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ON CONFORMAL MAPPINGS ONTO INCISED RADIAL SLIT DISKS

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A plane domain cannot always be mapped conformally onto a radial slit disk. But it was shown by Strebel [5] and Reich [3] that it can be mapped onto an incised radial slit disk. We are interested in the problem: under what circumstances do these incisions occur. In the present paper we shall show that the occurrence depends only on a property of a neighborhood of the "boundary element" corresponding to the incision. The concept of such a boundary element was introduced recently by the second author [8, 9]. We shall define it here in a somewhat different manner.

Statement of results.

1. Let Ω be a proper subdomain on the Riemann sphere, $\zeta \in \Omega$, and γ be a boundary component of Ω. For the sake of simplicity assume $\zeta \neq \infty$. Let

$$\mathcal{S}=\mathcal{S}(\Omega, \zeta, \gamma)$$

be the family of the functions f with the following properties: f is regular and univalent on Ω, $f(\zeta)=0$, $f'(\zeta)=1$, and $f(\gamma)$ is the other boundary component of $f(\Omega)$.

We introduce the quantity

$$R(\gamma)=R(\Omega, \zeta, \gamma)$$

by

$$\log R(\gamma)=\lim_{\varepsilon\to 0} (\log \varepsilon+2\pi\lambda(\Gamma_\varepsilon^*)).$$

Here $\varepsilon>0$ satisfies $\{z \mid |z-\zeta|\leqq\varepsilon\}\subset\Omega$, λ stands for extremal length, and Γ_ε^* is the family of the locally rectifiable open arcs in $\Omega-\{z \mid |z-\zeta|\leqq\varepsilon\}$ joining γ and the circle $|z-\zeta|=\varepsilon$. By an open arc in a domain we mean a continuous mapping c of the open interval (0, 1) into the domain; it is said to join sets E_0 and E_1 if the "tails" $T_0=\bigcap_{\tau>0}\overline{\{c(t) \mid 0<t<\tau\}}$ and $T_1=\bigcap_{\tau>0}\overline{\{c(t) \mid 1-\tau<t<1\}}$ belong to E_0 and E_1 respectively.

The quantity $R(\gamma)$ is called the *extremal radius* of γ by Strebel [5], and its reciprocal is referred to as the *capacity* $c_{0\gamma}$ of γ in Sario-Oikawa [4]. It satisfies

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$0<R(\gamma)\leqq\infty$. It depends on the reference point ζ. But the finiteness of $R(\gamma)$ is independent of ζ, and is equivalent to $\lambda(\Gamma_\varepsilon^*)<\infty$ for some, or equivalently all, ε ([4; p. 36]).

2. If $R(\gamma)<\infty$, we know the *unique existence* of the function

$$\varphi(z)=\varphi(z;\ \Omega, \zeta, \gamma)\in\mathcal{S}$$

with the following properties (Strebel [5], Reich [3], Oikawa [2], Suita [6], Sario-Oikawa [4; pp. 218-222]):

(a) $0\in\varphi(\Omega)\subset\{w\,|\,|w|<R(\gamma)\}$ and every component of $\partial\varphi(\Omega)-\varphi(\gamma)$ is either a point or a line segment on a ray $\arg w=$const.

(b) $\varphi(\gamma)$ consists of the circle $|w|=R(\gamma)$ and possibly of a number of line segments (called incisions) on rays $\arg w=$const.

(c) The complement of any compact subset of $\partial\varphi(\Omega)-\varphi(\gamma)$ is an extremal (minimal) radial slit plane.

(d) The angular measure of the incisions is zero and, if Γ_ε^i denotes the family of locally rectifiable open arcs in $\varphi(\Omega)-\{w\,|\,|w|\leqq\varepsilon\}$ joining $\varphi(\gamma)\cap\{w\,|\,|w|<R(\gamma)\}$ and $\{w\,|\,|w|=\varepsilon\}$, then $\lambda(\Gamma_\varepsilon^i)=\infty$.

