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A NOTE ON CERTAIN HYPERSURFACES OF SASAKIAN MANIFOLDS

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Introduction. Let $\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ be a Sasakian manifold and M^{2n} be a hypersurface of $\tilde{M}^{2n+1}$. It is known that M^{2n} cannot be an invariant hypersurface (Goldberg-Yano [1]). On the other hand, if M^{2n} is a non-invariant hypersurface (or more generally, if ξ is never tangent to M^{2n}), then M^{2n} admits a natural Kählerian structure (J, γ). This is a special case of the result of Goldberg-Yano [1]. Since the Kählerian structure is quite natural, one may conjecture that if the ambient Sasakian manifold is of constant ϕ-holomorphic sectional curvature, then $M^{2n}(J, \gamma)$ is of constant holomorphic sectional curvature under some conditions. The answer is affirmative if M^{2n} is totally geodesic in $\tilde{M}^{2n+1}$ (Theorem 3).

§1. Hypersurfaces of almost contact Riemannian manifolds.

Let $\tilde{M}=\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ be an almost contact Riemannian manifold, and let $M=M^{2n}$ be a hypersurface of $\tilde{M}$. Throughout this paper, we assume that ξ is never tangent to M. Then we have

$$(1) \qquad \phi X=JX+\alpha(X)\xi \qquad \text{for} \qquad X\in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the set of all vector fields on M and JX is the tangential part (with respect to ξ) of ϕX to M. We can see that J: $X\rightarrow JX$ and α: $X\rightarrow\alpha(X)$ are tensor fields of type (1, 1) and (0, 1), respectively, on M. If $\alpha\neq 0$ on M, then M is called a *non-invariant hypersurface*. If $\alpha=0$ on M, then M is called an *invariant hypersurface*.

Applying ϕ to the relation (1), we get

$$-X+\eta(X)\xi=J^2X+\alpha(JX)\xi,$$

which shows that

$$(2) \qquad J^2=-\text{identity},$$

$$(3) \qquad C\alpha=\eta|M,$$

where $C\alpha(X)=\alpha(JX)$. Thus the tensor field J is an almost complex structure on M.

Let $\tilde{\nabla}$ be the Levi-Civita connection of the Riemannian metric $\tilde{g}$. For X, Y

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$\in \mathfrak{X}(M)$, we have

(4) $$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi,$$

(5) $$\tilde{\nabla}_X \xi = -HX + \omega(X)\xi,$$

where $\nabla_X Y$ and $-HX$ are the tangential parts (with respect to ξ) of $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$, respectively, to M. We can see that ∇: $(X, Y) \to \nabla_X Y$ is a symmetric connection on M, h: $(X, Y) \to h(X, Y)$, H: $X \to HX$ and ω: $X \to \omega(X)$ are tensor fields of type (0, 2), (1, 1) and (0, 1), respectively, on M. h is symmetric and is called the *second fundamental form* of M (with respect to ξ). If $h=0$ on M, then M is called to be *totally geodesic*.

Let g be the induced metric: $g=\tilde{g}|M$. In general, the connection ∇ is not the Levi-Civita connection of g. Using (3), (4) and (5), we get

$$(\nabla_X g)(Y, Z) = h(X, Y)C\alpha(Z) + h(X, Z)C\alpha(Y).$$

Hence ∇ is the Levi-Civita connection of g if and only if $h(X, Y)C\alpha(Z)+h(X, Z)C\alpha(Y)=0$ for all vector fields X, Y and Z on M. In particular, if M is totally geodesic, then ∇ is the Levi-Civita connection of g. The converse is also true when $\tilde{M}$ is Sasakian, which will be shown later.

§2. Hypersurfaces of Sasakian manifolds.

