

CARATHÉODORY'S THEOREM ON BOUNDARY ELEMENTS OF AN ARBITRARY PLANE REGION

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1. Introduction. Let A be a simply connected hyperbolic region. By defining prime ends on the boundary of A we can get a compactification of A , denoted by \bar{A} and every conformal mapping of A is extended to a topological mapping of \bar{A} onto the compactification of the image domain [2]. Especially, if A is the interior of a Jordan curve, called a Jordan region, the realization (impression) of every prime end is a point on the curve and conversely every point on the curve represents a unique prime end. This is termed Carathéodory's theorem [1].

The purpose of the present note is to generalize this theorem to boundary elements of an arbitrary region. They were introduced as a generalization of a prime end by the author [6]. We shall define an almost Jordan region in terms of extremal length and show that the realizations of boundary elements are mutually disjoint continuums, each of which has a nonvoid connected intersection with its defining Jordan curve. Furthermore, we shall define a weak boundary element and prove that its realization on an almost Jordan region is always a point. The terminology of a weak boundary component was used by Sario [5] which was first investigated by Grötzsch [3]. A weak boundary element is related to a boundary point with vanishing extremal diameter introduced by Strebel [6] when the realization of the boundary component containing the point is a Jordan curve.

2. Extremal length. Let Ω be an arbitrary plane region and let Γ be a family of locally rectifiable curves in Ω . Let $P(\Gamma)$ denote the class of nonnegative metrics $\rho(z)|dz|$ such that

$$\int_{\gamma} \rho(z)|dz| \geq 1, \quad \gamma \in \Gamma$$

which is called an admissible class of metrics. The module of Γ is defined by

$$\text{mod } \Gamma = \inf_{\rho \in P(\Gamma)} \iint_{\Omega} \rho^2 dx dy$$

and its reciprocal is called the extremal length of Γ , denoted by $\lambda(\Gamma)$. If a curve family consists of the curves joining two sets, its extremal length is termed the extremal distance between these sets. A family of curves with vanishing module is said to be *exceptional*.

Received May 19, 1969.

3. Boundary elements. A *boundary part* of Ω is a decreasing sequence of open sets $\{A_n\}_{n=1}^{\infty}$ satisfying the following conditions:

I) the relative boundary of A_n consists of a finite number of Jordan curves either closed or open, both of whose end arcs tend to a common boundary components in the latter case,

II) $\Omega - \bar{A}_n$ is connected, and

III) $\cap A_n = \phi$.

Two boundary parts $\{A_n\}$ and $\{A'_n\}$ are said to be *equivalent*, if every A_n contains a A'_m and vice versa. We say that a boundary part $A' = \{A'_n\}$ lies on $A = \{A_n\}$, if the first half of the above conditions is satisfied. They are said to be *disjoint*, if $\bar{A}_n \cap \bar{A}'_n = \phi$ for a sufficiently large n . The set $I(A) = \cap \text{Cl}(A_n)$ is called the *realization* of a boundary part A , where $\text{Cl}(\ast)$ denotes the closure taken in the Riemann sphere.

A boundary part is called *elementary*, if the relative boundary of each A_n is a single curve. A boundary component is also an elementary boundary part. Consider an elementary boundary part $\xi = \{A_n\}$ such that $A_n \supset \bar{A}_{n+1}$. Clearly $A_n - \bar{A}_{n+1}$ is a region. ξ is called a *boundary element*, if the module of the family of curves dividing the relative boundaries of A_n and A_{n+1} is positive and if the extremal distance between ξ and a compact disc of Ω is infinite. The following facts are known:

i) Every two distinct boundary elements are disjoint;

ii) Let $\tilde{\Omega}$ be the union of Ω and the totality of its boundary elements. We define an open set of $\tilde{\Omega}$ by the union of an open set of Ω and all the boundary elements each of whose defining sequences has a member contained in the open set. Then $\tilde{\Omega}$ is a compact Hausdorff space and every conformal mapping between two regions is extended to a topological mapping between their compactifications.

As to these the readers are referred to [9]. It is easily verified that a boundary element coincides with a prime end in a simply connected region.

4. Weak boundary element. Let ξ be a boundary element with its defining sequence $\{A_n\}_{n=1}^{\infty}$ and let Γ_n be the family of curves dividing A_n and a compact disc in $\Omega - A_1$. Set $\Gamma_0 = \cup_n \Gamma_n$. If $\lambda(\Gamma_0) = 0$, ξ is called a *weak boundary element*. This is an analogy of a weak boundary component introduced by Sario [5].

5. Almost Jordan region. Suppose that Ω is contained in the interior of a Jordan curve J . Suppose furthermore that the curve J is a subset of the realization of its outer boundary component denoted by α and that the family of curves running within Ω and possessing at least one cluster point in $I(\alpha) - J$ is exceptional. Then we call the Ω an *almost Jordan region* with respect to J . We remark that the exceptionality of the above curve family is replaced by the fact that the extremal distance between a compact disc and the set $I(\alpha) - J$ is infinite in Ω . In fact this implies that the module of the family of curves starting from a point of the disc and clustering on the set vanishes and hence so does that of the subfamily of the first family composed of the curves passing through the disc. This property does not depend on the reference open sets with compact closure [7]. Let $\{\Omega_n\}$ be an

normal exhaustion of Ω . Then the first family is expressed as the union of a countable number of subfamilies of curves passing through Ω_n . Since the union of a countable number of exceptional families is exceptional, it is exceptional. The converse is obvious since the second family is a subfamily of the first.

