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TENSOR FIELDS AND CONNECTIONS IN CROSS-SECTIONS IN THE TANGENT BUNDLE OF ORDER 2

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The prolongations of tensor fields and connections given in a differentiable manifold M to its tangent bundle $T(M)$ have been studied in [1], [2], [5], [7]. If a vector field V is given in M, V determines a cross-section in $T(M)$ which is as an n-dimensional submanifold in $T(M)$. Yano [3] has recently studied the behavior of the prolongations of tensor fields and connections to $T(M)$ on the cross-sections determined by a vector field in M. On the other hand, the prolongations of tensor fields and connections in M to its tangent bundle $T_2(M)$ of order 2 are studied in [6]. If a vector field V is given in M, V determines a cross-section in $T_2(M)$. The main purpose of the present paper is to study the behavior of the prolongations of tensor fields and connections in M to $T_2(M)$ on the cross-section determined by a vector field in M.

In § 1 we first recall properties of the prolongations of tensor fields and connections in M to $T_2(M)$. In § 2 we study the cross-sections determined in $T_2(M)$ by vector fields given in M. § 3 will be devoted to the study of the prolongations of tensor fields given in M to $T_2(M)$ along the cross-sections and § 4 will be devoted to the study of the prolongations of connections given in M to $T_2(M)$ along the cross-sections.

§ 1. Prolongations of tensor fields and linear connections to the tangent bundle of order 2.

We shall recall, for the later use, some properties of the tangent bundle $T_2(M)$ of order 2 over a differentiable manifold M of dimension n, and those of prolongations of tensor fields and linear connections in M to $T_2(M)$ (cf. [6]).

The tangent bundle $T_2(M)$ of order 2 is the space of equivalence classes of mappings from the real line R into M, the equivalence relation being defined as follows: we say that two mappings F and G are equivalent to each other if, in a coordinate neighborhood U, they satisfy the conditions

$$F(0)=G(0)=p, \qquad \frac{dF^h}{dt}(0)=\frac{dG^h}{dt}(0), \qquad \frac{d^2F^h}{dt^2}(0)=\frac{d^2G^h}{dt^2}(0),$$

where $F^h(t)$ and $G^h(t)$ are the coordinates of $F(t)$ and $G(t)$ in U respectively. This

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definition of the equivalence does not depend on the choice of the local coordinates. We call this equivalence class containing F a 2-jet and denote it by $j_p{}^2(F)$. Namely the tangent bundle of order 2 over M is the space of all 2-jets of M and its bundle projection π_2: $T_2(M)\to M$ is defined by

$$\pi_2(j_p{}^2(F))=p.$$

Let (U, x^h) be a coordinate neighborhood with the local coordinate system (x^h). A system of local coordinates (x^h, y^h, z^h) can be introduced in $\pi_2{}^{-1}(U)$ in such a way that a 2-jet $j_p{}^2(F)$ $(p\in U)$ has coordinates as

$$x^h=F^h(0), \qquad y^h=\frac{dF^h}{dt}(0), \qquad z^h=\frac{d^2F^h}{dt^2}(0).$$

We call the local coordinate system (x^h, y^h, z^h) thus introduced in $\pi_2{}^{-1}(U)$ the *induced coordinate system* and sometimes denote them by (ξ^A),[1] i.e.,

$$\xi^i=x^i, \qquad \xi^{n+i}=y^i, \qquad \xi^{2n+i}=z^i. \tag{1.1}$$

Let (U, x^h) and $(U', x^{h'})$ be two coordinates neighborhoods of M related by coordinate transformation

$$x^{h'}=x^{h'}(x^h)$$

in $U\cap U'$. If we denote by (x^h, y^h, z^h) and $(x^{h'}, y^{h'}, z^{h'})$ the induced coordinates in $\pi_2{}^{-1}(U)$ and $\pi_2{}^{-1}(U')$ respectively, the coordinate transformation in $\pi_2{}^{-1}(U)\cap\pi_2{}^{-1}(U')$ is given by

$$x^{h'}=x^{h'}(x^h), \qquad y^{h'}=\frac{\partial x^{h'}}{\partial x^h}y^h,$$

$$z^{h'}=\frac{\partial x^{h'}}{\partial x^h}z^h+\frac{\partial^2 x^{h'}}{\partial x^j\partial x^i}y^jy^i$$