The properties (a) and (b) mean that the image domain $\varphi(\Omega)$ is an incised radial slit disk with radius $R(\gamma)$. To be precise let us define an *incision* as a closed line segment I on a ray $\arg w=$const such that $I\cap\{w\,|\,|w|<R(\gamma)\}$ is a component of $\varphi(\gamma)\cap\{w\,|\,|w|<R(\gamma)\}$. A point on the circle $|w|=R(\gamma)$ belonging to no incision will be referred to as a *periphery point*.

The properties (a) and (b) alone are insufficient to characterize the founctin φ. On the other hand some of the properties (a)–(d) can be deduced from others among them.

The property (c) can be stated in another form by means of the linear operator L_0 (Ahlfors-Sario [1; p. 168]). Namely the condition (c) is equivalent to the following:

(c′) For any closed analytic Jordan curve $\alpha\subset\Omega$ whose interior D is disjoint from $\gamma\cup\{\zeta\}$

$$L_0(\log|\varphi|)=\log|\varphi|$$

holds on $D\cap\Omega$ with respect to the operator L_0 acting from α into $D\cap\Omega$.

For details we refer to [4; pp. 209 ff].

3. We now introduce a boundary element of a plane domain. As a consequence of Theorem 1 below it will be evident that our concept of boundary element coincides with the one introduced by Suita [8, 9]. In particular, if the domain is simply connected and hyperbolic, we are dealing with Carathédory's prime end.

Given an arbitrary proper subdomain Ω of the Riemann sphere, we shall write Δ for a subdomain of Ω whose relative boundary $c_\Delta=\Omega\cap\partial\Delta$ is a locally rectifiable simple open arc. Consider a family (base of a filter) $\mathcal{D}$ of Δ's satisfying the following conditions:

(I) For any $\Delta_1, \Delta_2\in\mathcal{D}$, there exists a $\Delta_3\in\mathcal{D}$ with $\Delta_3\subset\Delta_1\cap\Delta_2$.

(II) $\cap_{\Delta\in\mathcal{D}}\Delta=\phi$.

(III) For any $\Delta\in\mathcal{D}$, the extremal length of the family $\Gamma_\Delta=\{c_{\Delta_1}|\Delta_1\subset\Delta,\ \Delta_1\in\mathcal{D}\}$ is finite.

(IV) A $\mathcal{D}$ with (I)—(III) and finer than $\mathcal{D}$ is always coarser than $\mathcal{D}$.

The terms finer and coarser are in the sense used commonly in the theory of filters. Two families $\mathcal{D}_1$ and $\mathcal{D}_2$ are said to be *equivalent* if each is finer than the other.

DEFINITIONS. An equivalence class e of families $\mathcal{D}$ with (I)—(IV) will be called a *boundary element* of the domain Ω. A domain $\Delta\in\mathcal{D}\in e$ is referred to as a *neighborhood* of e. The set

$$|e|=\bigcap_{\Delta\in\mathcal{D}}\bar{\Delta},$$

independent of the choice of $\mathcal{D}\in e$, will be called the *impression* of e. It is a connected closed set on $\partial\Omega$. If $|e|\subset\gamma$, we shall simply say that *e belongs to* γ.

The argument in §§9, 12 shows that if $R(\gamma)<\infty$ there exists a boundary element belonging to γ.

4. Suppose Ω, ζ, γ and φ are as in §2. Given an e, the family $\{\varphi(\Delta)|\Delta\in\mathcal{D}\}$ for $\mathcal{D}\in e$ satisfies (I)—(IV) and determines a boundary element of $\varphi(\Omega)$ independent of the choice of $\mathcal{D}$. It will be denoted by $\varphi(e)$, the *image* of e under φ.

THEOREM 1. *If e belongs to* γ, $|\varphi(e)|$ *is either an incision or a periphery point. This correspondence is one-to-one from the set of boundary elements belonging to* γ *onto the set of incisions and periphery points.*

THEOREM 2. *A necessary and sufficient condition for* $|\varphi(e)|$ *to be a periphery point is that e has a neighborhood* Δ *and a harmonic function v on* Δ *such that*

(i) $$v>0,$$

(ii) *for any closed analytic Jordan curve* $\alpha\subset\Delta$ *whose interior* Δ *is disjoint from* $c_\Delta\cup\gamma\cup\{\zeta\}$,

$$L_0v=v$$

holds on $D\cap\Delta$ *with respect to the operator* L_0 *acting from* α *into* $D\cap\Delta$,

(iii) $$\lim_{z\to e}v(z)=0.$$

As a consequence, the property of $|\varphi(e)|$ being an incision depends only on a neighborhood of e.

