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SUBMANIFOLDS OF CODIMENSION 2 OF A EUCLIDEAN SPACE

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The main purpose of the present paper is to generalize, to the case of submanifolds of codimension 2, a famous theorem of Liebmann [1] and Süss [3]: The only convex hypersurface of a Euclidean space with constant mean curvature is a sphere.

In § 1, we recall fundamental concepts and formulas for submanifolds of codimension 2 of a Euclidean space assuming that the mean curvature vector field never vanishes and taking it as the first normal to the submanifolds.

In § 2, we prove integral formulas for general submanifolds of codimension 2 of a Euclidean space.

§ 3 is devoted to the study of submanifolds whose mean curvature vector field is parallel with respect to the connection induced in the normal bundle.

In the last section 4, we study submanifolds which admit a normal vector field passing through a fixed point.

§ 1. Preliminaries.

Let E be an $(n+2)$-dimensional Euclidean space and X the position vector $\overrightarrow{\mathrm{OP}}$ representing a point P of E, O being the origin. Let S be an n-dimensional C^∞ differentiable closed and orientable manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and imbedded in E with C^∞ differentiable imbedding map i: $S \to E$, where and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. We identify S with the submanifold $i(S)$ and refer S sometimes as the submanifold of E.

Let

$$X = X(x^h) \tag{1.1}$$

be the parametric representation of S and put

$$X_i = \partial_i X, \qquad \partial_i = \partial/\partial x^i. \tag{1.2}$$

We assume that n linearly independent vectors $X_1, X_2, \cdots, X_n$ tangent to S give the positive orientation of S. If we put

$$g_{ji} = X_j \cdot X_i, \tag{1.3}$$

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the dot representing the inner product of vectors in E, then g_{ji} are the components of the metric tensor of S with the Riemannian metric induced from the Euclidean metric of E.

If we denote by ∇_i the operator of covariant differentiation with respect to the Christoffel symbols $\{{}^{h}_{ji}\}$ formed with the g_{ji}, then we see that the vectors of E

$$\nabla_j X_i = \partial_j X_i - \left\{ {h \atop j\,i} \right\} X_h \tag{1. 4}$$

are all normal to the submanifold S and consequently the so-called mean curvature vector

$$H = \frac{1}{n} g^{ji} \nabla_j X_i \tag{1. 5}$$

is an intrinsic normal vector field defined along S.

We assume that the mean curvature vector field H never vanishes along S and take the first unit normal C of S in the direction of the mean curvature vector field H. We choose the second normal D of S in such a way that $n+2$ vector fields $X_1, X_2, \cdots, X_n, C, D$ give the positive orientation of E.

Now, the equations of Gauss of S are

$$\nabla_j X_i = h_{ji} C + k_{ji} D, \tag{1. 6}$$

where h_{ji} and k_{ji} are components of the second fundamental tensors of S with respect to unit normals C and D respectively. Since C is in the direction of mean curvature vector field, we have

$$g^{ji} k_{ji} = 0. \tag{1. 7}$$

We note here that since C and D are intrinsic normal vector fields of S, h_{ji} and k^{ji} are also intrinsic tensor fields of S, and consequently

$$h^{ji}(\nabla_j X_i) = h^{ji} h_{ji} C + h^{ji} k_{ji} D \tag{1. 8}$$

and

$$k^{ji}(\nabla_j X_i) = k^{ji} h_{ji} C + k^{ji} k_{ji} D \tag{1. 9}$$

are also intrinsic normal vector fields of S, unless they do not vanish.

The equations of Weingarten are

$$\nabla_j C = -h_j{}^t X_t + l_j D \tag{1. 10}$$

and

$$\nabla_j D = -k_j{}^t X_t - l_j C, \tag{1. 11}$$

where

$$h_j{}^t = h_{ji} g^{it}, \qquad k_j{}^t = k_{ji} g^{it},$$

g^{it} being contravariant components of the metric tensor and l_j are the third fundamental tensor of S.

