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PSEUDO-UMBILICAL SUBMANIFOLDS OF CODIMENSION 2

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Dedicated to Professor Hiraku Tôyama on his sixtieth birthday

The purpose of the present paper is to study the so-called pseudo-umbilical submanifolds of codimension 2 in Euclidean and Riemannian manifolds. Our main results appear in Propositions 2.3, 3.2, 3.3, 4.1, 4.2 and 4.3.

In § 1, we reformulate formulas for submanifolds of a general Riemannian manifold and, in § 2, we specialize these formulas to those for submanifolds of codimension 2 of a Euclidean or a Riemannian manifold.

We study, in § 3, pseudo-umbilical submanifolds of codimension 2 in a space of constant curvature and, in § 4, those in a Euclidean space. In the last section 5, we prove, for the completeness, some of lemmas which are used in the paper.

§ 1. Formulas for submanifolds.

As we are going to study some special kinds of submanifolds, we would like first of all to reformulate formulas for submanifolds of a Riemannian manifold for the later use. Let M^n be an n-dimensional manifold[1] differentiablly immersed as a submanifold of an m-dimensional Riemannian manifold M^m, where $n<m$, and denote by x: $M^n \to M^m$ the immersion. Denote by B: $T(M^n) \to T(M^m)$ the differential of the mapping x, i.e., $B=dx$, where $T(M^n)$ and $T(M^m)$ are the tangent bundles of M^n and M^m respectively. On putting $T(M^n, M^m)=BT(M^n)$, the set of all vectors tangent to $x(M^n)$, we see by definition that B: $T(M^n) \to T(M^n, M^m)$ is an isomorphism, since $x: M^n \to M^m$ is an immersion. The set of all vectors normal to $x(M^n)$ forms a vector bundle $N(M^n, M^m)$ over $x(M^n)$, which is the normal bundle of $x(M^n)$. The vector bundle over M^n, which is induced by x from $N(M^n, M^m)$ is denoted by $N(M^n)$ and called the *normal bundle* of M^n with respect to the immersion x. We now denote by C: $N(M^n) \to N(M^n, M^m)$ the natural isomorphism.

We now introduce the following notations: $\mathcal{T}^r_s(M^n)$ is the space of all tensor fields of type (r, s), i.e., of contravariant degree r and covariant degree s, associated with $T(M^n)$. $\mathcal{T}(M^n)=\sum_{r,s}\mathcal{T}^r_s(M^n)$ is the space of all tensor fields associated with $T(M^n)$. $\mathcal{N}^r_s(M^n)$ and $\mathcal{N}(M^n)=\sum_{r,s}\mathcal{N}^r_s(M^n)$ denote the respective spaces associated

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1) Manifolds, mappings, functions, tensor fields and any other geometric objects we discuss are assumed to be differentiable and of class C^∞. We restrict ourselves only to connected submanifolds of dimension $n \geqq 2$.

with $N(M^n)$. $\mathcal{T}^r_s(M^n, M^m)$ and $\mathcal{N}^r_s(M^n, M^m)$ denote the corresponding spaces of tensor fields associated with $T(M^n, M^m)$ and $N(M^n, M^m)$ respectively. Thus $\mathcal{T}^0_0(M^n) = \mathcal{N}^0_0(M^n)$ is the space of all functions defined in M^n and $\mathcal{T}^0_0(M^n, M^m) = \mathcal{N}^0_0(M^n, M^m)$ is the space $\mathcal{T}^0_0(x(M^n))$ of all functions defined on $x(M^n)$. Any element f of $\mathcal{T}^0_0(x(M^n))$ is identified with $f\circ x$ which is an element of $\mathcal{T}^0_0(M^n)$. Denoting by $B(M^n, M^m)$ the restriction of $T(M^m)$ to $x(M^n)$, we see that $B(M^n, M^m)$ is the Whitney sum $T(M^n, M^m)\oplus N(M^n, M^m)$. $\mathcal{B}^r_s(M^n, M^m)$ denotes the space of all tensor fields of type (r, s) associated with $B(M^n, M^m)$.

The mapping B: $T(M^n)\to T(M^n, M^m)$ induces naturally an isomorphism of $\mathcal{T}(M^n)$ onto $\mathcal{T}(M^n, M^m)$, which is denoted also by B, in such a way that $B(fP+gQ)=fBP+gBQ$, $B(P\otimes S)=(BP)\otimes(BS)$ for $f, g\in\mathcal{T}^0_0(M^n)$, $P, Q, S\in\mathcal{T}(M^n)$. The mapping B thus introduced is called the *tangential mapping* of the immersion x: $M^n\to M^m$. The mapping C: $N(M^n)\to N(M^n, M^m)$ induces naturally an isomorphism of $\mathcal{N}(M^n)$ onto $\mathcal{N}(M^n, M^m)$, which is denoted also by C, in such a way that $C(fP+gQ)=fCP+gCQ$, $C(P\otimes S)=(CP)\otimes(CS)$ for $f, g\in\mathcal{T}^0_0(M^n)$, $P, Q, S\in\mathcal{N}(M^n)$. The mapping C thus introduced is called the *normal mapping* of the immersion x: $M^n\to M^m$.

We take an element $\bar{X}$ of $\mathcal{B}^1_0(M^n, M^m)$. For any point p of $x(M^n)$, there exists in M^m a neighborhood Ω containing p such that there exists in Ω a vector field $\tilde{X}$ which is an extension of $\bar{X}$ restricted to $\Omega'=x(U)\cap\Omega$ containing p, U being a certain neighborhood of M^n. Such an $\tilde{X}$ is called a *local extension* of $\bar{X}$ in Ω. Taking arbitrarily two elements $\bar{X}$ and $\bar{Y}$ of $\mathcal{T}^1_0(M^n, M^m)$ and local extensions $\tilde{X}$ of $\bar{X}$ and $\tilde{Y}$ of $\bar{Y}$ in a neighborhood Ω of M^m, we see that the restriction $[\tilde{X}, \tilde{Y}]_{M^n}$ of $[\tilde{X}, \tilde{Y}]$ to $x(M^n)$ is tangent to $x(M^n)$ and determined independently of the choice of the local extensions $\tilde{X}$ and $\tilde{Y}$. Thus $[\tilde{X}, \tilde{Y}]_{M^n}$ defines an element of $\mathcal{T}^1_0(M^n, M^m)$. If we put

$$[\bar{X}, \bar{Y}]=[\tilde{X}, \tilde{Y}]_{M^n} \tag{1.1}$$

for $\bar{X}, \bar{Y}\in\mathcal{T}^1_0(M^n, M^m)$, we have

$$[BX, BY]=B[X, Y] \tag{1.2}$$

for $X, Y\in\mathcal{T}^1_0(M^n)$.

If we denote by $(\ , \ )$ the inner product determined by the Riemannian metric $\tilde{G}$ of M^m and put

$$\langle X_1, X_2\rangle=(BX_1, BX_2), \qquad \langle N_1, N_2\rangle^*=(CN_1, CN_2) \tag{1.3}$$

for $X_1, X_2\in\mathcal{T}^1_0(M^n)$ and $N_1, N_2\in\mathcal{N}^1_0(M^n)$, then the inner product $\langle \ , \ \rangle$ determines in M^n a Riemannian metric g, which is called *induced metric* of the submanifold M^n, and the inner product $\langle \ , \ \rangle^*$ determines in $N(M^n)$ an element g^* of $\mathcal{N}^0_2(M^n)$, which is called the *induced metric* of $N(M^n)$.