In this section, we assume that $\tilde{M}=\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ is a Sasakian manifold; that is, the following holds good:

(6) $$(\tilde{\nabla}_U \phi)V = \eta(V)U - g(U, V)\xi, \qquad U, V \in \mathfrak{X}(\tilde{M}),$$

where $\mathfrak{X}(\tilde{M})$ is the set of all vector fields on $\tilde{M}$. It is known that (6) implies the followings:

(7) $$\tilde{\nabla}_U \xi = \phi U,$$

(8) $$d\eta(U, V) = g(\phi U, V).$$

(1), (5) and (7) imply

(9) $$H = -J \quad \text{and} \quad \omega = \alpha.$$

Using (1), (4) and (6), we get

(10) $$\tilde{\nabla}_X \phi Y = \{C\alpha(Y)X + J\nabla_X Y\} + \{\alpha(\nabla_X Y) - g(X, Y\}\xi.$$

On the other hand, using (1) and (7), we get

(11) $$\tilde{\nabla}_X \phi Y = (\nabla_X J)Y + J\nabla_X Y + \alpha(Y)JX + \{h(X, JY) + (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) + \alpha(X)\alpha(Y)\}\xi.$$

Comparing (10) and (11), we obtain

(12) $$(\nabla_X J)Y=\alpha(JY)X-\alpha(Y)JX,$$

(13) $$(\nabla_X \alpha)Y=-g(X, Y)-h(X, JY)-\alpha(X)\alpha(Y).$$

Now, we can calculate the Nijenhuis tensor of J:

$$\begin{aligned} N(X, Y) &= [JH, JY]-J[JX, Y]-J[X, JY]-[X, Y] \\ &= (\nabla_{JX}J)Y-(\nabla_{JY}J)X-J(\nabla_X J)Y+J(\nabla_Y J)X. \end{aligned}$$

Substituting (12) in the above equation, we get

$$\begin{aligned} N(X, Y) &= \alpha(JY)JX+\alpha(Y)X-\alpha(JX)JY-\alpha(X)Y \\ &\quad -\alpha(JY)JX-\alpha(Y)X+\alpha(JX)JY+\alpha(X)Y \\ &= 0. \end{aligned}$$

Hence J is a complex structure on M.

We put

(14) $$\gamma=g-C\alpha\otimes C\alpha.$$

Then, since ξ is not tangent to M at each point, γ is a Riemannian metric on M. Since we have

$$\begin{aligned} \gamma(JX, JY) &= g(JX, JY)-\alpha(J^2X)\alpha(J^2Y) \\ &= g(X, Y)-\eta(X)\eta(Y) \\ &= \gamma(X, Y), \end{aligned}$$

(J, γ) is a Hermitian structure on M.

We put

$$\Phi(U, V)=\tilde{g}(\phi U, V), \qquad U, V\in \mathfrak{X}(\tilde{M}),$$

$$\Omega(X, Y)=\gamma(JX, Y), \qquad X, Y\in \mathfrak{X}(M).$$

Then we get $\Omega(X, Y)=\Phi(X, Y)$ for any vector fields X and Y on M. Hence, since $\Phi=d\eta$ is closed, Ω is closed. Consequently, $M=M^{2n}(J, \gamma)$ is a Kählerian manifold. In particular, we have

(15) $$\bar{\nabla} J=0,$$

where $\bar{\nabla}$ is the Levi-Civita connection of γ.

THEOREM 1 (Goldberg-Yano [1]). *A hypersurface M^{2n} of a Sasakian manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ admits the Kählerian structure (J, γ) under the assumption that ξ is not tangent to M^{2n} at each point.*

§ 3. The Levi-Civita connection of the Kählerian metric γ.

In this section, we assume that $\tilde{M}=\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ is a Sasakian manifold and the induced connection ∇ is the Levi-Civita connection of the induced metric g.