Some properties of φ.

5. We list further properties of the mapping φ needed for the proofs of our theorems.

Let Ω, ζ, γ be as in § 1, and consider an exhaustion $0 \in \Omega_1 \subset \Omega_2 \subset \cdots \to \Omega$ towards γ. By this we mean that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and the relative boundary $\gamma_n = \Omega \cap (\partial\Omega_n)$ is a closed analytic Jordad curve in Ω_{n+1} separating γ from ζ. Write R_n for $R(\Omega_n, \zeta, \gamma_n)$ and $\varphi_n(z)$ for $\varphi(z; \Omega_n, \zeta, \gamma_n)$.

The image domain $\varphi_n(\Omega_n)$ is a radial slit disk and has no incision.

We know that $\lim_{n\to\infty} R_n = R(\gamma)$ and, if $R(\gamma) < \infty$, $\lim_{n\to\infty} \varphi_n = \varphi$. More specifically we shall need the following:

If $R(\gamma) < \infty$, *then* $\log (R_n/|\varphi_n(z)|)$ *increases with* n *and converges to* $\log(R(\gamma)/|\varphi(z)|)$ *as* $n \to \infty$.

6. If we consider the family Γ_ε^* in § 1 on the image domain $\varphi(\Omega)$, the identity $2\pi\lambda(\Gamma_\varepsilon^*) = \log (R(\gamma)/\varepsilon)$ is well-known. This result is generalized as follows (Suita [7; p. 443]):

Under the assumption of § 2, *let* S *be a sector of the form* $r < |w| < R(\gamma)$, $\theta_0 < \arg w < \theta_0 + \Theta$, *for some* r $(0 < r < R(\gamma))$, θ_0, *and* $\Theta > 0$. *Let* Γ_S *be the family of locally rectifiable open arcs in* $S \cap \varphi(\Omega)$ *joining* $\varphi(\gamma)$ *and* $|w| = r$. *Then*

$$\Theta\lambda(\Gamma_S) = \log \frac{R(\gamma)}{r}.$$

Some properties of the extremal length.

7. In addition to standard known properties of the extremal length we shall need some more.

LEMMA 1. *If every member of* Γ *passes through a point* z_0, *then* $\lambda(\Gamma) = \infty$.

Proof. Cover the complement of $\{z_0\}$ by a countable number of closed disks $K_1, K_2, \cdots$ which do not contain z_0. Let $\Gamma_n = \{c \in \Gamma | c \cap K_n \neq \phi\}$. It is well known that $\lambda(\Gamma_n) = \infty$. The lemma follows from: $\lambda(\Gamma)^{-1} \leqq \sum \lambda(\Gamma_n)^{-1} = 0$.

LEMMA 2. *Let* $\Gamma, \Gamma_1, \Gamma_2, \cdots$ *be such that, for any* $c_n \in \Gamma_n$, $n = 1, 2, \cdots$, *there exists a* $c \in \Gamma$ *with* $c \subset c_1 \cup c_2 \cup \cdots$. *Then*

$$\lambda(\Gamma)^{1/2} \leqq \sum_{n=1}^{\infty} \lambda(\Gamma_n)^{1/2}.$$

The case $\Gamma_m = \Gamma_n$ *for* $m \neq n$ *is not excluded.*

Proof. Use notations in Ahlfors-Sario [1; p. 220]. On considering all ρ with $A(\rho) = 1$ we have $\lambda(\Gamma)^{1/2} = \sup_\rho L(\Gamma; \rho) \leqq \sup_\rho \sum_n L(\Gamma_n; \rho) \leqq \sum_n \sup_\rho L(\Gamma_n; \rho) = \sum_n \lambda(\Gamma_n)^{1/2}$.