and its Jacobian matrix by

$$\begin{pmatrix} \dfrac{\partial x^{h'}}{\partial x^h}, & 0, & 0 \\ \dfrac{\partial^2 x^{h'}}{\partial x^h\partial x^s}y^s, & \dfrac{\partial x^{h'}}{\partial x^h}, & 0 \\ \dfrac{\partial^2 x^{h'}}{\partial x^h\partial x^s}z^s+\dfrac{\partial^3 x^{h'}}{\partial x^h\partial x^t\partial x^s}y^ty^s, & 2\dfrac{\partial^2 x^{h'}}{\partial x^h\partial x^s}y^s, & \dfrac{\partial x^{h'}}{\partial x^h} \end{pmatrix}. \tag{1.2}$$

1) The indices $A, B, C, D, \cdots$ and $i, j, k, \cdots$ run over the ranges $1, \cdots, 3n$ and $1, \cdots, n$, respectively.

We denote by $\mathcal{T}^r_s(M)$ the space of all tensor fields of type (r, s) in M. Especially, $\mathcal{T}^0_0(M)$, $\mathcal{T}^1_0(M)$ and $\mathcal{T}^0_1(M)$ are respectively the spaces of all functions, of all vector fields and of all 1-forms all defined in M. We denote also by $\mathcal{T}^r_s(T_2(M))$ the space of all tensor fields of type (r, s) in $T_2(M)$.

Prolongations of tensor fields. For any element f of $\mathcal{T}^0_0(M)$, its prolongations f^0, f^{I} and f^{II} to $T_2(M)$ are elements of $\mathcal{T}^0_0(T_2(M))$ and have respectively local expressions of the form

$$f^0\colon f(x^h), \qquad f^{\mathrm{I}}\colon y^i\partial_i f(x^h), \qquad f^{\mathrm{II}}\colon z^i\partial_i f(x^h)+y^j y^i \partial_j\partial_i f(x^h) \tag{1.3}$$

in the induced coordinate system (ξ^A), $f(x^h)$ being the local expression of f in (x^h), where $\partial_i=\partial/\partial x^i$.

For any element X of $\mathcal{T}^1_0(M)$, its prolongations X^0, X^{I} and X^{II} are elements of $\mathcal{T}^1_0(T_2(M))$ and have the following properties:

$$\begin{aligned}
&X^0 f^0=0, && X^0 f^{\mathrm{I}}=0, && X^0 f^{\mathrm{II}}=(Xf)^0,\\
&X^{\mathrm{I}} f^0=0, && X^{\mathrm{I}} f^{\mathrm{I}}=\frac{1}{2}(Xf)^0, && X^{\mathrm{I}} f^{\mathrm{II}}=(Xf)^{\mathrm{I}},\\
&X^{\mathrm{II}} f^0=(Xf)^0, && X^{\mathrm{II}} f^{\mathrm{I}}=(Xf)^{\mathrm{I}}, && X^{\mathrm{II}} f^{\mathrm{II}}=(Xf)^{\mathrm{II}},
\end{aligned} \tag{1.4}$$

f being an arbitrary element of $\mathcal{T}^0_0(M)$.

For any element ω of $\mathcal{T}^0_1(M)$, its prolongations ω^0, ω^{I} and ω^{II} are elements of $\mathcal{T}^0_1(T_2(M))$ and have the following properties:

$$\begin{aligned}
&\omega^0(X^0)=0, && \omega^0(X^{\mathrm{I}})=0, && \omega^0(X^{\mathrm{II}})=(\omega(X))^0,\\
&\omega^{\mathrm{I}}(X^0)=0, && \omega^{\mathrm{I}}(X^{\mathrm{I}})=\frac{1}{2}(\omega(X))^0, && \omega^{\mathrm{I}}(X^{\mathrm{II}})=(\omega(X))^{\mathrm{I}},\\
&\omega^{\mathrm{II}}(X^0)=(\omega(X))^0, && \omega^{\mathrm{II}}(X^{\mathrm{I}})=(\omega(X))^{\mathrm{I}}, && \omega^{\mathrm{II}}(X^{\mathrm{II}})=(\omega(X))^{\mathrm{II}},
\end{aligned} \tag{1.5}$$

X being an arbitrary element of $\mathcal{T}^1_0(M)$.