Now, as integrability conditions of (1. 6), (1. 10) and (1. 11), the equations of Gauss are

$$K_{kji}{}^{h}=h_k{}^{h}h_{ji}-h_j{}^{h}h_{ki}+k_k{}^{h}k_{ji}-k_j{}^{h}k_{ki}, \tag{1. 12}$$

where

$$K_{kji}{}^{h}=\partial_k\left\{ {h \atop j\,i} \right\}-\partial_j\left\{ {h \atop k\,i} \right\}+\left\{ {h \atop k\,t} \right\}\left\{ {t \atop j\,i} \right\}-\left\{ {h \atop j\,t} \right\}\left\{ {t \atop k\,i} \right\}$$

are components of the curvature tensor of S, from which

$$K_{ji}=h_t{}^{t}h_{ji}-h_j{}^{t}h_{ti}-k_j{}^{t}k_{ti} \tag{1. 13}$$

and

$$K=h_t{}^{t}h_s{}^{s}-h_s{}^{t}h_t{}^{s}-k_s{}^{t}k_t{}^{s}, \tag{1. 14}$$

K_{ji} and K being the Ricci tensor and the scalar curvature respectively.

The equations of Codazzi are

$$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}=0, \tag{1. 15}$$

$$\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}=0, \tag{1. 16}$$

which are also written as

$$\nabla_k h_j{}^{i}-\nabla_j h_k{}^{i}-l_k k_j{}^{i}+l_j k_k{}^{i}=0, \tag{1. 17}$$

$$\nabla_k k_j{}^{i}-\nabla_j k_k{}^{i}+l_k h_j{}^{i}-l_j h_k{}^{i}=0, \tag{1. 18}$$

from which, by contraction,

$$\nabla_t h_j{}^{t}=\nabla_j h_t{}^{t}+l_t k_j{}^{t}, \tag{1. 19}$$

$$\nabla_t k_j{}^{t}=-l_t h_j{}^{t}+l_j h_t{}^{t} \tag{1. 20}$$

respectively. Finally the equations of Ricci are

$$\nabla_k l_j-\nabla_j l_k+h_k{}^{t}k_{tj}-h_j{}^{t}k_{tk}=0. \tag{1. 21}$$

§ 2. Integral formulas.

We write the position vector X in the form

$$X=v^i X_i+\alpha C+\beta D, \tag{2. 1}$$

where v^i is a vector field on the submanifold S and α, β are functions on S. Differentiating (2. 1) covariantly, we find

$$X_j=(\nabla_j v^i)X_i+v^i(h_{ji}C+k_{ji}D)$$
$$+(\nabla_j\alpha)C+\alpha(-h_j{}^iX_i+l_jD)$$
$$+(\nabla_j\beta)D+\beta(-k_j{}^iX_i-l_jC),$$

from which

(2. 2) $$\nabla_j v^i=\delta_j^i+\alpha h_j{}^i+\beta k_j{}^i$$

or

(2. 3) $$\nabla_j v_i=g_{ji}+\alpha h_{ji}+\beta k_{ji},$$

where $v_i=g_{ih}v^h$, and

(2. 4) $$h_{ji}v^i+\alpha_j-\beta l_j=0,$$

(2. 5) $$k_{ji}v^i+\beta_j+\alpha l_j=0,$$

where $\alpha_j=\nabla_j\alpha$ and $\beta_j=\nabla_j\beta$.

From (2. 2), we have

(2. 6) $$\nabla_t v^t=n+\alpha h_t{}^t,$$

because of $k_t{}^t=0$, from which, integrating over S and applying Green's theorem, we find

(2. 7) $$\int_S(n+\alpha h_t{}^t)dS=0,$$

dS being the surface element of S, that is,

$$dS=\sqrt{g}\,dx^1\wedge dx^2\wedge\cdots\wedge dx^n,$$

where g is the determinant formed with g_{ji}.