LEMMA 3. *Let* E_1, E_2, E_3 *be sets on the closure of a domain* Ω. *Denote by* Γ_{jk} *the family of locally rectifiable arcs in* Ω *joining* E_j *and* E_k. *Then*

$$\lambda(\Gamma_{12})<\infty,\ \lambda(\Gamma_{23})<\infty \Rightarrow \lambda(\Gamma_{13})<\infty.$$

8. *Proof of Lemma* 3 will be divided into several steps. Given a closed disk $K\subset\Omega$, let $\Gamma_{jk}(K)=\{c\in\Gamma_{jk}|c\cap K\neq\phi\}$. We first show that *if* $\lambda(\Gamma_{jk}(K_0))=\infty$ *for some* K_0, *then the same is true for every* K.

Take a simply connected domain D_0 such that $K_0\cup K\subset D_0$ and $\bar{D}_0\subset\Omega$, and let $\Gamma_{jk}(D_0)=\{c\in\Gamma_{jk}|c\cap D_0\neq\phi\}$. It suffices to prove $\lambda(\Gamma_{jk}(D_0))=\infty$. For this purpose take a simply connected domain D such that $\bar{D}_0\subset D$ and $\bar{D}\subset\Omega$. Consider the family Γ of closed rectifiable curves in the doubly connected domain $D-\bar{D}_0$ separating its boundary components, and the family Γ^* of locally rectifiable open arcs in $D-K_0$ joining its boundary components. For any $c_0\in\Gamma_{jk}(D_0)$, $c_1\in\Gamma$ and $c_2\in\Gamma^*$, we can find a $c\in\Gamma_{jk}(K_0)$ such that $c\subset c_0\cup c_1\cup(-c_1)\cup c_2\cup(-c_2)$. Accordingly, by Lemma 2, $\lambda(\Gamma(K_0))^{1/2}\leqq\lambda(\Gamma(D_0))^{1/2}+2\lambda(\Gamma)^{1/2}+2\lambda(\Gamma^*)^{1/2}$. Since $\lambda(\Gamma)$ and $\lambda(\Gamma^*)$ are finite we conclude that $\lambda(\Gamma(D_0))=\infty$.

Next, by the same argument as in the proof of Lemma 1, we see that $\lambda(\Gamma_{jk})=\infty$ *if and only if* $\lambda(\Gamma_{jk}(K))=\infty$ *for some* K.

Let $\Gamma_j(K)$ be the family of locally rectifiable open arcs in $\Omega-K$ joining E_j and K. We shall prove that $\lambda(\Gamma_{jk}(K))=\infty$ *if and only if either* $\lambda(\Gamma_j(K))=\infty$ *or* $\lambda(\Gamma_k(K))=\infty$.

The if-part is evident. For the proof of the only-if part, let K' be a closed disk in Ω containing K in its interior. Let Γ be the family of closed rectifiable curves in $\mathrm{Int}(K')-K$ separating $\partial K'$ from K. For arbitrary $c_j\in\Gamma_j(K)$, $c_k\in\Gamma_k(K)$, $c_0\in\Gamma$, there exists a $c\in\Gamma_{jk}(K)$ such that $c\subset c_j\cup c_k\cup c_0$. By Lemma 2 we obtain $\lambda(\Gamma_{jk}(K))^{1/2}\leqq\lambda(\Gamma_j(K))^{1/2}+\lambda(\Gamma_k(K))^{1/2}+\lambda(\Gamma)^{1/2}$. Since $\lambda(\Gamma)<\infty$ we infer that either $\lambda(\Gamma_j(K))=\infty$ or $\lambda(\Gamma_k(K))=\infty$.

The assertion of Lemma 3 is now verified as follows: $\lambda(\Gamma_{12})<\infty$ and $\lambda(\Gamma_{23})<\infty$ imply $\lambda(\Gamma_{12}(K))<\infty$, $\lambda(\Gamma_{23}(K))<\infty$, so that $\lambda(\Gamma_1(K))<\infty$, $\lambda(\Gamma_3(K))<\infty$. Thus $\lambda(\Gamma_{13}(K))<\infty$ and therefore $\lambda(\Gamma_{13})<\infty$.