6. Results. We state

THEOREM 1. *Let Ω be an almost Jordan region with respect to a Jordan curve J . Then the intersection of the realization of every boundary element on its outer boundary component α with J is a continuum. Every two of distinct boundary elements on α have disjoint realizations if α is not a boundary element.*

Proof. If α is a boundary element, the assertion is trivial by definition. Otherwise, there are uncountable boundary elements on α equipollent to the points of the unit circle [9]. Let $\{A_n\}$ be a defining sequence of a boundary element ξ on α . It is sufficient to show that there exists an equivalent defining sequence $\{D_n\}$ such that the relative boundary of D_n is a Jordan arc with two distinct end points on J and that the closures of the relative boundaries of D_n and D_{n+1} are disjoint in the Riemann sphere. Indeed D_n is a subregion of a simply connected region divided from the interior of J by the relative boundary of D_n as a crosscut and $J \cap \text{Cl}(D_n)$ is equal to the arc between these end points. The first statement follows immediately. The second half is obvious, since $\bar{D}_n \cap \bar{D}'_n = \phi$ for an n from the property i) of boundary elements and $\text{Cl}(D_{n+1}) \cap \text{Cl}(D'_{n+1}) = \phi$, where $\{D_n\}$ and $\{D'_n\}$ are the defining sequences of two distinct boundary elements mentioned above.

To construct such a defining sequence, we consider the family of curves dividing the relative boundaries of A_{2n} and A_{2n+1} which has a finite extremal length by definition. The subfamily of curves clustering on $I(\alpha) - J$ is exceptional and so is those not having both end points, since each of their lengths is infinite [4]. After removing those families, the family of the remainder curves has the same extremal length. Then there exists a Jordan arc with end points on J and dividing the relative boundaries of A_{2n} and A_{2n+1} .

We denote by D_n the subregion divided from Ω by the arc and containing A_{2n+1} . Then $\{D_n\}$ denotes the same boundary element ξ , since $D_n - \bar{D}_{n+1}$ contains $A_{2n+1} - \bar{A}_{2n+2}$ which implies the first condition of a boundary element and others are obvious. We complete the proof by showing that the end points of the relative boundary of D_n are disjoint from those of D_{n+1} . Indeed, if they have a point a as a common end point, a metric $\rho_\epsilon |dz|$ with

$$\rho_\epsilon = \epsilon |z - a| \log |z - a|^{-1}$$

considered in $D_n - \bar{D}_{n+1}$ is admissible for the family of curves dividing the relative boundaries of D_n and D_{n+1} and tending to two points of J in both end arcs. The quantity

$$\iint \rho_\epsilon^2 dx dy$$

tends to zero as $\varepsilon \rightarrow 0$, which implies that the module of the family of curves dividing both the relative boundaries vanishes contrary to the definition of a boundary element.

As to a weak boundary element, we show

THEOREM 2. *Under the same notations as in Theorem 1, if ξ is a weak boundary element on α which is not a boundary element, then $I(\xi)$ is a point on J .*

Proof. Let $\xi = \{A_n\}$ be a weak boundary element on α . In this case we can construct a defining sequence $\{D_n\}$ as before and such that the length of relative boundary of D_n tends to zero. In fact let K be a compact disc in Ω . The metrics $\rho|dz|$ with $\rho = k$ for arbitrarily large positive k are never admissible for the family of curves with end points on J and dividing K and ξ . It follows that there exists a sequence of Jordan arcs of the above family whose lengths tend to zero. On the other hand the lengths of curves of its subfamily contained in $\Omega - \bar{J}_n$ have a positive lower bound which tends to zero as $n \rightarrow \infty$. Then we can make a desired defining sequence D_n from the subregions divided by the above crosscuts so that $D_n - \bar{D}_{n+1}$ contains some of the regions $A_\nu - \bar{A}_{\nu+1}$. We denote by D_n^* the simply connected region bounded by the relative boundary of D_n and by a subarc of J and containing D_n . Then we can conclude that the diameter of D_n^* tends to zero. Then $\{D_n\}$ contains a subsequence $\{D_{n_\nu}\}$ such that each point of $\text{Cl}(D_{n_\nu})$ tends to a point ζ on J . We can see that $I(\xi)$ is equal to ζ . For, if it contains another point, then it is a continuum not a point which contradicts the existence of a crosscut dividing K and it whose length is arbitrarily small.

7. Remarks. For example a minimal radial slit disc with a finite radius R is an almost Jordan region with respect to the circle $|z| = R$ [7] and so is a circular and radial slit disc [8]. When the interior of the circle is mapped onto a Jordan region, then the image of the above slit disc is an almost Jordan region.

Especially, if all the boundary elements on the outer boundary component of an almost Jordan region with respect to J are weak, then the realization of the outer boundary component coincides with J .

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