Let $\tilde{\nabla}$ be the Riemannian connection determined by $\tilde{G}$ in the enveloping manifold M^m, i.e., the torsionless affine connection satisfying $\tilde{\nabla}\tilde{G}=0$. Taking an element $\bar{X}$ of $\mathcal{T}^1_0(M^n, M^m)$ and an element $\bar{Y}$ of $\mathcal{B}^1_0(M^n, M^m)$ and arbitrary local extensions $\tilde{X}$ of $\bar{X}$ and $\tilde{Y}$ of $\bar{Y}$ in a neighborhood Ω of M^m, we can easily show that the restriction $(\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{M^n}$ of $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ to $x(M^n)$ is independent of the choice of the local extensions $\tilde{X}$ and $\tilde{Y}$. Therefore we can define an element $\tilde{\nabla}_{\bar{X}}\bar{Y}$ of $\mathscr{B}^1_0(M^n, M^m)$ by the equation

$$\tilde{\nabla}_{\bar{X}}\bar{Y}=(\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{M^n} \tag{1. 4}$$

for $\bar{X}\in\mathscr{T}^1_0(M^n, M^m)$ and $\bar{Y}\in\mathscr{B}^1_0(M^n, M^m)$. Thus, by virtue of (1. 1) and $\tilde{\nabla}_{\tilde{X}}\tilde{Y}-\tilde{\nabla}_{\tilde{Y}}\tilde{X}=[\tilde{X}, \tilde{Y}]$, we obtain

$$\tilde{\nabla}_{\bar{X}}\bar{Y}-\tilde{\nabla}_{\bar{Y}}\bar{X}=[\bar{X}, \bar{Y}] \tag{1. 5}$$

for $\bar{X}, \bar{Y}\in\mathscr{T}^1_0(M^n, M^m)$.

Taking an arbitrary element $\bar{X}$ of $\mathscr{B}^1_0(M^n, M^m)$, we denote by $\bar{X}^T$ its tangential component to $x(M^n)$ and by $\bar{X}^\perp$ its normal component to $x(M^n)$. Then we have a unique decomposition $\bar{X}=\bar{X}^T+\bar{X}^\perp$ for any element $\bar{X}$ of $\mathscr{B}^1_0(M^n, M^m)$, where $\bar{X}^T\in\mathscr{T}^1_0(M^n, M^m)$ and $\bar{X}^\perp\in\mathscr{N}^1_0(M^n, M^m)$.

In the remaining part of the paper, unless otherwise stated, X, Y and Z mean arbitrary elements of $\mathscr{T}^1_0(M^n)$ and N an arbitrary element of $\mathscr{N}^1_0(M^n)$. If we put

$$B(\nabla_X Y)=(\tilde{\nabla}_{BX}BY)^T, \tag{1. 6}$$

we have a unique element $\nabla_X Y$ of $\mathscr{T}^1_0(M^n)$ and can easily verify that $\nabla_{fX}Y=f\nabla_X Y$, $\nabla_X(fY)=f\nabla_X Y+(Xf)Y$ for $f\in\mathscr{T}^1_0(M^n)$. Thus the correspondence $(X, Y)\to\nabla_X Y$ determines in M^n an affine connection ∇ which coincides, as is well known, with the Riemannian connection determined by the induced metric g of M^n. That is to say, ∇ is torsionless and satisfies $\nabla g=0$. The affine connection ∇ thus introduced in M^n is called the *induced connection* of the submanifold M^n. If we put

$$CH(X, Y)=(\tilde{\nabla}_{BX}BY)^\perp, \tag{1. 7}$$

we have a unique element $H(X, Y)$ of $\mathscr{N}^1_0(M^n)$. It is easily verified that $H(fX, gY)=fgH(X, Y)$ for $f, g\in\mathscr{T}^0_0(M^n)$. Thus the correspondence $(X, Y)\to H(X, Y)$ determines an element H of $\mathscr{T}^0_2(M^n)\otimes\mathscr{N}^1_0(M^n)$, which is called the *Euler-Schouten tensor* or the *second fundamental tensor* of the submanifold M^n, or, that of the immersion x: $M^n\to M^m$. Combining (1. 6) and (1. 7), we obtain the following equation

$$\tilde{\nabla}_{BX}BY=B(\nabla_X Y)+CH(X, Y), \tag{1. 8}$$

which is *Gauss' equation* of the submanifold M^n.

If we put

$$C(\nabla^*_X N)=(\tilde{\nabla}_{BX}CN)^\perp \tag{1. 9}$$

for $N\in\mathscr{N}^1_0(M^n)$, we have a unique element $\nabla^*_X N$ of $\mathscr{N}^1_0(M^n)$ and can easily verify that $\nabla^*_{fX}N=f\nabla^*_X N$, $\nabla^*_X(fN)=f\nabla^*_X N+(Xf)N$ for $f\in\mathscr{T}^0_0(M^n)$. Thus the correspondence $(X, N)\to\nabla^*_X N$ defines in $N(M^n)$ a linear connection ∇^*, which is called the *induced connection* of $N(M^n)$ and satisfies $\nabla^* g^*=0$, g^* being the induced metric of $N(M^n)$. If we put

$$BK(X, N) = -(\tilde{\nabla}_{BX} CN)^T, \tag{1. 10}$$

we have a unique element $K(X, N)$ of $\mathcal{T}^1_0(M^n)$. It is easily verified that $K(fX, gN) = fgK(X, N)$ for $f, g \in \mathcal{T}^0_0(M^n)$. Thus the correspondence $(X, N) \to K(X, N)$ determines an element K of $\mathcal{T}^1_1(M^n) \otimes \mathcal{N}^1_0(M^n)$, which is called also the second fundamental tensor of the submanifold M^n (Cf. (1. 12)). Combining (1. 9) and (1. 10), we obtain the following equation

$$\tilde{\nabla}_{BX} CN = C(\nabla^*_X N) - BK(X, N), \tag{1. 11}$$

which is *Weingarten's equation* of the submanifold M^n.

Differentiating covariantly $(BY, CN) = 0$ along the submanifold $x(M^m)$ and taking account of $\tilde{\nabla}\tilde{G} = 0$, we find $(\tilde{\nabla}_{BX} BY, CN) + (BY, \tilde{\nabla}_{BX} CN) = 0$, from which

$$\langle H(X, Y), N \rangle^* = \langle K(X, N), Y \rangle \tag{1. 12}$$

by means of (1. 3), (1. 8) and (1. 11). On the other hand, taking account of (1. 2) and (1. 5), we have $\tilde{\nabla}_{BX} BY - \tilde{\nabla}_{BY} BX = B[X, Y]$. Substituting (1. 8) in this equation, we obtain

$$H(X, Y) = H(Y, X), \tag{1. 13}$$

i.e., the second fundamental tensor $H(X, Y)$ is symmetric with respect to X and Y.

We extend naturally the operations of the induced connections ∇ of M^n and ∇^* of $N(M^n)$ respectively to $\mathcal{T}(M^n)$ and to $\mathcal{N}(M^n)$ as derivations and denote the extended covariant differentiations also by the same symbols ∇ and ∇^* respectively. We shall now define a derivation ∇_X ($X \in \mathcal{T}^1_0(M^n)$) in $\mathcal{T}(M^n) \otimes \mathcal{N}(M^n)$ as follows: $\nabla_X(T \otimes U) = (\nabla_X T) \otimes U + T \otimes (\nabla^*_X U)$, T and U being arbitrary elements of $\mathcal{T}(M^n)$ and $\mathcal{N}(M^n)$ respectively. The derivation ∇_X thus introduced in $\mathcal{T}(M^n) \otimes \mathcal{N}(M^n)$ is the so-called van der Waerden-Bortolotti covariant differentiation along the submanifold M^n.