Proof of Theorem 1.

9. As the first step of the proof we shall show the following:

If E *is an incision or a singleton consisting of a periphery point, there exists a family* $\mathfrak{D}$ *of* Δ*'s which satisfies* (I)—(III) *and*

$$\bigcap_{\Delta\in\mathfrak{D}}\overline{\varphi(\Delta)}=E.$$

Without loss of generality we may assume that E is an interval $[r, R(\gamma)]$ on the real axis; here $0<r<R(\gamma)$ if E is an incision and $r=R(\gamma)$ otherwise. There exists a sequence of points $r_n\in\varphi(\Omega)$ on the real axis such that $0<r_1<r_2<\cdots$, $\lim r_n=r$. For a neighborhood of a point r_n we take $N_n=\{w||\log w-\log r_n|<\theta_n\}$ such that $\bar{N}_n\subset\varphi(\Omega)$. We may assume, on taking a subsequence if necessary, that $\theta_n/2>\theta_{n+1}$, $n=1, 2, \cdots$, and that the $\bar{N}_n$ are pairwise disjoint.

Consider the sectors $S_n=\{w|r_n<|w|<R(\gamma), \theta_n/2<\arg w<\theta_n\}$, $S'_n=\{w|r_n<|w|<R(\gamma),$

$-\theta_n<\arg w<-\theta_n/2\}$, and a quadrilateral $Q_n=\{w|\theta_n/2<|\log w-\log r_n|<\theta_n, |w|<r_n\}$. Since $\bar{Q}_n\subset\varphi(\Omega)$ we can take $a_n>0$ so small that the quadrilaterals $Q_n'=\{w\in S_n|r_n<|w|<r_n+a_n\}$ and $Q_n''=\{w\in S_n'|r_n<|w|<r_n+a_n\}$ are in $\varphi(\Omega)$. Let Q_n^* be the quadrilateral which is the union of Q_n, Q_n', Q_n'', and the common sides. Observe that the Q_n^*, $n=1, 2, \cdots$, are pairwise disjoint.

Let $\mathscr{D}_n$ be the family of those $\Delta\subset\Omega$ for which $\varphi(c_\Delta)$ is contained in $Q_n^*\cup S_n\cup S_n'$ and has tails T_0 and T_1 respectively is $\bar{S}_n$ and $\bar{S}_n'$. By the result quoted in § 6, we have $\lambda(\Gamma_{S_n})<\infty$, $\lambda(\Gamma_{S_n'})<\infty$. Clearly the following three families have finite extremal length: (1) arcs in Q_n^* joining sides on $|w|=r_n$, (2) arcs in Q_n' joining sides on $\arg w=\theta_n/2$, $\arg w=\theta_n$, (3) similar arcs in Q_n''. We conclude by Lemma 2 that $\lambda\{c_\Delta|\Delta\in\mathscr{D}_n\}=\infty$.

It is obvious that $\mathscr{D}=\cup_{n=1}^{\infty}\mathscr{D}_n$ has the required properties.

10. We shall say that Δ is *distinguished* if the tails T_0 and T_1 of $\varphi(c_\Delta)$ are singletons consisting of different periphery points.

Given a family $\mathscr{D}$ *with* (I)—(III) *and* $\cap_{\Delta\in\mathscr{D}}\overline{\varphi(\Delta)}\subset\gamma$, *the subfamily of the distinguished* Δ*'s in* $\mathscr{D}$ *satisfies* (I)—(III) *and is finer than* $\mathscr{D}$.

For the proof, take $\varepsilon>0$ with $K_0=\{z||z-\zeta|\leqq\varepsilon\}\subset\Omega$, and chose $\Delta_0\in\mathscr{D}$ with $K_0\cap\bar{\Delta}_0=\phi$. It suffices to show that the family $\mathscr{D}^*$ of all distinguished $\Delta\in\mathscr{D}$ with $\Delta\subset\Delta_0$ satisfies (I)—(III) and is finer than $\mathscr{D}$.