Taking arbitrarily two tensor fields P and Q in M, we have the following formulas:

$$\begin{aligned}
&(P\otimes Q)^0=P^0\otimes Q^0,\\
&(P\otimes Q)^{\mathrm{I}}=P^{\mathrm{I}}\otimes Q^0+P^0\otimes Q^{\mathrm{I}},\\
&(P\otimes Q)^{\mathrm{II}}=P^{\mathrm{II}}\otimes Q^0+2P^{\mathrm{I}}\otimes Q^{\mathrm{I}}+P^0\otimes Q^{\mathrm{II}}.
\end{aligned} \tag{1.6}$$

The prolongations P^0, P^{I} and P^{II} are called respectively the *0-th*, the *1-st* and the *2-nd lifts* of P, P being an arbitrary tensor field in M.

REMARK. Let $\tilde{X}$ and $\tilde{Y}$ be two vector fields in $T_2(M)$. If we have $\tilde{X}f^{II}=\tilde{Y}f^{II}$ for any element f of $\mathcal{T}^0_0(M)$, then we have $\tilde{X}=\tilde{Y}$. Generally speaking, any tensor field in $T_2(M)$ is completely determined by giving its values for the 2-nd lifts of vector fields arbitrarily given in M.

Let F be an element of $\mathcal{T}^1_1(M)$ and $P(t)$ a polynomial of t. Then we have

$$(P(F))^{II}=P(F^{II}). \tag{1.7}$$

We now note that the 2-nd lift of the identity tensor field I of type (1, 1) is also the identity tensor field in $T_2(M)$, which is also denoted by I in $T_2(M)$, that is to say, $I^{II}=I$. For example, if $F^2+I=0$, we have $(F^{II})^2+I=0$. Thus, we obtain

PROPOSITION. *If F is an almost complex structure in M, so is F^{II} in $T_2(M)$.*

We denote by N_F the Nijenhuis tensor of an element F of $\mathcal{T}^1_1(M)$. We have then

$$(N_F)^{II}=N_{F^{II}} \tag{1.8}$$

for $F\in\mathcal{T}^1_1(M)$.

Prolongations of linear connections. Let there be given a linear connection ∇ in M. Then there exists a unique linear connection ∇^{II} in $T_2(M)$ characterized by the equation

$$\nabla^{II}{}_{Y^{II}}X^{II}=(\nabla_Y X)^{II}, \tag{1.9}$$

X and Y being arbitrary elements of $\mathcal{T}^1_0(M)$. The connection ∇^{II} is called the *lift* of the given connection ∇. If we denote by T and R respectively the torsion and the curvature tensors of ∇, we have

$$\tilde{T}=T^{II}, \qquad \tilde{R}=R^{II}, \tag{1.10}$$

where $\tilde{T}$ and $\tilde{R}$ are the torsion and the curvature tensors of ∇^{II} respectively.

We have the following formulas:

$$\nabla^{II}{}_{Y^{II}}X^0=(\nabla_Y X)^0, \qquad \nabla^{II}{}_{Y^{II}}X^{I}=(\nabla_Y X)^{I} \tag{1.11}$$

for X, $Y\in\mathcal{T}^1_0(M)$.

Let there be given a pseudo-Riemannian metric g in M. Then g^{II} is a pseudo-Riemannian metric in $T_2(M)$. If we denote by ∇ the Riemannian connection determined by g, then its lift ∇^{II} is the Riemannian connection determined by g^{II} in $T_2(M)$.

§ 2. Cross-sections determined by vector fields.