For the divergence of $h_j{}^iv^j$, we have, using (1. 19) and (2. 2),

$$\nabla_i(h_j{}^iv^j)=(\nabla_j h_t{}^t+l_tk_j{}^t)v^j+h_j{}^i(\delta_i^j+\alpha h_i{}^j+\beta k_i{}^j),$$

that is,

$$\nabla_i(h_j{}^iv^j)=(\nabla_j h_t{}^t+l_tk_j{}^t)v^j+h_t{}^t+\alpha h^{ts}h_{ts}+\beta h^{ts}k_{ts},$$

(2. 8) $$\int_S[(\nabla_j h_t{}^t+l_tk_j{}^t)v^j+h_t{}^t+\alpha h^{ts}h_{ts}+\beta h^{ts}k_{ts}]dS=0.$$

For the divergence of $k_j{}^iv^j$, we have, using (1. 20) and (2. 2),

$$\nabla_i(k_j{}^i v^j) = -(l_t h_j{}^t - l_j h_t{}^t)v^j + k_j{}^i(\delta_i^j + \alpha h_i{}^j + \beta k_i{}^j),$$

that is,

$$\nabla_i(k_j{}^i v^j) = -(l_t h_j{}^t - l_j h_t{}^t)v^j + \alpha h^{ts} k_{ts} + \beta k^{ts} k_{ts},$$

from which, integrating over S, we find

$$\int_S [(l_t h_j{}^t - l_j h_t{}^t)v^j - \alpha h^{ts} k_{ts} - \beta k^{ts} k_{ts}] dS = 0 \tag{2. 9}$$

Also, for the divergence of $(h_t{}^i h_j{}^t + k_t{}^i k_j{}^t)v^j$, we have, using (1. 17), (1. 18), (1. 19), (1. 20) and (2. 1),

$$\begin{aligned}\nabla_i[(h_t{}^i h_j{}^t + k_t{}^i k_j{}^t)v^j] &= [\nabla_i(h_t{}^i h_j{}^t + k_t{}^i k_j{}^t)]v^j + (h_t{}^i h_j{}^t + k_t{}^i k_j{}^t)\nabla_i v^j \\ &= [(\nabla_t h_i{}^i + l_i k_t{}^i)h_j{}^t + h_t{}^i(\nabla_j h_i{}^t + l_i k_j{}^t - l_j k_i{}^t) \\ &\quad -(l_i h_t{}^i - l_t h_i{}^i)k_j{}^t + k_t{}^i(\nabla_j k_i{}^t - l_i h_j{}^t + l_j h_i{}^t)]v^j \\ &\quad + (h_t{}^i h_j{}^t + k_t{}^i k_j{}^t)(\delta_i^j + \alpha h_i{}^j + \beta k_i{}^j),\end{aligned}$$

that is,

$$\begin{aligned}\nabla_i[(h_t{}^i h_j{}^t + k_t{}^i k_j{}^t)v^j] &= \left[\frac{1}{2}\nabla_j(h^{ts}h_{ts} + k^{ts}k_{ts}) + h_j{}^t \nabla_t h_i{}^i + k_j{}^t l_t h_i{}^i\right]v^j + h^{ts}h_{ts} + k^{ts}k_{ts} \\ &\quad + \alpha(h_s{}^r h_t{}^s h_r{}^t + h_s{}^r k_t{}^s k_r{}^t) + \beta(h_s{}^r h_t{}^s k_r{}^t + k_s{}^r k_t{}^s k_r{}^t),\end{aligned} \tag{2. 10}$$

and for the divergence of $h_t{}^t h_j{}^i v^j$, we have, using (1. 19) and (2. 2),

$$\nabla_i(h_t{}^t h_j{}^i v^j) = (\nabla_i h_t{}^t)h_j{}^i v^j + h_t{}^t(\nabla_j h_i{}^i + l_i k_j{}^i)v^j + h_t{}^t h_j{}^i(\delta_i^j + \alpha h_i{}^j + \beta k_i{}^j),$$

that is,

$$\nabla_i(h_t{}^t h_j{}^i v^j) = (\nabla_i h_t{}^t)h_j{}^i v^j + \frac{1}{2}(\nabla_i h_t{}^t h_s{}^s) + h_t{}^t l_i k_j{}^i v^j + h_t{}^t(h_i{}^i + \alpha h^{sr}h_{sr} + \beta h^{sr}k_{sr}). \tag{2. 11}$$