Let $\mathscr{D}_0=\{\Delta\in\mathscr{D}|\Delta\subset\Delta_0\}$. As is well known, if $\mathscr{D}_1$ is the family of all $\Delta\in\mathscr{D}_0$ such that the tail T_0 or T_1 of $\varphi(c_\Delta)$ consists of more than one point, then $\lambda(\{c_\Delta|\Delta\in\mathscr{D}_1\})=\infty$. Next let $\mathscr{D}_2$ be the family of those $\Delta\in\mathscr{D}_0-\mathscr{D}_1$ such that T_0 or T_1 of $\varphi(c_\Delta)$ lies on an incision. By the property (d) in § 2 and Lemma 3 the family of arcs in Δ_0 joining *any* disk in Δ_0 and the incisions has infinite extremal length. By an argument similar to that in the proof of Lemma 1 we then conclude that $\lambda(\{c_\Delta|\Delta\in\mathscr{D}_2\})=\infty$.

Now let $\mathscr{D}^*=\mathscr{D}_0-\mathscr{D}_1-\mathscr{D}_2$. It clearly satisfies (I)—(III) and is finer than $\mathscr{D}$. Every distinguished $\Delta\in\mathscr{D}_0$ belongs to $\mathscr{D}^*$. Conversely every $\Delta\in\mathscr{D}^*$ is distinguished, for, if not, the T_0 and T_1 of $\varphi(c_\Delta)$ consist of the same point and therefore $\lambda(\{c_{\Delta_1}|\Delta_1\subset\Delta\})=\infty$ by Lemma 1, contradicting the condition (III). Thus this $\mathscr{D}^*$ is what we set out to obtain.

11. Suppose $\mathscr{D}$ satisfies (I)—(III) and is such that

$$E=\bigcap_{\Delta\in\mathscr{D}}\overline{\varphi(\Delta)}$$

is an incision or a singleton consisting of a periphery point. Let w_0 be the point such that $\{w_0\}=E\cap\{w||w|=R(\gamma)\}$.

For any distinguished $\Delta\in\mathscr{D}$, the end points of $\varphi(c_\Delta)$ determine two closed arcs on the circle $|w|=R(\gamma)$. Let A_Δ be the one for which Δ is contained in the interior of the closed Jordan curve $\varphi(c_\Delta)\cup A_\Delta$. Evidently $w_0\in A_\Delta$.

We infer that w_0 *is not an end point of the arc* A_Δ.

Suppose w_0 is an end point of A_Δ. If $\mathscr{D}^*$ is the family of distinguished $\Delta\in\mathscr{D}$, then $\{w_0\}=\cap_{\Delta\in\mathfrak{D}^*}\bar{A}_\Delta$. Thus, for every $\Delta_1\in\mathfrak{D}^*$, $\Delta_1\subset\Delta$, w_0 is an end point of A_{Δ_1}. By Lemma 1 we obtain $\lambda(\{c_{\Delta_1}|\Delta_1\in\mathfrak{D}^*, \Delta_1\subset\Delta\})=\infty$, in contradiction of the condition (III) for $\mathfrak{D}^*$.

12. We are ready to prove Theorem 1.

First we shall show that, given a boundary element e with $|e|\subset\gamma$, $|\varphi(e)|$ is either an incision or a singleton consisting of a periphery point. It suffices to verify that $A=|\varphi(e)|\cap\{|w|\,|w|=R(\gamma)\}$ is a singleton, for $|\varphi(e)|$ is known to be a connected subset of $\varphi(\gamma)$. Take a $\mathfrak{D}\in e$ arbitrarily and let $\mathfrak{D}^*$ be the family of distinguished $\Delta\in\mathfrak{D}$. It satisfies (I)−(IV) and $\mathfrak{D}^*\in e$. Clearly $A\subset A_\Delta$ for every $\Delta\in\mathfrak{D}^*$. Now suppose A is not a singleton but an arc. By (d) in §2, we can find a periphery point $w_0\in A$ different from the end points of A. By the method used in §9, we can construct a $\tilde{\mathfrak{D}}$ which satisfies (I)−(III), is finer than $\mathfrak{D}$, and is such that

$$\bigcap_{\Delta\in\tilde{\mathfrak{D}}}\overline{\varphi(\Delta)}=\{w_0\}.$$

$\mathfrak{D}^*$ cannot be finer than $\mathfrak{D}$; this contradicts the condition (IV) for $\mathfrak{D}^*$. We conclude that A is a singleton.