We have by virtue of (1. 8) and (1. 11) the following equations

$$\begin{aligned}
&\tilde{\nabla}_{BX} \tilde{\nabla}_{BY} BZ = \tilde{\nabla}_{BX}\{B(\nabla_Y Z) + CH(Y, Z)\} \\
&= B\{\nabla_X \nabla_Y Z - K(X, H(Y, Z))\} + C\{\nabla^*_X(H(Y, Z)) + H(X, \nabla_Y Z)\} \\
&= B\{\nabla_X \nabla_Y Z - K(X, H(Y, Z))\} \\
&\quad + C\{H(X, \nabla_Y Z) + H(\nabla_X Y, Z) + H(Y, \nabla_X Z) + (\nabla_X H)(Y, Z)\}.
\end{aligned} \tag{1. 14}$$

and

$$\begin{aligned}
\tilde{\nabla}_{[BX, BY]} BZ &= \tilde{\nabla}_{B[X, Y]} BZ = B(\nabla_{[X, Y]} Z) + CH([X, Y], Z) \\
&= B(\nabla_{[X, Y]} Z) + C\{H(\nabla_X Y, Z) - H(\nabla_Y X, Z)\}
\end{aligned} \tag{1. 15}$$

because of (1. 2) and $[X, Y] = \nabla_X Y - \nabla_Y X$. Therefore, denoting by L and R respectively the curvature tensors of the enveloping manifold M^m and the submanifold

M^n, we have, by definition,

$$L(BX, BY)BZ=\tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ-\tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ-\tilde{\nabla}_{[BX,BY]}BZ,$$

$$R(X, Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z,$$

which imply together with (1. 14) and (1. 15) the *equation of Gauss-Codazzi*

$$\begin{aligned} L(BX, BY)BZ=&B\{R(X, Y)Z-K(X, H(Y, Z))+K(Y, H(X, Z))\} \\ &+C\{(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)\} \end{aligned} \tag{1. 16}$$

along the submanifold M^n. Denoting by R^* the curvature tensor of the induced connection ∇^* of $N(M^n)$, we have, by definition,

$$R^*(X, Y)N=\nabla^*_X\nabla^*_Y N-\nabla^*_Y\nabla^*_X N-\nabla^*_{[X,Y]}N. \tag{1. 17}$$

Taking account of this equation, we have by a similar device the *equation of Codazzi-Ricci*

$$\begin{aligned} L(BX, BY)CN=&C\{R^*(X, Y)N-H(X, K(Y, N))+H(Y, K(X, N))\} \\ &-B\{(\nabla_X K)(Y, N)-(\nabla_Y K)(X, N)\} \end{aligned} \tag{1. 18}$$

along the submanifold M^n. The equations (1. 16) and (1. 18) are called the *structure equations* of the submanifold M^n.

Let $X_1, X_2, \cdots, X_n$ be n mutually orthogonal local unit vector fields in M^n. Then an element $\mathrm{Tr}\, H$ of $\mathcal{N}^1_0(M^n)$, called the *trace* of H, is defined by the equation

$$\mathrm{Tr}\, H=\sum_{j=1}^{n} H(X_j, X_j).$$

On putting

$$A=\frac{1}{n}\mathrm{Tr}\, H, \tag{1. 19}$$

we call A the *mean curvature vector* of the submanifold M^n or that of the immersion x: $M^n\to M^m$. The length $\alpha=|A|$ of A is called the *mean curvature* of the submanifold M^n.

Umbilical submanifolds. When there exists an element P of $\mathcal{N}^1_0(M^n)$ such that $|P|=\sqrt{\langle P, P\rangle^*}=1$ and

$$H(X, Y)=\alpha\langle X, Y\rangle P, \tag{1. 20}$$

for any X and Y, α being a certain non-negative element of $\mathcal{T}^0_0(M^n)$, the submanifold M^n is said to be *umbilical.* In (1. 20), α is the mean curvature and the mean curvature vector is $A=\alpha P$.

Pseudo-umbilical submanifolds. We now assume that the mean curvature vector A vanishes nowhere in M^n. Then we have an element $P=A/|A|$ of $\mathcal{N}_0^1(M^n)$ such that $|P|=1$. If, in such a case, we have

$$\langle H(X, Y), P\rangle^* = \alpha\langle X, Y\rangle, \tag{1. 21}$$

or equivalently

$$K(X, P)=\alpha X, \tag{1. 21$'$}$$

α being an element of $\mathcal{T}_0^0(M^n)$ and positive everywhere in M^n, then the submanifold M^n is said to be *pseudo-umbilical*. We now see, taking account of (1. 19) and (1. 21), that the mean curvature vector is given by

$$A=\alpha P,$$

where α is the mean curvature. We know from (1. 20) and (1. 21) that any umbilical submanifold is pseudo-umbilical if its mean curvature α vanishes nowhere in M^n.

Submanifolds of a submanifold. Let M^s be a submanifold of dimension s immersed in an n-dimensional Riemannian manifold M^n with immersion $\bar{x}: M^s \to M^n$ $(s<n)$. Moreover, we assume that M^n is a submanifold in a Riemannian manifold M^m of dimension m with immersion x: $M^n \to M^m$ $(n<m)$. Then M^s is a submanifold in M^m and $\hat{x}=x\bar{x}$ its immersion. We denote by $B, \bar{B}$ and $\hat{B}$ respectively the tangential mappings of $x, \bar{x}$ and $\hat{x}$. Then we have $\hat{B}=B\bar{B}$. The normal mappings of $x, \bar{x}$ and $\hat{x}$ are denoted respectively by $C, \bar{C}$ and $\hat{C}$ and the second fundamental tensors of $x, \bar{x}$ and $\hat{x}$ respectively by $H, \bar{H}$ and $\hat{H}$. Thus we now have Gauss' equations for $X, Y \in \mathcal{T}_0^1(M^s)$

$$\begin{aligned} \tilde{\nabla}_{\hat{B}X}\hat{B}Y &= \hat{B}(\nabla_X Y)+\hat{C}\hat{H}(X, Y), \\ \bar{\nabla}_{\bar{B}X}\bar{B}Y &= \bar{B}(\nabla_X Y)+\bar{C}\bar{H}(X, Y) \end{aligned} \tag{1. 22}$$

along $\hat{x}(M^s)$ and $\bar{x}(M^s)$ respectively, where $\tilde{\nabla}$ is the Riemannian connection in the enveloping manifold M^m and $\bar{\nabla}$ and ∇ denote the induced connections of M^n and M^s respectively. On the other hand, taking account of $\hat{B}=B\bar{B}$, we have

$$\tilde{\nabla}_{\hat{B}X}\hat{B}Y = B(\bar{\nabla}_{\bar{B}X}\bar{B}Y)+CH(\bar{B}X, \bar{B}Y). \tag{1. 23}$$

Combining (1. 22) and (1. 23), we find

$$\hat{C}\hat{H}(X, Y)=B\bar{C}\bar{H}(X, Y)+CH(\bar{B}X, \bar{B}Y) \tag{1. 24}$$

for $X, Y \in \mathcal{T}_0^1(M_s)$. Thus we have from (1. 24)

PROPOSITION 1. 1. *Let M^s be a submanifold immersed in M^n $(s<n)$ and M^n an umbilical submanifold immersed in M^m $(n<m)$. Then M^s is pseudo-umbilical in M^n if and only if so is M^s also in M^m.*

When the mean curvature vector A vanishes identically in a submanifold, the submanifold is said to be *minimal*. We have from (1. 24)

PROPOSITION 1. 2. *Let M^s be a submanifold immersed in M^n $(s<n)$, which is a submanifold immersed in M^m $(n<m)$. Then M^s is minimal in M^n if and only if the mean curvature vector of M^s with respect to M^m is orthogonal to M^n everywhere along M^s.*

§2. Submanifolds of codimension 2.