Let there be given a vector field V in M. Denote by φ_P: $I \to M$ the orbit of V passing through a point P of M in such a way that $\varphi_P(0)=P$, where I is an interval $(-\varepsilon, \varepsilon)$, ε being a certain positive number. If we denote by $\gamma_V(P)$ the 2-jet $j^2_P(\varphi_P)$, we set that the correspondence $P \to \gamma_V(P)$ defines a mapping γ_V: $M \to T_2(M)$ such that $\pi_2 \circ \gamma_V$ is the identity mapping, i.e., that γ_V: $M \to T_2(M)$ is a cross-section in $T_2(M)$. The submanifold $\gamma_V(M)$ imbedded in $T_2(M)$ is called the *cross-section* determined by the vector field V. If U is a coordinate neighborhood in M the cross-section $\gamma_V(M)$ is expressed locally in $\pi_2^{-1}(U)$ by equations

$$x^h = x^h, \qquad y^h = V^h(x^i), \qquad z^h = V^k(x^i)\partial_k V^h(x^i) \tag{2. 1}$$

with respect to the induced coordinate system (ξ^A), where $V = V^h(x^i)\partial_h$ is the local expression of V in U. We denote the equations (2. 1) by

$$\xi^A = \xi^A(x^i), \tag{2. 2}$$

i.e., $\xi^h = x^h$, $\xi^{n+h} = V^h$, $\xi^{2n+h} = V^k\partial_k V^h$.

Taking account of (1. 3) and (2. 1), we have along $\gamma_V(M)$ the equations

$$f^{II} = (\mathcal{L}_V{}^2 f)^0, \qquad f^I = (\mathcal{L}_V f)^0, \qquad f^0 = f^0 \tag{2. 3}$$

for $f \in \mathcal{T}^0_0(M)$, where $\mathcal{L}_V$ denotes the Lie derivation with respect to V and $\mathcal{L}_V{}^2 = \mathcal{L}_V \mathcal{L}_V$.

If we put $B_i{}^A = \partial_i \xi^A$, we get along $\gamma_V(M)$ n local vector fields B_i tangent to the cross-section which have the components of the form

$$(B_i{}^A) = \begin{pmatrix} \delta_i{}^h \\ \partial_i V^h \\ (\partial_i V^k)(\partial_k V^h) + V^k \partial_i \partial_k V^h \end{pmatrix} \tag{2. 4}$$

with respect to the induced coordinate system (ξ^A). For an element X of $\mathcal{T}^1_0(M)$ with local expression $X = X^i\ \partial/\partial x^i$, we denote by BX the vector field with components $B_i{}^A X^i$, which is defined globally along $\gamma_V(M)$ by virtue of (1. 2). The mapping B_p: $T_p(M) \to T_\sigma(T_2(M))$ $(\sigma = \gamma_V(p))$ defined by the correspondence $X_p \to (BX)_\sigma$, is the differential mapping $\gamma_V{}'$ of the cross-section mapping γ_V: $M \to T_2(M)$. Thus B_p: $T_p(M) \to T_\sigma(T_2(M))$ is an isomorphism and $B_p(T_p(M))$ is the tangent space of the cross-section $\gamma_V(M)$ at the point $\sigma = \gamma_V(p)$.

We consider along the cross-section $\gamma_V(M)$ n local vector fields C_i and n local vector fields $D_{\bar{i}}$ along $\gamma_V(M)$, which have respectively components of the form

$$(2.5)\qquad (C_i{}^A)=\begin{pmatrix} 0 \\ \frac{1}{2}\delta_i{}^h \\ \partial_i V^h \end{pmatrix},\qquad (D_{\bar{i}}{}^A)=\begin{pmatrix} 0 \\ 0 \\ \delta_i{}^h \end{pmatrix}$$

in the induced coordinate system (ξ^A). For an element X of $\mathcal{T}^1_0(M)$ with local experession $X=X^i\partial_i$, we denote by CX and DX the vector fields with components $C_i{}^A X^i$ and $D_{\bar{i}}{}^A X^i$ respectively. Then according to (1. 2), CX and DX are defined along $\gamma_V(M)$. We now defined two mappings C_p: $T_p(M)\to T_\sigma(T_2(M))$ and D_p: $T_p(M)\to T_\sigma(T_2(M))$ $(\sigma=\gamma_V(p))$ respectively by

$$(2.6)\qquad C_pX_p=(CX)_\sigma,\qquad D_pX_p=(DX)_\sigma$$

X being an arbitrary element of $\mathcal{T}^1_0(M)$. It is easily verified by virtue of (2. 5) that the two mappings C_p and D_p defined by (2. 6) are isomorphisms of $T_p(M)$ into $T_\sigma(T_2(M))$ $(\sigma=\gamma_V(\mathrm{P}))$.