We want to calculate the Levi-Civita connection $\bar{\nabla}$ of the Kählerian metric γ on M. Let A be the vector field on M defined by

$$\alpha(X)=\gamma(A, X), \qquad X\in \mathfrak{X}(M).$$

According to the definition of the Levi-Civita connection, we get

$$2\gamma(\bar{\nabla}_X Y, Z)=2g(\nabla_X Y, Z)-(*), \tag{16}$$

where

$$(*)=X\cdot\{\alpha(JY)\alpha(JZ)\}+Y\cdot\{\alpha(JX)\alpha(JZ)\}-Z\cdot\{\alpha(JX)\alpha(JY)\}$$
$$+\alpha(J[X, Y])\alpha(JZ)+\alpha(J[Z, X])\alpha(JY)+\alpha(J[Z, Y])\alpha(JX).$$

Using (12) and (13), we get

$$X\cdot\{\alpha(JY)\alpha(JZ)\}=\{h(X, Y)-g(X, JY)-\alpha(JX)\alpha(Y)+\alpha(J\nabla_X Y)\}\alpha(JZ)$$
$$+\{h(X, Z)-g(X, JZ)-\alpha(JX)\alpha(Z)+\alpha(J\nabla_X Z)\}\alpha(JY).$$

On the other hand, (14) implies

$$g(X, JY)+\alpha(JX)\alpha(Y)=\gamma(X, JY). \tag{17}$$

Hence we get

$$X\cdot\{\alpha(JY)\alpha(JZ)\}=\{h(X, Y)-\gamma(X, JY)+\alpha(J\nabla_X Y)\}\alpha(JZ)$$
$$+\{h(X, Z)-\gamma(X, JZ)+\alpha(J\nabla_X Z)\}\alpha(JY).$$

Thus (*) becomes

$$(*)=2\{h(X, Y)\alpha(JZ)+\alpha(J\nabla_X Y)\alpha(JZ)+\gamma(JX, Z)\alpha(JY)+\gamma(JY, Z)\alpha(JX)\}.$$

Consequently, (16) becomes

$$\gamma(\bar{\nabla}_X Y, Z)=\gamma(\nabla_X Y, Z)+\alpha(J\nabla_X Y)\alpha(JZ)-h(X, Y)\alpha(JZ)$$
$$-\alpha(J\nabla_X Y)\alpha(JZ)-\gamma(JX, Z)\alpha(JY)-\gamma(JY, Z)\alpha(JX)$$
$$=\gamma(\nabla_X Y, Z)+h(X, Y)\gamma(JA, Z)-\alpha(JY)\gamma(JX, Z)-\alpha(JX)\gamma(JY, Z).$$

Thus we get

$$\bar{\nabla}_X Y=\nabla_X Y+h(X, Y)JA-\alpha(JY)JX-\alpha(JX)JY. \tag{18}$$

Since (12) implies

$$\nabla_X JY = \alpha(JY)X - \alpha(Y)JX + J\nabla_X Y,$$

(18) implies

$$\bar{\nabla}_X JY = \alpha(JY)X + J\nabla_X Y + h(X, JY)JA + \alpha(JX)Y. \tag{19}$$

On the other hand, (15) and (18) imply

$$\bar{\nabla}_X JY = J\nabla_X Y - h(X, Y)A + \alpha(JY)X + \alpha(JX)Y. \tag{20}$$

Comparing (19) and (20), we get

$$h(X, JY)JA = -h(X, Y)A. \tag{21}$$

THEOREM 2. *A non-invariant hypersurface M^{2n} of a Sasakian manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ is totally geodesic if and only if the induced connection ∇ given by (4) is the Levi-Civita connection of the induced metric g under the assumption that ξ is never tangent to M^{2n}.*

Proof. Since the hypersurface is non-invariant, the vector fields A and JA are linearly independent at each point. Hence (21) implies that $h(X, Y)=0$ for all vector fields X and Y, showing M^{2n} to be totally geodesic in $\tilde{M}^{2n+1}$. Q.E.D.