In this section we study more in dedail formulas stated in §1 for submanifolds of codimension 2. Let M^n be a submanifold of codimension 2 immersed in an $(n+2)$-dimensional Riemannian manifold M^{n+2} and let its immersion be denoted by $x: M^n \to M^{n+2}$. We assume that the normal bundle $N(M^n)$ is orientable, i.e. that there exist in $\mathcal{N}_0^1(M^n)$ two elements P and Q such that $|P|=|Q|=1$ and $\langle P, Q\rangle^*=0$. Then CP and CQ are two unit vector fields defined globally along $x(M^n)$, normal to $x(M^n)$ and mutually orthogonal. Thus the second fundamental tensor H of M^n has the form

$$H(X, Y)=h(X, Y)P+h'(X, Y)Q, \tag{2. 1}$$

h and h' being elements of $\mathcal{T}_2^0(M^n)$. As direct consequences of (1. 13), two tensor fields h and h' are symmetric. If we now define two elements k and k' of $\mathcal{T}_1^1(M^n)$ by the equations

$$k(X)=K(X, P), \qquad k'(X)=K(X, Q), \tag{2. 2}$$

then we have from (1. 12), (2. 1) and (2. 2)

$$h(X, Y)=\langle k(X), Y\rangle, \qquad h'(X, Y)=\langle k'(X), Y\rangle. \tag{2. 3}$$

There exists an element θ of $\mathcal{T}_1^0(M^n)$ such that

$$\nabla_X^* P=\theta(X)Q, \qquad \nabla_X^* Q=-\theta(X)P. \tag{2. 4}$$

This θ is called the *third fundamental tensor* of the submanifold M^n. We have directly from (1. 17) and (2. 4)

$$R^*(X, Y)P=d\theta(X, Y)Q, \qquad R^*(X, Y)Q=-d\theta(X, Y)P, \tag{2. 5}$$

R^* being the curvature tensor of the induced connection ∇^* of $N(M^n)$, where $d\theta$ denote the exterior differential of the 1-form θ. Differentiating (2. 1) and (2. 2) covariantly along M^n and taking account of (2. 5), we have respectively

$$\nabla_X H=(\nabla_X h-\theta(X)h')P+(\nabla_X h'+\theta(X)h)Q \tag{2. 6}$$

and

$$(\nabla_X K)(Y, P)=(\nabla_X k)Y-\theta(X)k'(Y), \quad (\nabla_X K)(Y, Q)=(\nabla_X k')Y+\theta(X)k(Y). \tag{2. 7}$$

We now assume the submanifold M^n to be pseudo-umbilical. Then we can choose P in such a way that $P=A/|A|$, A being the mean curvature vector of M^n.

Thus, taking account of (1. 21), (1. 21)′ and (2. 1), we find

$$h(X, Y)=\alpha\langle X, Y\rangle, \qquad k(X)=\alpha X, \tag{2. 8}$$

where $\alpha=|A|\neq 0$. According to (1. 20), (2. 1) and (2. 8), we have

$$\frac{1}{n}\operatorname{Tr} h=\alpha, \qquad \operatorname{Tr} h'=0, \tag{2. 9}$$

where Tr h and Tr h' denote respectively the *traces* of h and h', for example, Tr $h=\sum_{j=1}^{n} h(X_j, X_j)$ for an orthonormal local basis $\{X_1, X_2, \cdots, X_k\}$ of $\mathcal{T}_0^1(M^n)$.

For pseudo-umbilical submanifolds, we restrict ourselves only to such a P that defined by $P=A/|A|$. Comparing (1. 20) and (2. 1) with $P=A/|A|$, we have

LEMMA 2. 1. *A pseudo-umbilical submanifold M^n of codimension 2 is umbilical, if and only if $h'=0$ holds identically in M^n. In such a case, we have $h(X, Y)=\alpha\langle X, Y\rangle$, α being the mean curvature.*

Substituting (2. 5), (2. 6), (2. 7) and (2. 8) in (1. 16) and (1. 18), we have rsepectively

$$\begin{aligned} &L(BX, BY)BZ \\ &=B\{R(X, Y)Z+\alpha^2(\langle X, Z\rangle Y-\langle Y, Z\rangle X)-(h'(X, Z)k'(Y)-h'(Y, Z)k'(X))\} \\ &\quad+\{\langle X, Z\rangle d\alpha(Y)-\langle Y, Z\rangle d\alpha(X)+\theta(X)h'(Y, Z)-\theta(Y)h'(X, Z)\}CP \\ &\quad+\{(\nabla_X h')(Y, Z)-(\nabla_Y h')(X, Z)-\alpha(\langle X, Z\rangle\theta(Y)-\langle Y, Z\rangle\theta(X))\}CQ \end{aligned} \tag{2. 10}$$

and

$$\begin{aligned} &L(BX, BY)CP \\ &=-B\{d\alpha(X)Y-d\alpha(Y)X+\theta(X)k'(Y)-\theta(Y)k'(Y)\}+\{d\theta(X, Y)\}CQ, \end{aligned} \tag{2. 11}$$

$d\alpha$ being the exterior differential of the function α. These are the *structure equations* of the pseudo-umbilical submanifold M^n of codimension 2.

In general, when there is given a submanifold M^n immersed in a Riemannian manifold M^m with immersion x: $M^n\to M^m$ $(n<m)$, for any two vector fields $\bar{X}$ and $\bar{Y}$ tangent to $x(M^n)$, the tensor field $L(\bar{X}, \bar{Y})$, L being the curvature tensor of the ambient manifold M^m, defines a linear endomorphism of the tangent space of $x(M^m)$ at each point p of $x(M^n)$. This linear endomorphism $L(\bar{X}, \bar{Y})$ is called the *curvature transformation* of the submanifold M^n determined by $\bar{X}$ and $\bar{Y}$ at p.

We here assume that, for our submanifold M^n of codimension 2, all curvature transformations of M^n preserve the tangent space $T_p(x(M^n))$ at each point p of $x(M^n)$. Then we have from (2. 10) the following equations:

$$\langle X, Z\rangle d\alpha(Y)-\langle Y, Z\rangle d\alpha(X)+\theta(X)h'(Y, Z)-\theta(Y)h'(X, Z)=0, \tag{2. 12}$$

$$(\nabla_X h')(Y, Z)-(\nabla_Y h')(X, Z)-\alpha\{\langle X, Z\rangle\theta(Y)-\langle Y, Z\rangle\theta(X)\}=0. \tag{2. 13}$$

Suppose that $\theta \neq 0$ at p. If we suppose that $d\alpha=0$ at a point p of M^n, we have, from (2.12), $\theta(X)h'(Y, Z)-\theta(Y)h'(X, Z)=0$, which implies $h'(X, Y)=\mu\theta(X)\theta(Y)$ at p, μ being a certain number. Thus, taking account of (2.9), we have at p

$$h'=0. \tag{2.14}$$

Conversely, suppose that $h'=0$ holds at p. Substituting (2.14) in (2.12), we have $\langle X, Z\rangle d\alpha(Y)-\langle Y, Z\rangle d\alpha(X)=0$, which implies that the equation

$$d\alpha=0 \tag{2.15}$$

holds at p. Thus we have

LEMMA 2.2. *Let M^n be a pseudo-umbilical submanifold of codimension* 2 *immersed in a Riemannian manifold M^{n+2}. Assume that all curvature transformations of M^n preserve the tangent space $T_q(x(M^n))$ at each point q of $x(M^n)$* and *$\theta \neq 0$ at p. Then, $d\alpha=0$ at a point p of M^n if and only if $h'=0$ at p.*

Let our submanifold M^n of codimension 2 satisfy the assumption stated in Lemma 2.2. We first assume that the mean curvature α is constant in M^n, i.e., that $d\alpha=0$ holds identically in M^n. Then by means of Lemma 2.2 we have identically $h'=0$ or $\theta=0$. Substituting $h'=0$ in (2.13), we obtain $\langle X, Z\rangle\theta(Y)-\langle Y, Z\rangle\theta(X)=0$, from which $\theta=0$. We next assume that $\theta=0$ and substitute this equation in (2.12). Then we obtain $d\alpha=0$. Summing up, we have, by means of Lemma 2.1,

LEMMA 2.3. *Let M^n be a pseudo-umbilical submanifold of codimension* 2 *immersed in a Riemannian manifold M^{n+2}. Assume that all curvature transformations of M^n preserve the tangent space $T_q(x(M^n))$ of $x(M^n)$ at each point q of $x(M^n)$. In this case, the following three conditions* (a), (b) *and* (c) *are equivalent to each other*:

(a) *M^n is umbilical, i.e., $h'=0$ holds identically,*

(b) *the mean curvature α is constant, i.e., $d\alpha=0$ holds identically,*

(c) *$\tilde{\nabla}_{BX}CP$ is tangent to $x(M^n)$, P being defined by $P=A/|A|$, where A is the mean curvature vector, i.e., $\theta=0$ holds identically.*

For any submanifold M^n immersed in a space of constant curvature, all of its curvature transformations preserve the tangent space of $x(M^n)$ at each point of $x(M^n)$. Thus we have

PROPOSITION 2.1. *For any pseudo-umbilical submanifold of codimension* 2 *immersed in a space of constant curvature, the three conditions* (a), (b) *and* (c) *stated in Lemma* 2.3 *are equivalent to each other.*

As a consequence of Proposition 2.1 and Lemma 5.2, which will be proved in §5, we have

PROPOSITION 2.2. *Let M^n be a complete pseudo-umbilical submanifold of codimension* 2 *immersed in an* $(n+2)$*-dimensional Euclidean space E^{n+2}. If M^n*

satisfies one of the three conditions (a), (b) *and* (c) *stated in Lemma* 2. 3, *then* M^n *is necessarily an n-dimensional natural sphere* S^n *in* E^{n+2}.

