}_A$, $\bar{\omega}_{AB}$ and $\bar{\Omega}_{AB}$ are the basic forms, the connection forms and the curvature forms respectively. Putting $\omega_A = \tilde{\iota}^* \bar{\omega}_A$ and $\omega_{AB} = \tilde{\iota}^* \bar{\omega}_{AB}$, we get

$$\omega_\alpha = 0 \qquad \text{for} \quad \alpha = n+1, \cdots, n+m.$$

Hence we obtain

$$\omega_{i\alpha} = \sum A_{\alpha ij} \omega_j, \qquad A_{\alpha ij} = A_{\alpha ji}, \qquad i, j = 1, \cdots, n.$$

The mean curvature H of $\iota(N)$ is defined by

$$H=\left[\sum_{\alpha}(\text{trace }(A_{\alpha ij}))^2\right]^{1/2}.$$

In the local argument, we identify the elements of N with those of $\iota(N)$. N is by definition a minimal submanifold if $H=0$ is satisfied at every point of it. Since $A_{\alpha ij}$ can be considered as a symmetric matrics, we can choose a frame at a point $p\in N$ such that

$$(A_{n+1ij})|_p=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

is satisfied with respect to a fixed unit normal vector e_{n+1} at p, where λ_i is called the eigenvalue of the second fundamental form with respect to e_{n+1} at p.

§ 2. Basic lemma.

Let N be a compact Riemannian manifold of dimension n immersed isometrically in a complete and non-compact Riemannian manifold M of dimension $n+m$. There is a sequence of points $\{x_k\}$, $x_k\in M$ such that $d(\iota(N), x_k)>k$ by non-compactness of M. Let $\Gamma_k=\{\gamma_k(t)\}$ $(0\leqq t\leqq a_k)$ be a shortest geodesic such that $\Gamma_k\in G(\iota(N), x_k)$ and put $p_k=\gamma_k(0)\in\iota(N)$. Then we can choose a sequence of the family of geodesics $\{\Gamma_k\}$ in such a way that both $\{p_k\}$ and $\{\gamma_k'(0)\}$ converge to a point $p\in\iota(N)$ and a unit tangent vector $v\in M_p$ respectively. We see that the geodesic Γ defined by $\gamma(t)=\exp_p tv$ is a ray from the compact subset $\iota(N)$ to the point at infinity. Then we shall prove the following

LEMMA 2. 1. *Assume that the curvature of M is non-negative everywhere. Then there exist a point $p\in\iota(N)$ and a unit normal vector e_{n+1} where all of eigenvalues of the second fundamental from $\sum A_{n+1ij}\omega_i\omega_j$ with respect to e_{n+1} do not have different sign.*

Proof. Let Γ be a ray from $\iota(N)$ to ∞ whose starting point is $\gamma(0)=p\in\iota(N)$. We shall prove that the second fundamental form with respect to the tangent vector to Γ at p has non-negative eigenvalues. We note that for arbitrary 1-parameter variation α: $[0, l]\times(-\varepsilon, \varepsilon)\to M$ along any subarc $\Gamma\,|\,[0, l]$ such that $\alpha(0, s)\in N$, $\alpha(t, 0)=\gamma(t)$ and $\alpha(l, s)=\gamma(l)$ for all $s\in(-\varepsilon, \varepsilon)$ and all $t\in[0, l]$, we obtain $L''(0)\geqq 0$. Let $(x, e_1, \cdots, e_{n+m})$ be a frame field defined locally in a neighborhood of p such that $e_{n+1}(p)=\gamma'(0)$. Then for any unit tangent vector $v\in N_p$, we denote by $A_{n+1}v$ the vector $\sum A_{n+1ij}v_ie_j$ where $v=\sum v_ie_i(p)$. It suffices to show that

$$\langle A_{n+1}v, v\rangle\geqq 0$$

holds for any unit tangent vector $v\in N_p$. Let v be a fixed unit tangent vector and X the unit parallel vector field along Γ defined by $X(0)=v$, Suppose that the function $t \to K(X(t), \gamma'(t))$ is identically zero for all $t \geqq 0$. Then we see that X is a Jacobi field along Γ and the vector field Y defined by $Y(t) = ((l-t)/l)X(t)$ is also a Jacobi field along it such that $Y(0)=X(0)$, $Y(l)=0$ for any fixed $l>0$. Therefore we obtain a 1-parameter variation α: $[0, l] \times (-\varepsilon, \varepsilon) \to M$ along $\Gamma | [0, l]$ such that $\alpha(0, s) \in N$, $\alpha(t, 0) = \gamma(t)$ and $\alpha(l, s) = \gamma(l)$ for all $s \in (-\varepsilon, \varepsilon)$ and $t \in [0, l]$ whose variation vector field is the Jacobi field Y defined above. Then we get

$$L''(0) = \int_0^l (\langle Y', Y' \rangle - \langle R(Y, \gamma')\gamma', Y \rangle) dt + \langle A_{n+1} v, v \rangle \geqq 0$$

where the inequality follows from $\Gamma | [0, l] \in G(\iota(N), \gamma(l))$. Hence we have

$$\frac{1}{l} + \langle A_{n+1} v, v \rangle \geqq 0$$

for all $l>0$. Therefore we obtain $\langle A_{n+1} v, v \rangle \geqq 0$.