§4. Hypersurfaces of Sasakian manifolds of constant ϕ-holomorphic sectional curvature.

In this section, we assume that $\tilde{M}=\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ is a Sasakian manifold and that $M=M^{2n}$ is a totally geodesic hypersurface of $\tilde{M}$. The purpose of this section is to show that if $\tilde{M}$ is of constant ϕ-holomorphic sectional curvature k, then M is of constant holomorphic sectional curvature $k+3$.

As stated at the end of §1, $h=0$ implies that the induced connection ∇ is the Levi-Civita connection of the induced metric, and hence we may use some results of §3. (4) and (18) imply

$$\tilde{\nabla}_Y Z = \bar{\nabla}_Y Z + \alpha(JZ)JY + \alpha(JY)JZ.$$

Hence we get

$$\begin{aligned}\tilde{\nabla}_X\tilde{\nabla}_Y Z = \bar{\nabla}_X\bar{\nabla}_Y Z &+ \alpha(J\bar{\nabla}_Y Z)JX + \alpha(JX)J\bar{\nabla}_Y Z \\ &+ \{(\bar{\nabla}_X\alpha)(JZ) + \alpha(J\bar{\nabla}_X Z)\}JY + \{(\bar{\nabla}_X\alpha)(JY) + \alpha(J\bar{\nabla}_X Y)\}JZ \\ &+ \alpha(JZ)\{J\bar{\nabla}_X Y - \alpha(Y)JX - \alpha(JX)Y\} + \alpha(JY)\{J\bar{\nabla}_X Z - \alpha(Z)JX - \alpha(JX)Z\}.\end{aligned}$$

Thus we get the following:

$$\begin{aligned}\tilde{R}(X, Y)Z = \bar{R}(X, Y)Z &+ (\bar{\nabla}_X\alpha)(JZ)JY - (\bar{\nabla}_Y\alpha)(JZ)JX \\ &+ \{(\bar{\nabla}_X\alpha)(JY) - (\bar{\nabla}_Y\alpha)(JX)\}JZ \\ &+ \alpha(JZ)\{-\alpha(Y)JX - \alpha(JX)Y + \alpha(X)JY + \alpha(JY)X\} \\ &- \alpha(JY)\alpha(Z)JX + \alpha(JX)\alpha(Z)JY,\end{aligned}$$

where $\tilde{R}$ and $\bar{R}$ are curvature tensors of $\tilde{g}$ and γ, respectively. Using (12) and (13), we get

$$X\cdot\alpha(JZ)=(\nabla_X\alpha)(JZ)+\alpha((\nabla_XJ)Z)+\alpha(J\nabla_XZ)$$
$$=-\alpha(X)\alpha(JZ)-g(X, JZ)+\alpha(X)\alpha(JZ)-\alpha(JX)\alpha(Z)$$
$$+\alpha(J\{\bar{\nabla}_XZ+\alpha(JZ)JX+\alpha(JX)JZ\})$$
$$-\gamma(X, JZ)+\alpha(J\bar{\nabla}_XZ)-\alpha(X)\alpha(JZ)-\alpha(JX)\alpha(Z).$$

On the other hand, we have

$$X\cdot\alpha(JZ)=(\bar{\nabla}_X\alpha)(JZ)+\alpha(J\bar{\nabla}_XZ).$$

Hence we get

$$(\bar{\nabla}_X\alpha)(JZ)=-\gamma(X, JZ)-\alpha(X)\alpha(JZ)-\alpha(JX)\alpha(Z),$$

and hence

$$(\bar{\nabla}_X\alpha)(JY)-(\bar{\nabla}_Y\alpha)(JX)=2\gamma(JX, Y).$$

Consequently, we obtain

$$\tilde{R}(X, Y)Z=\bar{R}(X, Y)Z+\gamma(JX, Z)JY-\gamma(JY, Z)JX+2\gamma(JX, Y)JZ \tag{22}$$
$$+\alpha(JZ)\{\alpha(JY)X-\alpha(JX)Y).$$