In Proposition 2. 2, we mean by an n-dimensional *natural sphere* S^n in an m-dimensional Euclidean space E^m an n-dimensional sphere S^n lying naturally on an $(n+1)$-dimensional plane E^{n+1} imbedded in E^m $(n<m)$.

By a similar device, we can prove the following proposition by means of Lemma 5. 1, which will be proved in §5.

PROPOSITION 2. 3. *Let* M^n *be a complete pseudo-umbilical submanifold of codimension* 2 *immersed in an* $(n+2)$*-dimensional sphere* S^{n+2} $(\subset E^{n+3})$. *If* M^n *satisfies one of the three conditions* (a), (b) *and* (c) *stated in Lemma* 2. 3, *then* M^n *is the intersection of* S^{n+2} *and a plane* E^{n+1} *of codimension* 2, *which does not pass the origin of* S^{n+2}.

§3. Pseudo-umbilical submanifolds of codimension 2 in a space of constant curvature.

Let M^n be a pseudo-umbilical submanifold of codimension 2 immersed in an $(n+2)$-dimensional space M^{n+2} of constant curvature c. The curvature tensor L of M^{n+2} has, by definition, the form

$$L(\tilde{X}, \tilde{Y})\tilde{Z}=c\{(\tilde{Y}, \tilde{Z})\tilde{X}-(\tilde{X}, \tilde{Z})\tilde{Y}\}$$

for $\tilde{X}, \tilde{Y}, \tilde{Z}\in \mathcal{T}_0^1(M^{n+2})$, from which we have

$$L(BX, BY)BZ=cB\{\langle Y, Z\rangle X-\langle X, Z\rangle Y\}, \qquad L(BX, BY)CP=0$$

for $X, Y, Z\in \mathcal{T}_0^1(M^n)$. Substituting these in (2. 10) and (2. 11), we have the equations (2. 12), (2. 13) and

$$R(X, Y)Z=(\alpha^2+c)\{\langle Y, Z\rangle X-\langle X, Z\rangle Y\}+\{h'(Y, Z)k'(X)-h'(X, Z)k'(Y)\}, \tag{3. 1}$$

$$d\theta=0. \tag{3. 2}$$

Taking the trace in (2. 12) with respect to Y and Z, we have

$$(n-1)d\alpha(X)+h'(l, X)=0, \tag{3. 3}$$

l being an element of $\mathcal{T}_0^1(M^n)$ such that

$$\theta(X)=\langle l, X\rangle. \tag{3. 4}$$

Substituting $Z=l$ in (2. 12) and using (3. 3), we have

$$(n-2)(\theta(X)d\alpha(Y)-\theta(Y)d\alpha(X))=0,$$

from which, if $n\geqq 3$,

$$\theta(X)d\alpha(Y)-\theta(Y)d\alpha(X)=0. \tag{3. 5}$$

We now assume that $d\alpha\neq 0$ holds everywhere in M^n and $n\geqq 3$. Then (3. 5) implies

$$\theta=\gamma d\alpha, \tag{3.6}$$

γ being a certain function in M^n, where $\gamma\neq 0$ holds everywhere in M^n because of Lemma 2.3. Thus we have equivalently

$$d\alpha=\beta\theta, \qquad \beta=\frac{1}{\gamma}. \tag{3.6$'$}$$

Substituting (3.6)$'$ in (2.12), we obtain

$$\theta(Y)\{\beta\langle X, Z\rangle-h'(X, Z)\}-\theta(X)\{\beta\langle Y, Z\rangle-h'(Y, Z)\}=0,$$

from which $h'(X, Z)-\beta\langle X, Z\rangle=\lambda\theta(X)\theta(Z)$, λ being a certain function in M^n, because of $\theta=\gamma d\alpha\neq 0$. Therefore, taking account of Tr $h'=0$, we have $\lambda=-n\beta/|\theta|^2$ and hence

$$h'(X, Z)=\beta\{\langle X, Z\rangle-n\varphi(X)\varphi(Z)\}, \tag{3.7}$$

where $\varphi=\theta/|\theta|$ and $|\theta|$ denotes the length of θ. Substituting (3.7) in (3.1), we obtain

$$\begin{aligned} R(X, Y)Z=&(\alpha^2+\beta^2+c)\{\langle Y, Z\rangle X-\langle X, Z\rangle Y\} \\ &-n\beta^2\{(\varphi(Y)X-\varphi(X)Y)\varphi(Z)+(\varphi(X)\langle Y, Z\rangle-\varphi(Y)\langle X, Z\rangle)e\}, \end{aligned} \tag{3.8}$$

e being defined by $e=l/|l|$. We easily see from (3.8) that the Ricci tensor S and the curvature scalar r of M^n have respectively the following forms:

$$S(X, Y)=\{(n-1)(\alpha^2+c)-\beta^2\}\langle X, Y\rangle-n(n-2)\beta^2\varphi(X)\varphi(Y), \tag{3.9}$$

$$r=n(n-1)(\alpha^2+c-\beta^2). \tag{3.10}$$

Denoting by $\sigma_p(\xi, \eta)$ the sectional curvature of M^n corresponding to two vectors ξ and η tangent to M^n at a point p of M^n, we have by means of (3.8)

$$\sigma_p(\xi, \eta)=(\alpha^2+\beta^2+c)+n\beta^2(\langle e, \xi\rangle^2+\langle e, \eta\rangle^2), \tag{3.11}$$

if $|\xi|=|\eta|=1$ and $\langle\xi, \eta\rangle=0$. Thus the formulas (3.8)$\sim$(3.11) hold, provided $n\geqq 3$, when $d\alpha\neq 0$ holds everywhere in M^n.

On the other hand, if we assume that $d\alpha=0$ at a point q, we have $h'=0$ at q because of Lemma 2.2. Thus, substituting $h'=0$ in (3.1), we have $R(X, Y)Z=\alpha^2\{\langle Y, Z\rangle X-\langle X, Z\rangle Y\}$, from which

$$\sigma_q(\xi, \eta)=\alpha^2+c \tag{3.11$'$}$$

if $d\alpha=0$ at q. Therefore, taking account of (3.11) and (3.11)$'$, we have

PROPOSITION 3.1. *Let M^n be a complete pseudo-umbilical submanifold of codimension 2 immersed in a space M^{n+2} of constant curvature c and denote by α the mean curvature. If there exists a positive constant δ such that $\alpha^2+c>\delta^2>0$, and, if $n\geqq 3$, then M^n is necessarily compact.*

As a corollary to Proposition 3.1, we have

PROPOSITION 3.2. *Any complete pseudo-umbilical submanifold M^n of codimension 2 immersed in a space M^{n+2} of positive constant curvature is necessarily* *compact, if* $n \geqq 3$.

