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FIXED POINT THEOREM FOR AMENABLE SEMIGROUP OF NONEXPANSIVE MAPPINGS

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1. Introduction.

Let K be a subset of a Banach space B. A mapping s of K into B is said to be *nonexpansive* if for each pair of elements x and y of K, we have $||sx-sy||\leqq||x-y||$.

Kakutani [5] and Markov [7] proved the following theorem: Let K be a compact convex subset of a locally convex linear topological space B and S be a commuting family of linear continuous mappings of K into itself. Then S has a common fixed point in K.

Day [2] showed that this is true even if S is an amenable semigroup.

On the other hand, de Marr [3] proved a fixed point theorem for a family of nonlinear mappings: Let K be a nonempty compact convex subset of a Banach space B. If S is a nonempty commutative family of nonexpansive mappings of K into itself, then the family S has a common fixed point in K.

The question naturally arises as to whether this is true if one considers an amenable semigroup of nonexpansive mappings.

In this paper, we shall show that the answer is affirmative.

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2. Amenable semigroup.

Let S be an abstract semigroup and $m(S)$ be the space of all bounded real valued functions of S, where $m(S)$ has the supremum norm. An element $\lambda\in m(S)^*$ (the dual space of $m(S)$) is *mean* on $m(S)$ if $\lambda(e)=||\lambda||=1$ where e denotes the constant 1 function on S. A mean λ is *left* [*right*] *invariant* if $\lambda(l_sf)=\lambda(f)$ [$\lambda(r_sf)=\lambda(f)$] for all $f\in m(S)$ and $s\in S$, where the left [right] translation l_s [r_s] of $m(S)$ by s is given by $(l_sf)(s')=f(ss')$ [$(r_sf)(s')=f(s's)$]. An *invariant mean* is a left and a right invariant mean. A semigroup that has a left invariant mean [right invariant mean] is called *left amenable* [*right amenable*]. A semigroup that has an invariant mean is called *amenable*.

Let M be a nonempty compact Hausdorff space and $C(M)$ be the space of bounded continuous real valued functions on M. The norm will be the supremum

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norm. Let S be a semigroup of continuous mappings of M into M and define a mapping U_s for each s in S from $C(M)$ into $C(M)$ by attaching to each $f \in C(M)$, the function $U_s f$ on M such that $(U_s f)(x) = f(sx)$ for each x in M.

We shall prove the following Lemma by using Day's fixed point theorem [2].

LEMMA. *Let M be a nonempty compact Hausdorff space and S be an amenable semigroup of continuous mappings of M into M. Then there exists $L^* \in C(M)^*$ (the dual space of $C(M)$) such that $L^*(e) = \|L^*\| = 1$ where e is the constant 1 function on M and $L^*(U_s f) = L^*(f)$ for all $f \in C(M)$ and $s \in S$.*

Proof. Let $K[C(M)] = \{L \in C(M)^* : L(e) = \|L\| = 1\}$. Since U_s for each s in S is a linear mapping of $C(M)$ into itself such that $U_s(e) = e$ and $\|U_s\| = 1$, a mapping U_s^* that is given by $(U_s^* L)(f) = L(U_s f)$ for all $L \in C(M)^*$ and $f \in C(M)$ is a weak*-continuous affine mapping of $K[C(M)]$ into itself.

If $\{U_s^* : s \in S\}$ is an amenable semigroup, from Day's fixed point theorem [2], there exists $L^* \in K[C(M)]$ such that $(U_s^* L^*)(f) = L^*(f)$ for all $f \in C(M)$.

We shall show that $\{U_s^* : s \in S\}$ is an amenable semigroup.

Since the mapping σ of S onto $\{U_s^* : s \in S\}$ that is given by $\sigma(s) = U_s^*$ for each s in S is a homomorphism, $\{U_s^* : s \in S\}$ is an amenable semigroup from [1]. Q.E.D.

3. Main theorem.

THEOREM. *Let K be a nonempty compact convex subset of a Banach space B and S be an amenable semigroup of nonexpansive mappings of K into K. Then there exists an element z in K such that $sz = z$ for each s in S.*

Proof. By using Zorn's lemma, we can find a minimal nonempty compact convex set $X \subset K$ such that X is invariant under each s in S. If X consists of a single point, then the theorem is proved.

By using Zorn's lemma again, we can find a minimal nonempty compact set $M \subset X$ such that M is invariant under each s in S.

We will now that $M = \{sx : x \in M\}$ for each s in S.

Since the semigroup of restrictions of all mappings s in S to M is amenable [1], by Lemma there exists an element L^* in $K[C(M)]$ such that $L^*(U_s f) = L^*(f)$ for all $f \in C(M)$. The Riesz theorem asserts that to the element L^*, there corresponds a unique probability measure m on M such that

$$L^*(f) = \int_M f \, dm$$

for each f in $C(M)$.

Since M is a compact metric space and m is a probability measure on M, there exists a unique closed set $F \subset M$ called *support* of m satisfying (i) $m(F) = 1$, (ii) if D is any closed set such that $m(D) = 1$, then $F \subset D$. Moreover F is the set of all point $x \in M$ having the property that $m(G) > 0$ for each open set G containing x.

It is obvious that F is contained in $s(M)$ for each s in S, since each s in S is a measurable transformation of M into M and hence $m(sM)=m(M)=1$.

Let χ_F be the characteristic function of the closed subset F in M. Since for each s in S

$$1=m(F)=\int_M \chi_F(x)dm$$

$$=\int_M \chi_F(sx)dm=m(s^{-1}F),$$

it is clear that F is contained in $s^{-1}(F)$ for each s in S. Therefore F is invariant under each s in S.

If M contains more than one point, there exists an element u in the closed convex hull of M such that

$$\rho=\sup\{\|u-x\|: x\in M\}<\delta(M)$$

where $\delta(M)$ is the diameter of M.

Let us define

$$X_0=\bigcap_{x\in M}\{y\in X: \|x-y\|\leqq\rho\},$$

then X_0 is the nonempty closed convex proper subset of X such that $s(X_0)\subset X_0$ for each s in S. This is a contradiction to the minimality of X. Therefore M contains only one point which is a common fixed point for the semigroup of nonexpansive mappings of K into itself. Q.E.D.

COROLLARY (de Marr [3]). *Let K be a compact convex subset of a Banach space B and S be a family of commutative nonexpansive mappings of K into itself. Then S has a common fixed point in K.*

Proof. Since a commutative semigroup is an amenable semigroup, Corollary is obvious from Theorem.

REMARK. Theorem is true even if S is a left amenable semigroup. We can discuss the above by using purely metric methods [6] [8].

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