Putting

$$N_\sigma{}^{(1)}=C_pT_p(M),\qquad N_\sigma{}^{(2)}=D_pT_p(M)\qquad (\sigma=\gamma_V(p)),$$

we have the following direct sum representation of $T_\sigma(T_2(M))$:

$$T_\sigma(T_2(M))=T_\sigma(\gamma_V(M))+N_\sigma{}^{(1)}+N_\sigma{}^{(2)}.$$

The $3n$ local vector fields B_i, $C_{\bar{i}}$ and $D_{\bar{i}}$ along $\gamma_V(M)$ are expressed respectively by

$$(2.7)\qquad B_i=B\partial_i,\qquad C_{\bar{i}}=C\partial_i,\qquad D_{\bar{i}}=D\partial_i$$

and form a local family of frames $\{B_i, C_{\bar{i}}, D_{\bar{i}}\}$ along $\gamma_V(M)$, which are called the *adapted frames* of $\gamma_V(M)$. The n local vector fields B_i span $T_\sigma(\gamma_V(M))$, $C_{\bar{i}}$ span $N_\sigma{}^{(1)}$ and $D_{\bar{i}}$ span $N_\sigma{}^{(2)}$, all at $\sigma\in\gamma_V(M)$.

Taking account of (2. 4), (2. 5) and (2. 7), we have along $\gamma_V(M)$

$$(2.9)\qquad \begin{aligned} X^{II}&=BX+2C(\mathcal{L}_VX)+D(\mathcal{L}_V{}^2X),\\ X^{I}&=\qquad\quad\; CX\qquad\;\; +D(\mathcal{L}_VX),\\ X^{0}&=\qquad\qquad\qquad\qquad\; DX, \end{aligned}$$

or equivalently

$$X^{\mathrm{II}}=(X^i)^0B_i+2(\mathcal{L}_VX^i)^0C_{\bar{i}}+(\mathcal{L}_V{}^2X^i)^0D_{\bar{\bar{i}}},$$

$$\text{(2. 10)}\qquad X^{\mathrm{I}}=\qquad (X^i)^0C_{\bar{i}}+(\mathcal{L}_VX^i)^0D_{\bar{\bar{i}}},$$

$$X^0=\qquad (X^i)^0D_{\bar{\bar{i}}}$$

for any element X of $\mathcal{T}_0^1(M)$ with local expression $X=X^i\partial_i$.

§ 3. Prolongations of tensor fields in the cross-sections.

Let there be given a vector field $\tilde{X}$ along $\gamma_V(M)$. Putting

$$\tilde{X}=\tilde{X}^iB_i+\tilde{X}^{\bar{i}}C_{\bar{i}}+\tilde{X}^{\bar{\bar{i}}}D_{\bar{\bar{i}}}$$

we call $(\tilde{X}^\alpha)=(\tilde{X}^i, \tilde{X}^{\bar{i}}, \tilde{X}^{\bar{\bar{i}}})$ [3] the components of $\tilde{X}$ in the adapted frame. Similarly, for any tensor field $\tilde{T}$ of type (1. 2) along $\gamma_V(M)$, we denote by

$$(\tilde{T}_{\beta\gamma}{}^\alpha)=(\tilde{T}_{ji}{}^h, \tilde{T}_{ji}{}^{\bar{h}}, \tilde{T}_{ji}{}^{\bar{\bar{h}}}, \cdots, \tilde{T}_{\bar{\bar{j}}\bar{\bar{i}}}{}^{\bar{\bar{h}}})$$

its components in the adapted frame. Thus by means of (2. 10), the lifts X^0, X^{I} and X^{II} have along $\gamma_V(M)$ components of the form

$$\text{(3. 1)}\qquad (X^{0\alpha})=\begin{pmatrix}0\\0\\X^h\end{pmatrix},\qquad (X^{\mathrm{I}\alpha})=\begin{pmatrix}0\\X^h\\\mathcal{L}_VX^h\end{pmatrix},\qquad (X^{\mathrm{II}\alpha})=\begin{pmatrix}X^h\\2\mathcal{L}_VX^h\\\mathcal{L}_V{}^2X^h\end{pmatrix}$$

in the adapted frame, where X is a vector field in M with local expression $X=X^i\partial_i$. In (3. 1) we have identified the 0-th lift $(X^h)^0$, $(\mathcal{L}_VX^h)^0$ and $(\mathcal{L}_V{}^2X^h)^0$ respectively with functions X^h, $\mathcal{L}_VX^h$ and $\mathcal{L}_V{}^2X^h$. In the sequel we sometimes use such identification.