Next, suppose that there is a point $\gamma(t_0)$ at which $K(X(t_0), \gamma'(t_0)) > 0$. As is stated in Lemma 1 in [1], the differential equation given by

$$\varphi'' + K(X(t), \gamma'(t)) \cdot \varphi = 0, \qquad t \geqq t_0$$

with the initial condition $\varphi(t_0) = 1$, $\varphi'(t_0) = 0$ has a zero point $\varphi(t_0 + \tau_1)$ for some $\tau_1 > 0$. It follows from the definition of φ that

$$\int_{t_0}^{t_0+\tau_1} (\langle (\varphi X)', (\varphi X)' \rangle - K(\varphi X, \gamma') \langle \varphi X, \varphi X \rangle) dt = 0.$$

Let Y be the vector field along $\Gamma | [0, t_0 + \tau_1]$ defined by

$$Y(t) = \begin{cases} X(t) & (0 \leqq t \leqq t_0), \\ \varphi X(t) & (t_0 \leqq t \leqq t_0 + \tau_1). \end{cases}$$

We can construct a 1-parameter variation α: $[0, t_0 + \tau_1] \times (-\varepsilon, \varepsilon) \to M$ associated with Y such that $\alpha(0, s) \in N$, $\alpha(t, 0) = \gamma(t)$, $\alpha(t_0 + \tau_1, s) = \gamma(t_0 + \tau_1)$. Then we have

$$\begin{aligned} 0 \leqq L''(0) &= \int_0^{t_0+\tau_1} (\langle Y', Y' \rangle - K(Y, \gamma') \langle Y, Y \rangle) dt + \langle A_{n+1} v, v \rangle \\ &< \int_{t_0}^{t_0+\tau_1} (\langle Y', Y' \rangle - K(Y, \gamma') \langle Y, Y \rangle) dt + \langle A_{n+1} v, v \rangle \\ &= \langle A_{n+1} v, v \rangle. \end{aligned}$$

Then the proof is completed.

LEMMA 2.2. *Let N be a compact Riemannian manifold immersed minimally in an $(n+1)$-dimensional complete and non-compact Riemannian manifold of non-negative curvature. Let Γ be a ray from N to ∞. Then we have $K(X, \gamma'(t)) = 0$ for any tangent vector $X \in M_{\gamma(t)}$ and any $t \geqq 0$.*

Proof. Putting $\gamma(0)=p$ and $e_{n+1}(p)=\gamma'(0)$, we have $\lambda_i \geqq 0$ for $i=1, \cdots, n$ by Lemma 2. 1, where λ_i is the eigen-value of the second fundamental form at p. Since N is minimal, we have $\sum \lambda_i=0$, from which $\lambda_i=0$ holds for $i=1, \cdots, n$. Suppose that there is a point $\gamma(t_0)$ at which $K(X_0, \gamma'(t_0))>0$ holds for some $X_0 \in M_{\gamma(t_0)}$. We can construct a vector field Y along $\Gamma \mid [0, t_0+\tau_1]$ and the 1-parameter variation α as is stated in the proof of Lemma 2. 1. Then we have $L''(0)<\langle A_{n+1}X(0), X(0)\rangle =0$, which is a contradiction.

§ 3. Some consequences.

Making use of Lemma 2. 1 we obtain the following

PROPOSITION 3. 1. *Every compact Riemannian manifold immersed isometrically in a complete and non-compact Riemanuian manifold of positive cuvvature has at least one point and a unit noumal vector e at the point where the second fundamental form with respect to e is positive definit. Especially the index of relative nullity is zero at the point.*

Now Lemma 2. 2 shows the existence of a point on a compact Riemannian manifold N minimally immersed in a complete and non-compact $(n+1)$-dimensional Riemannian manifold of non-negative curvature at which all the eigenvalues of the second fundamental form are zero, i.e., geodesic at the point. Then we obtain

PROPOSITION 3. 2. *Let N be an n-dimensional compact Riemannian manifold minimally immersed in an $(n+1)$-dimensional complete and non-compact Riemannian manifold M of non-negative curvature. Suppose that the square of the norm of the second fundamental form is constant everywhere. Then N is totally geodesic.*

As a straightforward consequences of Lemmas 2. 1 and 2. 2 we have Theorems A and B. Theorem A implies that if a complete Riemannian manifold M of positive Ricci curvature admits a compact minimal hypersurface, then M is neccessarily compact.

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