We are now going to obtain the conformal curvature tensor $\mathfrak{C}$ of a pseudo-umbilical submanifold of codimension 2 immersed in a space of constant curvature c. We first assume that $d\alpha \neq 0$ and $n \geqq 3$. Defining an element $\mathfrak{D}$ of $\mathcal{T}_2^0(M^n)$ by the equation

$$\mathfrak{D}(X, Y) = -\frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} \langle X, Y \rangle$$

and substituting (3. 9) and (3. 10) in this, we obtain

$$\mathfrak{D}(X, Y) = -\frac{1}{2}(\alpha^2 + \beta^2 + c)\langle X, Y\rangle + n\beta^2 \varphi(X)\varphi(Y). \tag{3. 12}$$

The conformal curvature tensor of M^n is, by definition, an element $\mathfrak{C}$ of $\mathcal{T}_3^1(M^n)$ given by

$$\mathfrak{C}(X, Y)Z = R(X, Y)Z + \mathfrak{D}(Y, Z)X - \mathfrak{D}(X, Z)Y + \mathfrak{E}(X)\langle Y, Z\rangle - \mathfrak{E}(Y)\langle X, Z\rangle, \tag{3. 13}$$

$\mathfrak{E}$ being an element of $\mathcal{T}_1^1(M^n)$ defined by $\langle \mathfrak{E}(X), Y\rangle = \mathfrak{D}(X, Y)$. If we substitute (3. 8) and (3. 12) in (3. 13), we have

$$\mathfrak{C}(X, Y)Z = 0,$$

i.e., $\mathfrak{C} = 0$. That is to say, M^n is conformally flat, provided $n \geqq 4$, if $d\alpha \neq 0$. However, as was mentioned above, the formulas (3. 8) and (3. 12) with $\beta = 0$ hold at any point q where $d\alpha = 0$. Thus we have $\mathfrak{C} = 0$ at such a point q, if $n \geqq 4$. Therefore we have

PROPOSITION 3. 3. *Any n-dimensional pseudo-umbilical submanifold* M^n *of codimension* 2 *immersed in a space* M^{n+2} *of constant curvature is conformally flat, if* $n \geqq 4$.

We shall now study more in detail properties of pseudo-umbilical submanifold M^n of codimension 2 in a space of constant curvature. We assume that the mean curvature α satisfies the condition $d\alpha \neq 0$ everywhere in M^n. Substituting (3. 6) in (3. 2), we have $d\gamma \wedge d\alpha = 0$, which means that β (or equivalently γ) is a function $\beta(\alpha)$ depending only on α. If we substitute (3. 7) in (2. 13) and take account of (3. 6)′, we obtain

$$\begin{aligned} &\beta\beta'\theta(X)\{\langle Y, Z\rangle - n\varphi(Y)\varphi(Z)\} - n\beta\{(\nabla_X\varphi)(Y)\varphi(Z) + (\nabla_X\varphi)(Z)\varphi(Y)\} \\ &-\beta\beta'\theta(Y)\{\langle X, Z\rangle - n\varphi(X)\varphi(Z)\} + n\beta\{(\nabla_Y\varphi)(X)\varphi(Z) + (\nabla_Y\varphi)(Z)\varphi(X)\} \\ &\qquad -\alpha\{\langle X, Z\rangle\theta(Y) - \langle Y, Z\rangle\theta(X)\} = 0, \end{aligned} \tag{3. 14}$$

where $\beta' = d\beta/d\alpha$. Putting $Z = e$ in (3. 14) and taking account of $\varphi(X) = \langle X, e\rangle$ and $\varphi = \theta/|\theta|$, we have

$$(\nabla_X\varphi)(Y) - (\nabla_Y\varphi)(X) = 0 \tag{3. 15}$$

because of $\varphi(e) = 1$ and $(\nabla_Y\varphi)(e) = 0$ which is a direct consequence of $\varphi(e) = 1$ and

$\varphi(X)=\langle X, e\rangle$. Therefore, if we put $Y=e$ in (3. 15), we obtain

$$\nabla_e\varphi=0, \quad \text{or equivalently,} \quad \nabla_e e=0, \tag{3. 16}$$

which shows that the unit vector field e generates geodesics. Next, substituting $X=e$ in (3. 14) and taking account of (3. 6), we have

$$\nabla_Y e=-\mu\{Y-\varphi(Y)e\}, \tag{3. 17}$$

$$\mu=\frac{(\beta\beta'+\alpha)|\theta|}{n\beta}, \qquad \beta'=\frac{d\beta}{d\alpha}. \tag{3. 17$'$}$$

That is to say, the unit vector field e is torse-forming. Summing up, we have

LEMMA 3. 1. *Let M^n be a pseudo-umbilical submanifold of codimension 2 in a space of constant curvature. If $d\alpha\neq 0$ holds everywhere in M^n, α being the mean curvature, then the unit vector field e, which is proportional to the gradient vector of α, generates geodesics and is torse-forming.*

§4. Pseudo-umbilical submanifolds of codimension 2 of a Euclidean space.

We are going to study in detail properties of pseudo-umbilical submanifolds of codimension 2 immersed in a Euclidean space. We first have the following Proposition 4. 1, as a corollary to Proposition 3. 1.

PROPOSITION 4. 1. *Let M^n be a complete pseudo-umbilical submanifold of codimension 2 immersed in a Euclidean space E^{n+2}. If there exists a positive number δ such that $\alpha>\delta>0$, α being the mean curvature, then M^n is necessarily compact.*

Let M^n be a pseudo-umbilical submanifold of codimension 2 immersed in a Euclidean space E^{n+2} and assume that $d\alpha\neq 0$ holds everywhere in M^n, α being the mean curvature. For a certain constant c, a connected component of a submanifold defined in M^n by the equation $\alpha=c$ is denoted by M_c^{n-1}, which is $(n-1)$-dimensional because of $d\alpha\neq 0$. Denoting by $\bar{x}$: $M_c^{n-1}\to M^n$ the immersion of M_c^{n-1} into M^n and by $\hat{x}$: $M_c^{n-1}\to E^{n+2}$ the immersion of M_c^{n-1} into E^{n+2}, we have $\hat{x}=x\bar{x}$ where x: $M^n\to E^{n+2}$ is the immersion of M^n into E^{n+2}. We denote the tangential mappings of the immersions x, $\bar{x}$ and $\hat{x}$ by B, $\bar{B}$ and $\hat{B}$ respectively, where we have $\hat{B}=B\bar{B}$. The normal mappings of x, $\bar{x}$ and $\hat{x}$ are respectively denoted by C, $\bar{C}$ and $\hat{C}$. The second fundamental tensors of the immersions x, $\bar{x}$ and $\hat{x}$ are respectively denoted by H, $\bar{H}$ and $\hat{H}$. We denote by $\langle\!\langle\ ,\ \rangle\!\rangle$ and $\langle\!\langle\ ,\ \rangle\!\rangle^*$ respectively the inner products induced in $T(M_c^{n-1})$ and in $N(M_c^{n-1})$, $N(M_c^{n-1})$ being the normal bundle over M_c^{n-1} with respect to the immersion $\bar{x}$: $M_c^{n-1}\to M^n$. Taking account of (2. 8) and (3. 7), we have, by means of $\varphi(\bar{B}U)=\varphi(\bar{B}W)=0$,

$$h(\bar{B}U, \bar{B}W)=\alpha\langle\bar{B}U, \bar{B}W\rangle=\alpha\langle\!\langle U, W\rangle\!\rangle,$$

$$h'(\bar{B}U, \bar{B}W)=\beta\langle\bar{B}U, \bar{B}W\rangle=\beta\langle\!\langle U, W\rangle\!\rangle$$

for $U, W \in \mathcal{T}_0^1(M_c^{n-1})$, which imply together with (2. 1)

$$H(\bar{B}U, \bar{B}W) = \alpha\langle\langle U, W\rangle\rangle P + \beta\langle\langle U, W\rangle\rangle Q \tag{4. 1}$$

for $U, W \in \mathcal{T}_0^1(M_c^{n-1})$. Substituting $Y = \bar{B}U$ in (3. 17), we obtain

$$\nabla_{\bar{B}U} e = -\mu \bar{B}U \tag{4. 2}$$

because of $\varphi(\bar{B}U) = 0$. Denoting by $\bar{N}$ the element of $\mathcal{N}_0^1(M_c^{n-1}, \bar{x})$, which is the normal bundle of the immersion $\bar{x}$: $M_c^{n-1} \to M^n$, such that $\bar{C}\bar{N} = e$ along M_c^{n-1}, we have from (4. 2)

$$\bar{H}(U, W) = \mu\langle\langle U, W\rangle\rangle \bar{N} \tag{4. 3}$$

for $U, W \in \mathcal{T}_0^1(M_c^{n-1})$.