Now, suppose $\tilde{M}$ is of constant ϕ-holomorphic sectional curvature k (Ogiue [2]):

$$4\tilde{R}(X, Y)Z=(k+3)\{\tilde{g}(Y, Z)X-\tilde{g}(X, Z)Y\}$$
$$+(k-1)\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+\tilde{g}(X, Z)\eta(Y)\xi$$
$$-\tilde{g}(Y, Z)\eta(X)\xi+\tilde{g}(\phi Y, Z)\phi X+\tilde{g})\phi Z, X)\phi Y-2\tilde{g}(\phi X, Y)\phi Z\}.$$

Then, since we have

$$\tilde{g}(\phi Y, Z)\phi X=g(JY+\alpha(Y)\xi, Z)(JX+\alpha(X)\xi)$$
$$=\gamma(JY, Z)(JX+\alpha(X)\xi),$$

we get

$$4\tilde{R}(X, Y)Z=(k+3)\{g(Y, Z)X-g(X, Z)Y\} \tag{23}$$
$$+(k-1)\{\alpha(JX)\alpha(JZ)Y-\alpha(JY)\alpha(JZ)X$$
$$+\gamma(JY, Z)JX+\gamma(JZ, X)JY-2\gamma(JX, Y)JZ\}$$
$$+(k-1)\{g(X, Z)\alpha(JY)-g(Y, Z)\alpha(JX)$$
$$+\gamma(JY, Z)\alpha(X)+\gamma(JZ, X)\alpha(Y)-2\gamma(JX, Y)\alpha(Z)\}\xi.$$

Comparing (22) and (23), we get

$$(24)\quad \begin{aligned}&(k-1)\{g(X, Z)\alpha(JY)-g(Y, Z)\alpha(JX)\\&+\gamma(JY, Z)\alpha(X)+\gamma(JZ, X)\alpha(Y)-2\gamma(JX, Y)\alpha(Z)\}=0\end{aligned}$$

and

$$(25)\quad \begin{aligned}4\tilde{R}(X, Y)Z=4\{&\gamma(JY, Z)JX-\gamma(JX, Z)JY-2\gamma(JX, Y)JZ\\&+\alpha(JZ)[\alpha(JX)Y-\alpha(JY)X]\}\\&+(k+3)\{g(Y, Z)X-g(X, Z)Y\}\\&+(k-1)\{\alpha(JX)\alpha(JZ)Y-\alpha(JY)\alpha(JZ)X\\&+\gamma(JY, Z)JX-\gamma(JX, Z)JY-2\gamma(JX, Y)JZ\}.\end{aligned}$$

(25) becomes

$$\begin{aligned}4\bar{R}(X, Y)Z&=(k+3)\{\gamma(JY, Z)JX-\gamma(JX, Z)JY-2\gamma(JX, Y)JZ\\&\qquad+\gamma(Y, Z)X-\gamma(X, Z)Y\}\\&=(k+3)\{X\wedge Y+JX\wedge JY-2\gamma(JX, Y)J\}Z,\end{aligned}$$

where $X\wedge Y$ denotes the endomorphism $Z\rightarrow\gamma(Y, Z)X-\gamma(X, Z)Y$. Hence $M^{2n}(J, \gamma)$ is of constant holomorphic sectional curvature $k+3$.

THEOREM 3. *Let M^{2n} be a hypersurface of a Sasakian manifold $\tilde{M}^{2n+1}(\phi, \xi, \eta, \tilde{g})$ of constant ϕ-holomorphic sectional curvature k. Suppose ξ is not tangent to M^{2n} at each point and M^{2n} is totally geodesic in $\tilde{M}^{2n+1}$. Then the Kählerian manifold $M^{2n}(J, \gamma)$ is of constant holomorphic sectional curvature $k+3$.*

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