Thus, subtracting (2. 11) from (2. 10) and integrating over S, we find

$$\begin{aligned}\int_S \Big[&\frac{1}{2}\{\nabla_i(h^{ts}h_{ts} + k^{ts}k_{ts} - h_t{}^t h_s{}^s)\}v^i + h^{ts}h_{ts} + k^{ts}k_{ts} - h_t{}^t h_s{}^s \\ &+ \alpha(h_s{}^r h_t{}^s h_r{}^t + h_s{}^r k_t{}^s k_r{}^t - h_t{}^t h^{sr}h_{sr}) + \beta(h_s{}^r h_t{}^s k_r{}^t + k_s{}^r k_t{}^s k_r{}^t - h_t{}^t h^{sr}k_{sr})\Big]dS = 0\end{aligned} \tag{2. 12}$$

Thus, taking account of (1. 14), we get

$$\begin{aligned}\int_S \left[\frac{1}{2}(\nabla_i K)v^i + K\right]dS - \int_S [&\alpha(h_s{}^r h_t{}^s h_r{}^t + h_s{}^r k_t{}^s k_r{}^t - h_t{}^t h^{sr}h_{sr}) \\ &+ \beta(h_s{}^r h_t{}^s k_r{}^t + k_s{}^r k_t{}^s k_r{}^t - h_t{}^t h^{sr}k_{sr})]dS = 0.\end{aligned} \tag{2. 13}$$

§3. The case in which the mean curvature vector H is parallel with respect to the connection induced in the normal bundle.

Take an arbitrary vector field along S

$$V=\lambda C+\mu D,$$

which is normal to S. Then

$$\nabla_i V=-(\lambda h_i{}^t+\mu k_i{}^t)X_t+(\partial_i\lambda-l_i\mu)C+(\partial_i\mu+l_i\lambda)D,$$

and consequently, we define the connection $'\nabla$ induced in the normal bundle by

$$'\nabla_i\lambda=\partial_i\lambda-l_i\mu, \qquad '\nabla_i\mu=\partial_i\mu+l_i\lambda.$$

If $'\nabla_i\lambda=0$, $'\nabla_i\mu=0$, that is, if $\nabla_i V$ is tangent to the submanifold S, then we say that V is parallel with respect to the connection induced in the normal bundle.

In this section, we assume that the mean curvature vector H is parallel with respect to the connection induced in the normal bundle.

From

$$\nabla_i H=\nabla_i\left(\frac{1}{n}g^{ts}\nabla_t X_s\right)=\frac{1}{n}\nabla_i(h_t{}^tC)$$

$$=\frac{1}{n}[-h_t{}^th_i{}^sX_s+(\nabla_i h_t{}^t)C+h_t{}^tl_iD],$$

we find

(3. 1) $$h_t{}^t=\text{constant}\neq 0, \quad l_i=0.$$

Thus, from (2. 8), we get

(3. 2) $$\int_S[h_t{}^t+\alpha h^{ts}h_{ts}+\beta h^{ts}k_{ts}]dS=0.$$

Thus subtracting (2. 7) multiplied by $(1/n)h_s{}^s$ from (3. 2), we obtain

(3. 3) $$\int_S\left[\alpha\left(h^{ts}h_{ts}-\frac{1}{n}h_t{}^th_s{}^s\right)+\beta h^{ts}k_{ts}\right]dS=0$$

or

(3. 4) $$\int_S\left[\alpha\left\{\left(h^{ts}-\frac{1}{n}h_v{}^vg^{ts}\right)\left(h_{ts}-\frac{1}{n}h_u{}^ug_{ts}\right)+k^{ts}k_{ts}\right\}+\beta h^{ts}k_{ts}-\alpha k^{ts}k_{ts}\right]dS=0.$$

Thus, if

$$\alpha>0$$

and

$$\beta h^{ts}k_{ts}-\alpha k^{ts}k_{ts}\geqq 0,$$

that is, the vector field $k^{ji}(\nabla_j X_i)$ vanishes, or vectors

$$k^{ji}(\nabla_j X_i)=h^{ts}k_{ts}C+k^{ts}k_{ts}D$$

and

$$X-v^i X_i=\alpha C+\beta D$$

form the positive orientation of the normal bundle, or if

$$\alpha<0$$

and

$$\beta h^{ts}k_{ts}-\alpha k^{ts}k_{ts}\leqq 0,$$

that is, the vector field $k^{ji}(\nabla_j X_i)$ vanishes or vector fields $k^{ji}(\nabla_j X_i)$ and $X-v^i X_i$ form the negative orientation of the normal bundle, we have

$$h_{ji}-\frac{1}{n}h_t{}^t g_{ji}=0, \qquad k_{ji}=0,$$

that is, S is totally umbilical.