If we substitute (4. 1) and (4. 3) in (1. 24), we obtain

$$\hat{C}\hat{H}(U, W) = \alpha\langle\langle U, W\rangle\rangle CP + \beta\langle\langle U, W\rangle\rangle CQ + \mu\langle\langle U, W\rangle\rangle B\bar{C}\bar{N} \tag{4. 4}$$

for $U, W \in \mathcal{T}_0^1(M_c^{n-1})$, which shows that the immersion $\hat{x}$: $M_c^{n-1} \to E^{n+2}$ is umbilical.

Taking account of (1. 11), (2. 4) and $\theta(\bar{B}U) = 0$, we have

$$\tilde{\nabla}_{\hat{B}U} CP = \tilde{\nabla}_{B\bar{B}U} CP = -BK(\bar{B}U, P) = -\alpha B\bar{B}U = -\alpha \hat{B}U, \text{ i.e.,}$$

$$\tilde{\nabla}_{\hat{B}U} CP = -\alpha \hat{B}U \tag{4. 5}$$

for $U \in \mathcal{T}_0^1(M_c^{n-1})$. Similarly we have

$$\tilde{\nabla}_{\hat{B}U} CQ = -\beta \hat{B}U \tag{4. 6}$$

for $U \in \mathcal{T}_0^1(M_c^{n-1})$. Putting $Y = \bar{B}U$ and $e = \bar{C}\bar{N}$ in (3. 17) and taking account of $\varphi(\bar{B}U) = 0$, we obtain $\nabla_{\bar{B}U} \bar{C}\bar{N} = -\mu \bar{B}U$ for $U \in \mathcal{T}_0^1(M_c^{n-1})$, which implies together with (1. 8)

$$\tilde{\nabla}_{\hat{B}U} B\bar{C}N = -\mu \hat{B}U + CH(\bar{B}U, \bar{C}\bar{N}).$$

However, since we have $H(\bar{B}U, \bar{C}\bar{N}) = 0$ because of (2. 8) and (3. 7), we have

$$\tilde{\nabla}_{\hat{B}U} B\bar{C}\bar{N} = -\mu \hat{B}U \tag{4. 7}$$

for $U \in \mathcal{T}_0^1(M_c^{n-1})$. We now have the following identity

$$\tilde{\nabla}_{\hat{B}W} \tilde{\nabla}_{\hat{B}U} B\bar{C}N - \tilde{\nabla}_{\hat{B}U} \tilde{\nabla}_{\hat{B}W} B\bar{C}N - \tilde{\nabla}_{[\hat{B}W, \hat{B}U]} B\bar{C}N = 0$$

for $U, W \in \mathcal{T}_0^1(M_c^{n-1})$, because the enveloping manifold is Euclidean. Substituting (4. 7) in the identity above and taking account of $[\hat{B}W, \hat{B}U] = \hat{B}[W, U]$, we obtain $d\mu(W)U - d\mu(U)W = 0$, from which $d\mu = 0$, because U and W are arbitrary. On the other hand, α is constant along M_c^{n-1} and hence so are β, β'. Therefore, taking account of (3. 6)′ and (3. 17)′, we see that the length $|d\alpha|$ of $d\alpha$ is constant along M_c^{n-1}.

According to (4. 4), the mean curvature vector $\hat{A}$ of the immersion $\hat{x}$: $M_c^{n-1} \to E^{n+2}$ has the form

(4. 8) $$\hat{C}\hat{A}=\alpha CP+\beta CQ+\mu B\bar{C}\bar{N}.$$

α, β and μ being constant along M_c^{n-1}, we have, from (4. 5), (4. 6), (4. 7) and (4. 8),

(4. 9) $$\tilde{\nabla}_{\hat{B}U}\hat{C}\hat{P}=-\nu\hat{B}U$$

for $U\in\mathfrak{T}_0^1(M_c^{n-1})$, where we have put

(4. 10) $$\hat{P}=\frac{\hat{A}}{|\hat{A}|},\qquad \nu=\frac{\alpha^2+\beta^2}{\sqrt{\alpha^2+\beta^2+\mu^2}}>0.$$

Thus, taking account of (4. 9) and Lemma 5. 1 which will be proved in §5, we have

LEMMA 4. 1. *Let M^n be a pseudo-umbilical submanifold of codimension 2 immersed in E^{n+2}. Assume that $d\alpha\neq 0$ holds everywhere in M^n, α being the mean curvature. If a connected component M_c^{n-1} of a submanifold defined by $\alpha=c$, c being a certain constant, is complete, and, if $n\geqq 3$, then M_c^{n-1} is an $(n-1)$-dimensional natural sphere S^{n-1} with radius $1/\nu$ in E^{n+2}* (see (4. 10)). *The length $|d\alpha|$ of $d\alpha$ (or equivalently $|\theta|$) is constant along each M_c^{n-1}.*

Let $\mathfrak{g}$ be the family of orthogonal trajectories of M_c^{n-1}'s. Then, by virtue of Lemma 3. 1, each element of $\mathfrak{g}$ is a geodesic. On the other hand, according to Lemma 4. 1, $|d\alpha|$ is constant along each M_c^{n-1}. Thus, taking certain consecutive numbers c and c', we see that M_c^{n-1} and $M_{c'}^{n-1}$ cut off a geodesic-arc of the same length from each of geodesics belonging to $\mathfrak{g}$. Therefore, combining Lemmas 3. 1 and 4. 1, we have

PROPOSITION 4. 2. *Let M^n be a pseudo-umbilical submanifold of codimension 2 immersed in an Euclidean space E^{n+2}. Assume that $d\alpha\neq 0$ holds everywhere in M^n, α being the mean curvature. If each of connected components M_c^{n-1} of submanifold defined by $\alpha=c$, c being constant, is complete, and, if $n\geqq 3$, then M_c^{n-1} is an $(n-1)$-dimensional natural sphere S_c^{n-1} in E^{n+2}, M^n is generated by a family $\mathfrak{F}$ of such spheres S_c^{n-1} $(=M_c^{n-1})$ and the orthogonal trajectories of $\mathfrak{F}$ are geodesics, whose unit tangent vectors e form a torse-forming vector field. Any two consecutive S_c^{n-1} and $S_{c'}^{n-1}$ cut off a geodesic-arc of the same length from each of orthogonal trajectories of $\mathfrak{F}$.*

Let M^n be a pseudo-umbilical submanifold of codimension 2 in E^{n+2}. We assume that $d\alpha\neq 0$ holds in a coordinate neighborhood U of M^n, α being the mean curvature. Then we can choose in U a system of local coordinates $(x^1, x^2, \cdots, x^n)$ in such a way that the equation $x^1=$const. represents in U an $(n-1)$-dimensional submanifold M_c^{n-1} defined by $\alpha=c$, the variable x^1 indicates the arc length along any geodesic, which is an orthogonal trajectory of the family $\mathfrak{F}$ of the submanifolds M_c^{n-1}'s and the equations $x^a=$const. $(a=2, \cdots, n)$ respresent in U an orthogonal trajectory of the family $\mathfrak{F}$ of M_c^{n-1}. If we now follow classical notations, we see, from the proof of Lemma 4. 1, that the line element ds^2 of the submanifold M^n

has in U the following form:

$$ds^2=(dx^1)^2+\rho(x^1)^2 d\sigma^2 \qquad (\rho(x^1)>0) \tag{4.11}$$

with respect to such local coordinates $(x^1, x^2, \cdots, x^n)$,

$$d\sigma^2=\sum_{a,b=2}^{n}\gamma_{ba}(x^2, \cdots, x^n)dx^b dx^a$$

denoting the line element of an $(n-1)$-dimensional space of constant curvature 1. Moreover, the mean curvature α is in U a function depending only on the variable x^1 (Cf. Lemma 4.1).

