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A FIXED POINT THEOREM FOR NONEXPANSIVE MAPPINGS IN METRIC SPACE

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Let X be a metric space with a metric d. A mapping U of X into X is said to be *nonexpansive* if for each pair x, y of elements in X, $d(Ux, Uy) \leqq d(x, y)$.

Recently several fixed point theorems for nonexpansive mappings in Banach space have been derived by Belluce and Kirk [1], [2], Browder [3], de Marr [4] and Kirk [6].

In this paper we shall prove a fixed point theorem for nonexpasive mappings in metric space under certain conditions.

1. Notations and definitions.

Let X be a metric space with metric d. For a subset A of X, the diameter of A is denoted by $\delta(A)$, that is,

$$\delta(A) = \sup\{d(x, y) : x, y \in A\},$$

and for a point $p \in A$, we define

$$\rho(p, A) = \sup\{d(p, x) : x \in A\}.$$

A point $p \in A$ is called *a nondiametral point* of A if $\rho(p, A) < \delta(A)$. A subset of X is said to be *admissible* (cf. Dunford and Schwartz [5] p. 459) if it is an intersection of closed spheres.

Throughout this paper, $S(x, r)$ denotes the closed sphere of center x and radius r.

2. Fixed point theorem.

LEMMA. *If a bounded metric space X satisfies the following coditions* (1) *and* (2), *then every nonexpansive mapping U of X into X has a fixed point.*

(1) *If a family of closed spheres has finite intersection property, then the intersection of the family is nonempty.*

(2) *Each admissible subset which contains more than one point contains a nondiametral point.*

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Proof. Let Φ be the family of all nonempty admissible subsets invariant under U. By boundedness of X, $X\in\Phi$ and hence Φ is a nonempty family. Φ is considered to be partially ordered by usual set inclusion. Let $\{A_i: i\in I\}$ be a totally ordered subfamily of Φ. We show that the intersection A of all A_i's is an element of Φ. It is obvious that A is admissible and mapped into itself by U. Since every A_i is admissible, it can be written in the form

$$A_i=\cap\{S(x_j, r_j): j\in J_i\}$$

where J_i is an index set. We can assume that J_{i_1} and J_{i_2} are disjoint whenever i_1 and i_2 are distinct. Therefore

$$A=\cap\{S(x_j, r_j): j\in J\}$$

where $J=\cup\{J_i, i\in I\}$.

Now we consider the family $\{S(x_j, r_j): j\in J\}$ and take arbitrary finite elements $S(x_{j_1}, r_{j_1})$, $S(x_{j_2}, r_{j_2}), \cdots, S(x_{j_n}, r_{j_n})$ from it. Every $S(x_j, r_j)\ (j\in J)$ contain some $A_i\ (i\in I)$. Thus

$$\bigcap_{k=1}^{n} A_{i_k}\subset \bigcap_{k=1}^{n} S(x_{j_k}, r_{j_k}).$$

Since the family $\{A_i: i\in I\}$ is totally ordered, $\cap_{k=1}^{n}A_{i_k}$ is nonempty, and so is $\cap_{k=1}^{n}S(x_{j_k}, r_{j_k})$. This shows that the family $\{S(x_j, r_j): j\in J\}$ has finite intersection property and $A=\cap\{S(x_j, r_j): j\in J\}$ is nonempty because of the condition (1). It is evident that A is a lower bound of the family $\{A_i: i\in I\}$. Therefore by Zorn's lemma, Φ has a minimal element F.

We can put $F=\cap\{S(x_\gamma, r_\gamma): \gamma\in\Gamma\}$ and define

$$r=\inf\{\rho(y, F): y\in F\},$$

$$F_c=\{x\in F: \rho(x, F)=r\}.$$

We prove $F_c\in\Phi$ as follows. It is easy to see that

$$F_c=[\cap\{S(x_\gamma, r_\gamma): \gamma\in\Gamma\}]\cap[\cap\{S(x, r+1/n): x\in F, n=1, 2, \cdots\}].$$

This shows that F_c is admissible. We consider the family

$$\{S(x_\gamma, r_\gamma)\ (\gamma\in\Gamma), S(x, r+1/n)\ (x\in F, n=1, 2, \cdots)\}.$$

In order that it has finite intersection property, it is sufficient that $\cap_{k=1}^{m}S(x_k, r+1/n_k)$ contains a point of F for any $x_1, x_2, \cdots, x_m\in F$ and any positive integers $n_1, n_2, \cdots, n_m$. Let

$$n=\max\{n_1, n_2, \cdots, n_m\}.$$

By the definition of r, there exists some $x\in F$ such that

$$\rho(x, F) \leqq r+1/n.$$

Since $x_k \in F$ and $d(x, x_k) \leqq r+1/n \leqq r+1/n_k$ for $k=1, 2, \cdots, m$, then $x \in \bigcap_{k=1}^{m} S(x_k, r+1/n_k)$. Hence F_c is nonempty by the condition (1).