Thus, from the equations of Weingarten (1. 10) and (1. 11), we have

(3. 5) $$\nabla_j C=-\frac{1}{n}h_t{}^t X_j$$

and

(3. 6) $$\nabla_j D=0,$$

respectively, from which

(3. 7) $$C+\frac{1}{n}h_t{}^t X=A,$$

(3. 8) $$D=B,$$

A and B being fixed vectors. From (3. 7) we obtain

$$X=rA-rC,$$

r being a constant equal to $n/h_t{}^t$, which says that the point X is in a constant distance r from a fixed point rA. From (3. 8), we have

$$\nabla_i(X\cdot D)=0,$$

from which

$$X\cdot D=\text{constant},$$

that is, X lies in a hyperplane.

Thus, S being closed, S must be an n-dimensional sphere. Thus we have

THEOREM 3. 1. *Let S be a closed and orientable submanifold of differentiability class C^∞ of codimension 2 of an $(n+2)$-dimensional Euclidean space. If*

(i) *the mean curvature vector of S never vanishes and is parallel with respect to the connection induced in the normal bundle,*

(ii) $\alpha>0$, $(\alpha<0)$,

(iii) *the vector field $k^{ji}(\nabla_j X_i)$ vanishes, or this vector field and the vector field $X-v^iX_i=\alpha C+\beta D$ form the positive (negative) orientation of the normal bundle,*

then S is a sphere of codimension 2.

(See Okumura [2], Yano and Okumura [4].)

§ 4. The case in which there exists a normal passing through a fixed point.

In this section, we assume that there exists a normal passing through a fixed point, that is, there exist two scalar functions λ and μ such that

$$\nabla_j(X+\lambda C+\mu D)=0, \tag{4. 1}$$

from which

$$X_j+(\nabla_j\lambda)C+\lambda(-h_j{}^iX_i+l_jD)+(\nabla_j\mu)D+\mu(-k_j{}^iX_i-l_jC)=0,$$

and consequently

$$g_{ji}-\lambda h_{ji}-\mu k_{ji}=0, \tag{4. 2}$$

$$\nabla_j\lambda-\mu l_j=0, \tag{4. 3}$$

$$\nabla_j\mu+\lambda l_j=0. \tag{4. 4}$$

From (4. 2), we find

$$n-\lambda h_t{}^t=0, \tag{4. 5}$$

$$h_t{}^t-\lambda h^{ts}h_{ts}-\mu h^{ts}k_{ts}=0, \tag{4. 6}$$

$$\lambda h^{ts}k_{ts}+\mu k^{ts}k_{ts}=0, \tag{4. 7}$$

Equation (4. 5) shows that λ and $h_t{}^t$ never vanish.

From (4. 3) and (4. 4), we find

$$\lambda\nabla_i\lambda+\mu\nabla_i\mu=0,$$

that is,

$$\lambda^2+\mu^2=\text{const.} \tag{4. 8}$$

(I) We first assume that the mean curvature of the submanifold is constant, that is,

$$h_t{}^t=\text{const.}\neq 0. \tag{4.9}$$

Then we have, from (4. 5)

$$\lambda=\text{const.}\neq 0, \tag{4.10}$$

and consequently (4. 8) shows

$$\mu=\text{const.} \tag{4.11}$$

Hence, from (4. 4),

$$l_i=0. \tag{4.12}$$

Thus equations (2. 8) and (2. 9) reduce to

$$\int_S (h_t{}^t+\alpha h^{ts}h_{ts}+\beta h^{ts}k_{ts})\,dS=0 \tag{4.13}$$

and

$$\int_S (\alpha h^{ts}k_{ts}+\beta k^{ts}k_{ts})dS=0 \tag{4.14}$$

respectively.