According to (4.11), U is conformal to the Pythagrean product $R\times V^{n-1}$, where R and V^{n-1} denote respectively a line segment and an $(n-1)$-dimensional Riemannian space of constant curvature 1. Consequently, U is conformally flat. Thus we have

LEMMA 4.2. *Let M^n be a pseudo-umbilical submanifold of codimension 2 immersed in E^{n+2}. Assume that $d\alpha\neq 0$ holds everywhere in M^n. Then, if $n\geqq 3$, M^n is conformally flat.*

We consider a 3-dimensional pseudo-umbilical submanifold M^3 of codimension 2 in E^5. Denoting by $'M^3$ the set of all points at which $d\alpha\neq 0$, where α is the mean curvature, and taking account of Lemma 4.2, we see that $'M^3$ is conformally flat. Thus the element $'\mathfrak{C}$ of $\mathcal{T}_3^0(M^3)$ defined by

$$\begin{aligned}'\mathfrak{C}(X, Y, Z)=&\frac{1}{n-2}\{(\nabla_X S)\langle Y, Z\rangle-(\nabla_Y S)\langle X, Z\rangle\}\\ &-\frac{1}{2(n-1)(n-2)}\{\langle Y, Z\rangle\nabla_X r-\langle X, Z\rangle\nabla_Y r\},\end{aligned} \tag{4.12}$$

S and r being respectively the Ricci tensor and the curvature scalar of M^3, vanishes identically in $'M^3$. Putting $''M^3=M^3-'M^3$, we see by means of Proposition 2.2 that each connected component of the open kernel of $''M^3$ is a piece of a 3-dimensional natural sphere S^3 in E^5, and hence, as is well known, that the tensor $'\mathfrak{C}$ defined by (4.12) vanishes in the open kernel of $''M^3$. Therefore, taking account of the continuity of $'\mathfrak{C}$, we see that $'\mathfrak{C}$ vanishes identically in M^3. That is to say, M^3 should be conformally flat. Thus, taking account of Proposition 3.3, we have

PROPOSITION 4.3. *Any n-dimensional pseudo-umbilical submanifold of codimension 2 immersed in a Euclidean space E^{n+2} is conformally flat if $n\geqq 3$.*

We can prove Proposition 4.3 only by using Lemma 4.2. By a similar device as that used in the proof of Proposition 4.3, we can prove

PROPOSITION 4.4. *Any n-dimensional pseudo-umbilical submanifold of codimension 2 immersed in an $(n+2)$-dimensional spere S^{n+2} is conformally flat, if $n\geqq 3$.*

§5. Umbilical submanifolds immersed in a Euclidean space.

For the completeness, we shall prove the following

LEMMA 5.1. *Let M^n be an n-dimensional, complete, umbilical submanifold with non-zero mean curvature α, immersed in an m-dimensional Euclidean space E^m $(n<m)$. If the unit vector field P in $N(M^n)$, such that $A=\alpha P$ is the mean curvature vector, is parallel in $N(M^n)$, i.e., if $\nabla_X^* P=0$ for $X\in \mathcal{T}_0^1(M^n)$, and, if $n\geqq 2$, then the mean curvature α is necessarily constant and M^n is an n-dimensional natural sphere S^n in E^m.*

In Lemma 5.1, we mean by an n-dimensional *natural sphere* S^n in E^m a sphere lying naturally in an $(n+1)$-dimensional plane E^{n+1} of E^m.

Proof. Putting $L=0$ in (1.16), we have

$$(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)=0. \tag{5.1}$$

Since M^n is umbilical, we have

$$H(X, Y)=\alpha\langle X, Y\rangle P. \tag{5.2}$$

Substituting (5.2) in (5.1) and taking account of $\nabla_X^* P=0$, we have $\nabla_X \alpha=0$, from which we see that α is constant.

Denoting by x: $M^n\to E^m$ the immersion of M^n, we can express the position vector indicating the point $x(p)$, $p\in M^n$, also by $x(p)$ and the correspondence $p\to x(p)$ can be regarded as a differentiable function denoted by x, which takes vectors in E^m as its values.

Taking account of (5.2) and $\nabla_X^* P=0$, we have from (1.11)

$$\tilde{\nabla}_{BX} N_1=-\alpha\tilde{\nabla}_{BX} x \tag{5.3}$$

for $X\in \mathcal{T}_0^1(M^n)$, N_1 being defined by $N_1=CP$, because we have $BX=\tilde{\nabla}_{BX} x$ for any $X\in \mathcal{T}_0^1(M^n)$. This reduces to

$$\tilde{\nabla}_{BX}\left(x+\frac{1}{\alpha} N_1\right)=0,$$

because α is a non-zero constant. Thus the point $p_0=x+(1/\alpha)N_1$ is fixed. Therefore $x(M^n)$ lies on a hypersphere S^{m-1} with center p_0 and with radius $1/\alpha$.

Taking an element Q of $\mathcal{N}_0^1(M^n)$ such that $\langle Q, P\rangle^*=0$, we have $K(X, Q)=0$ because of (5.2). Thus we have from (1.11)

$$\tilde{\nabla}_{BX} N_2=C(\nabla_X Q)$$

for $X\in \mathcal{T}_0^1(M^n)$, N_2 being defined by $N_2=CQ$. Differentiating $(N_1, N_2)=0$ along $x(M)$ and taking account of (5.3), we find

$$(\tilde{\nabla}_{BX} N_2, N_1)=0.$$

Therefore, denoting by D_x the set of all vectors N at a point x of $x(M^n)$ such that N is normal to $x(M^n)$ and orthogonal to N_1, we see that $\{D_x | x \in x(M^n)\}$ forms a 1-dimensional distribution D which is parallel in E^m. Thus there exists a unique $(n+1)$-dimensional plane E^{n+1}, which is orthogonal to all of D_x and passing through the point $p_0 = x + (1/\alpha)N_1$. Since $N_1 = \alpha\overrightarrow{xp_0}$ is orthogonal to D_x at each point x of $x(M^n)$, each point x should belongto E^{n+1}. Consequently, $x(M^n)$ lies on E^{n+1}.

Summing up, $x(M^n)$ is contained in the natural sphere $S^n = S^{m-1} \cap E^{n+1}$. Therefore $x(M^n)$ coincides with S^n, because $x(M^n)$ is complete. The radius of S^n is obviously equal to $1/\alpha$. Hence we have proved Lemma 5. 1.

Combining Lemmas 2. 1, 2. 3 and 5. 1, we have

LEMMA 5. 2. *Let M^n be a complete umbilical submanifold of codimension* 2, *with non-zero mean curvature α, immersed in an $(n+2)$-dimensional Euclidean space E^{n+2}. Then the mean curvature α is necessarily constant and M^n is an n-dimensional natural sphere S^n in E^{n+2}.*

When M^n is a hypersurface in E^{n+1}, its normal bundle $N(M^n)$ is a 1-dimensional vector bundle. Then any unit vector field P in $N(M^n)$ satisfies the condition $\nabla^*_X P = 0$ for $X \in \mathcal{T}^1_0(M^n)$. Thus, taking account of Lemma 5. 1, we have the following well known

LEMMA 5. 3. *Let M^n be a complete umbilical submanifold of codimension* 1, *with non-zero mean curvature α, immersed in an $(n+1)$-dimensional Euclidean space E^{n+1}. Then the mean curvature α is necessarily constant and M^n is an n-dimensional natural sphere S^n in E^{n+1}.*

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