Next, to prove that the correspondence stated in Theorem 1 is onto, it suffices to show that the family $\mathfrak{D}$ constructed in §9 satisfies (IV). Let $\tilde{\mathfrak{D}}$ meet (I)−(III) and be finer than $\mathfrak{D}$. We may assume that every $\tilde{\Delta}\in\tilde{\mathfrak{D}}$ is distinguished. Clearly

$$\bigcap_{\tilde{\Delta}\in\tilde{\mathfrak{D}}}\overline{\varphi(\tilde{\Delta})}=|\varphi(e)|$$

holds. Thus every $\tilde{\Delta}\in\tilde{\mathfrak{D}}$ has the property stated in §11, so that it is possible to find a $\Delta\in\mathfrak{D}$ with $\Delta\subset\tilde{\Delta}$. We infer that $\mathfrak{D}$ is finer than $\tilde{\mathfrak{D}}$; this shows that $\mathfrak{D}$ satisfies (IV).

Finally, the correspondence is one-to-one. In fact, given e and $\tilde{e}$ with $|\varphi(e)|=|\varphi(\tilde{e})|$, take $\mathfrak{D}\in e$ and $\tilde{\mathfrak{D}}\in\tilde{e}$ consisting only of distinguished Δ's. The reasoning used in the above paragraph shows that one is finer than the other, that is $e=\tilde{e}$.

Proof of Theorem 2.

13. The necessity is evident. A Δ with $\zeta\notin\Delta$ and the restriction v of $\log|\varphi|$ to Δ qualify.

To prove the sufficiency we may assume without loss of generality that the given Δ is distinguished and such that $\zeta\notin\bar{\Delta}$, and v is defined and positive on $\Delta\cup c_\Delta$. Since Δ is distinguished the function

$$u(z)=\log\frac{R(\gamma)}{|\varphi(z)|}$$

has vanishing limit as z tends to γ along c_Δ. Therefore, given $\varepsilon>0$, there exists a compact set $C_\varepsilon\subset\Omega$ such that $u<\varepsilon$ on $c_\Delta\cap(\Omega-C_\varepsilon)$. On the other hand, on the set

$c_\Delta \cap C_\varepsilon$, we have $\min v > 0$ and $\max u < \infty$, so that there exists a constant M_ε such that $u < M_\varepsilon v$ on $c_\Delta \cap C_\varepsilon$. As a consequence

$$u < M_\varepsilon v + \varepsilon \quad \text{on } c_\Delta.$$

Consider an exhaustion $\zeta \in \Omega_n \uparrow \Omega$ towards γ. Since $u_n(z) = \log(R(\gamma_n)/|\varphi_n(z)|)$ increases with n, $u_n < M_\varepsilon v + \varepsilon$ on $c_\Delta \cap \Omega_n$. This inequality holds on $\gamma_n \cap \Delta$ as well, for $u_n = 0$ there. Furthermore, we have the identity $L_0(u_n - M_\varepsilon v) = u_n - M_\varepsilon v$, where the operator L_0 acts from $(c_\Delta \cap \Omega_n) \cup (\gamma_n \cap \Delta)$ into $\Delta \cap \Omega_n$. By the maximum-principle for L_0 we obtain $u_n < M_\varepsilon v + \varepsilon$ in $\Delta \cap \Omega_n$. On letting $n \to \infty$ we deduce

$$u < M_\varepsilon v + \varepsilon \quad \text{on } \Delta.$$

As $z \to e$, $\lim u \leqq \varepsilon$ and, therefore, $\lim u \leqq 0$. The inequality $\lim u \geqq 0$ is trivial and a fortiori

$$\lim u = 0$$

as $z \to e$. This shows that $|\varphi(e)|$ is not an incision.

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