Next we show that F_c is mapped into itself by U. If $x \in F_c$, then by the property of U,

$$d(Ux, Uy) \leqq d(x, y) \leqq \rho(x, F)=r \quad \text{for all} \quad y \in F,$$

and hence $U(F) \subset S(Ux, r)$. Since $U(F) \subset F$, it is easy to see that

$$U(F \cap S(Ux, r)) \subset F \cap S(Ux, r).$$

Evidently $F \cap S(Ux, r)$ is a nonempty admissible subset. Thus

$$F \cap S(Ux, r) \in \Phi.$$

By the minimality of F, $F \subset S(Ux, r)$. This shows $\rho(Ux, F)=r$, and we conclude $Ux \in F_c$.

We have proved that $F_c \in \Phi$, and hence $F_c=F$ by minimality of F. We use the fact $F_c=F$ to show that F contains only one point. If F contains more than one point, then by the condition (2), F has a nondiamental point x_0, that is, $\rho(x_0, F) < \delta(F)$. Hence

$$\begin{aligned} \delta(F_c) &= \sup\{d(y, z): y, z \in F_c\} \leqq \sup\{\rho(y, F_c): y \in F_c\} \\ &\leqq \sup\{\rho(y, F): y \in F_c\} = r = \inf\{\rho(x, F): x \in F\} \\ &\leqq \rho(x_0, F) < \delta(F). \end{aligned}$$

This shows $\delta(F_c) < \delta(F)$, but it contradicts $F_c=F$.

We have seen that F contains only one point. It is evident that the point is fixed by U.

THEOREM. *Let X be a bounded metric space, and suppose that X satisfies the conditions* (1) *and* (2) *of Lemma.*

If $\mathcal{F}$ is a finite commuting family of nonexpansive mappings of X into X, then $\mathcal{F}$ has a common fixed point.

Proof. Let Φ be a family of all nonempty admissible subsets invariant under each $U \in \mathcal{F}$. By the same method in the proof of Lemma, we can find a minimal element F of Φ.

Let $\mathcal{F}=\{U_1, U_2, \cdots, U_n\}$ and $W=\{x \in F: U_1 U_2 \cdots U_n x=x\}$. We can apply Lemma to the nonexpansive mapping $U_1 U_2 \cdots U_n$ of F into F, and we get $W \neq \phi$.

It is shown that $U_i(W)=W$ for $i=1, 2, \cdots, n$. In fact, if $x \in W$, then $U_i x = U_i U_1 U_2 \cdots U_n x = U_1 U_2 \cdots U_n U_i x$. Conversely if $x \in W$, then $U_1 U_2 \cdots U_{i-1} U_{i+1} \cdots U_n x \in W$ and $x = U_1 U_2 \cdots U_n x = U_i U_1 U_2 \cdots U_{i-1} U_{i+1} \cdots U_n x$.

Let K be the least admissible set containing W. Since F is admissible, $K \subset F$.

If we assume that K contains more than one point, then, by the condition (2), K must contain a point x_0 such that

$$\rho(x_0, K)=r<\delta(K).$$

We put

$$C=F\cap[\cap\{S(z, r): z\in K\}].$$

Then C is a nonempty admissible subset of F. We note that C can be written in the following form

$$C=F\cap[\cap\{S(z, r): z\in W\}].$$

For, if $d(x, z)\leqq r$ for all $z\in W$, then $W\subset S(x, r)$ and since K is the least admissible set containing W, it follows $K\subset S(x, r)$, that is, $d(x, z)\leqq r$ for all $z\in K$. Thus for any $c\in C$ and $w\in W$,

$$d(U_i c, U_i w)\leqq d(c, w)\leqq r.$$

Since $U_i(W)=W$, we get $U_i(C)\subset C$.

We have showed that C is a nonempty admissible subset invariant under each U_ι. Therefore $C=F$ by the minimality of F. Thus

$$\delta(K)=\delta(F\cap K)=\delta(C\cap K)\leqq r<\delta(K),$$

but this is a contradiction.

We conclude that K contains only one point and this point is the desired fixed point.

COROLLARY. *If* X *is a compact metric space which satisfies the condition* (2) *of Lemma, then every commuting family of nonexpansive mappings of X into X has a common fixed point.*

Proof. X satisfies all the conditions of Theorem.

Let $\{U_\iota: i\in I\}$ be a commuting family of nonexpansive mappings. We define $F_\iota=\{x: U_i x=x\}$. It is easy to see that F_ι is a closed subset. We consider the family $\{F_\iota: i\in I\}$ of closed subsets. Its family has finite intersection property because of Theorem.

Since X is compact, there exists a point which is contained in all the F_i's. Its point is fixed by all the U_ι's.

NOTE. Let X be a subset of a Banach space. A mapping U of X into X is said to be *nonexpansive* if $||Ux-Uy||\leqq||x-y||$ for any $x, y\in X$. A convex subset X of a Banch space is called to have *normal structure* if each bounded convex subset of X which contains more than one point, has a nondiametral point.

If X is a bounded, weakly compact, convex subset of a Banach space and X has normal structure, than X satisfies all the conditions of Theorem.

In case X is a compact convex subset of a Banach space, we can apply Corollary to X.

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