Since $\lambda\neq 0$, we have, from (4. 2),

$$h^{ts}=\frac{1}{\lambda}(g^{ts}-\mu k^{ts})$$

Substituting this into (4. 14), we find

$$\int_S (\lambda\beta-\mu\alpha)k^{ts}k_{ts}\,dS=0, \tag{4.15}$$

by virtue of $g^{ts}k_{ts}=0$. Thus, if $\lambda\beta-\mu\alpha$ has constant sign, that is, the concurrent vector

$$\lambda C+\mu D$$

and the projection of X on the normal space

$$\alpha C+\beta D$$

form the positive orientation or the negative orientation in the normal bundle, then we have

$$k^{ts}k_{ts}=0,$$

or

$$k_{ji}=0. \tag{4.16}$$

Thus, we have, from (4. 2)

$$h_{ji}=\frac{1}{\lambda}g_{ji}. \tag{4.17}$$

From (4. 16) and (4. 17) we can conclude that S is a sphere of codimension 2. Thus we have

THEOREM 4. 1. *Let S be a closed and orientable submanifold of differentiability class C^∞ of codimension 2 of an $(n+2)$-dimensional Euclidean space. If*

(i) *there exists a normal vector field passing through a fixed point,*

(ii) *the mean curvature is a constant different from zero,*

(iii) *the concurrent normal vector field and the projection of the position vector field on the normal space form always the positive orientation or the negative orientation of the normal bundle,*

then the submanifold S is a sphere of codimension 2.

(II) We next assume that the scalar curvature of the submanifold is constant, that is,

$$K=h_t{}^t h_s{}^s-h_t{}^s h_s{}^t-k_t{}^s k_s{}^t=\text{const}. \tag{4.18}$$

We then have, from (2. 13),

$$\int_S [K-\alpha(h_s{}^r h_t{}^s h_r{}^t+h_s{}^r k_t{}^s k_r{}^t-h_t{}^t h^{sr}h_{sr})-\beta(h_s{}^r h_t{}^s k_r{}^t+k_s{}^r k_t{}^s k_r{}^t-h_t{}^t h^{sr}k_{sr})]dS=0. \tag{4.19}$$

On the other hand, we have, from (4. 5),

$$h_t{}^t=\frac{n}{\lambda}, \tag{4.20}$$

and consequently, (4. 19) can be written as

$$\begin{aligned}\int_S\Big[[K-\alpha\Big(h_s{}^r h_t{}^s h_r{}^t+h_s{}^r k_t{}^s k_r{}^t-\frac{n}{\lambda}h^{sr}h_{sr}\Big)\\ -\beta\Big(h_s{}^r h_t{}^s k_r{}^t+k_s{}^r k_t{}^s k_r{}^t-\frac{n}{\lambda}h^{sr}k_{sr}\Big)\Big]dS=0.\end{aligned} \tag{4.21}$$

From (2. 7) and (4. 20), we have

$$\int_S\Big(1+\frac{\alpha}{\lambda}\Big)dS=0,$$

from which, K being a constant,

$$\int_S\Big[K+\frac{\alpha}{\lambda}(h_t{}^t h_s{}^s-h_t{}^s h_s{}^t-k_t{}^s k_s{}^t)\Big]dS=0. \tag{4.22}$$

Thus, subtracting (4. 22) from (4. 21), we obtain

$$\begin{aligned}(4.23)\qquad &\int_S \alpha\left[h_s{}^r h_t{}^s h_r{}^t + h_s{}^h k_t{}^s k_r{}^t - \frac{1}{\lambda}\{(n+1)h^{sr}h_{sr} + k^{sr}k_{sr} - h_t{}^t h_s{}^s\}\right]dS \\ &+\int_S \beta\left[h_s{}^r h_t{}^s k_r{}^t + k_s{}^r k_t{}^s k_r{}^t - \frac{n}{\lambda}h^{sr}k_{sr}\right]dS=0.\end{aligned}$$

On the other hand, from (4. 2), we have

$$h_s{}^r = \frac{1}{\lambda}(\delta_s^r - \mu k_s{}^r),$$

λ being different from zero, from which

$$h_t{}^t = \frac{n}{\lambda},$$

$$h_t{}^t h_s{}^s = \frac{n^2}{\lambda^2},$$

$$h^{sr}h_{sr} = \frac{1}{\lambda^2}(n+\mu^2 k^{sr}k_{sr}),$$

$$h^{sr}k_{sr} = -\frac{\mu}{\lambda}k^{sr}k_{sr},$$

$$h_s{}^r h_t{}^s h_r{}^t = \frac{1}{\lambda^3}(n+3\mu^2 k^{sr}k_{sr} - \mu^3 k_s{}^r k_t{}^s k_r{}^t),$$

$$h_s{}^r h_t{}^s k_r{}^t = \frac{\mu}{\lambda^2}(-2k^{sr}k_{st} + \mu k_s{}^r k_t{}^s k_r{}^t),$$

$$h_s{}^r k_t{}^s k_r{}^t = \frac{1}{\lambda}(k^{sr}k_{sr} - \mu k_s{}^r k_t{}^s k_r{}^t).$$

Substituting these into (4. 23), we find

$$\begin{aligned}&\int_S \alpha\left[\frac{1}{\lambda^3}(n+3\mu^2 k^{sr}k_{sr} - \mu^3 k_s{}^r k_t{}^s k_r{}^t) + \frac{1}{\lambda}(k^{sr}k_{sr} - \mu k_s{}^r k_t{}^s k_r{}^t)\right. \\ &\qquad\qquad \left. - \frac{1}{\lambda}\left\{\frac{1}{\lambda^2}(n+1)(n+\mu^2 k^{sr}k_{sr}) + k^{sr}k_{sr} - \frac{n^2}{\lambda^2}\right\}\right]dS \\ &+\int_S \beta\left[\frac{\mu}{\lambda^2}(-2k^{sr}k_{sr} + \mu k_s{}^r k_t{}^s k_r{}^t) + k_s{}^r k_t{}^s k_r{}^t + \frac{n\mu}{\lambda^2}k^{sr}k_{sr}\right]dS=0\end{aligned}$$

or

$$-\int_S \alpha\left[\frac{(n-2)\mu^2}{\lambda^3}k^{sr}k_{sr}+\frac{\mu}{\lambda^3}(\lambda^2+\mu^2)k_s{}^r k_t{}^s k_r{}^t\right]dS$$

$$+\int_S \beta\left[\frac{\mu}{\lambda^2}(n-2)k^{sr}k_{sr}+\frac{1}{\lambda^2}(\lambda^2+\mu^2)k_s{}^r k_t{}^s k_r{}^t\right]dS=0,$$

or, $\lambda^2+\mu^2$ being a constant,

$$(n-2)\int_S \frac{\lambda\beta-\mu\alpha}{\lambda^3}\mu k^{sr}k_{sr}dS+(\lambda^2+\mu^2)\int_S \frac{\lambda\beta-\mu\alpha}{\lambda^3}k_s{}^r k_t{}^s k_r{}^t dS=0$$

Thus, if $n>2$, $k_s{}^r k_t{}^s k_r{}^t=0$ and $\lambda\mu(\lambda\beta-\mu\alpha)$ has constant sign, we have

$$k_{ji}=0$$

and consequently

$$h_{ji}=\frac{1}{\lambda}g_{ji}$$

and S is a sphere of codimension 2. Thus we have

THEOREM 4. 2. *Let S be a closed and orientable submanifold of differentiability class C^∞ of codimension 2 in an $(n+2)$-dimensional Euclidean space, $(n>2)$. If*

(i) *there exists a normal vector field passing through a fixed point,*

(ii) *the scalar curvature of the submanifold is constant and $k_s{}^r k_t{}^s k_r{}^t=0$,*

(iii) *the vector fields $\lambda C+\mu D$ and $\alpha C+\beta D$ are situated in such a way that*

$$\lambda\mu(\lambda\beta-\mu\alpha)$$

has constant sign,

then S is a sphere of codimension 2.

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