Let there be given an element ω of $\mathcal{T}_1^0(M)$ with local expressions $\omega=\omega_idx^i$. Then its lifts ω^0, ω^{I} and ω^{II} have respectively components of the form

$$(\omega^0{}_\beta)=(\omega_i,\ 0,\ 0),$$

$$\text{(3. 2)}\qquad (\omega^{\mathrm{I}}{}_\beta)=\left(\mathcal{L}_V\omega_i,\ \frac{1}{2}\omega_i,\ 0\right),$$

$$(\omega^{\mathrm{II}}{}_\beta)=(\mathcal{L}_V{}^2\omega_i,\ \mathcal{L}_V\omega_i,\ \omega_i)$$

in the adapted frame. In fact by virtue of (2. 3), (3. 1) and (1. 5), we have along $\gamma_V(M)$, for example,

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3) We use Greek indices $\alpha, \beta, \cdots$ to represent the components in the adapted frame.

$$\omega^{II}{}_i X^i + 2\omega^{II}{}_{\bar{i}}(\mathcal{L}_V X^i) + \omega^{II}{}_{\bar{\bar{i}}}(\mathcal{L}_V{}^2 X^i)$$
$$= \mathcal{L}_V{}^2(\omega_i X^i)$$
$$= (\mathcal{L}_V{}^2 \omega_i) X^i + 2(\mathcal{L}_V \omega_i)(\mathcal{L}_V X^i) + \omega_i(\mathcal{L}_V{}^2 X^i)$$

for arbitrary element X of $\mathcal{T}_0^1(M)$ with local expression $X = X^i \partial_i$, and there exists an element X of $\mathcal{T}_0^1(M)$ such that at a given point, for any given values $(a^h, b^{\bar{h}}, c^{\bar{\bar{h}}})$,

$$X^h = a^h, \qquad \mathcal{L}_V X^h = b^{\bar{h}} \qquad \text{and} \qquad \mathcal{L}_V{}^2 X^h = c^{\bar{\bar{h}}}$$

hold. The other relations stated in (3. 2) are obtained similarly.

Taking account of (1. 6), (3. 1) and (3. 2) we find components of 0-th, 1-st and 2-nd lifts of any tensor field in M with respect to the adapted frame. For example, for an element h of $\mathcal{T}_2^0(M)$ we have

(3. 3)
$$(h^0{}_{\beta\alpha}) = \begin{pmatrix} h_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (h^I{}_{\beta\alpha}) = \begin{pmatrix} \mathcal{L}_V h_{ji} & \frac{1}{2} h_{ji} & 0 \\ \frac{1}{2} h_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$(h^{II}{}_{\beta\alpha}) = \begin{pmatrix} \mathcal{L}_V{}^2 h_{ji} & \mathcal{L}_V h_{ji} & h_{ji} \\ \mathcal{L}_V h_{ji} & \frac{1}{2} h_{ji} & 0 \\ h_{ji} & 0 & 0 \end{pmatrix},$$

h_{ji} being the components of h. For an element F of $\mathcal{T}_1^1(M)$,

(3. 4)
$$(F^0{}_\beta{}^\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_i{}^h & 0 & 0 \end{pmatrix}, \qquad (F^I{}_\beta{}^\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ F_i{}^h & 0 & 0 \\ \mathcal{L}_V F_i{}^h & \frac{1}{2} F_i{}^h & 0 \end{pmatrix},$$
$$(F^{II}{}_\beta{}^\alpha) = \begin{pmatrix} F_i{}^h & 0 & 0 \\ 2\mathcal{L}_V F_i{}^h & F_i{}^h & 0 \\ \mathcal{L}_V{}^2 F_i{}^h & \mathcal{L}_V F_i{}^h & F_i{}^h \end{pmatrix},$$

$F_i{}^h$ being the components of F. For an element S of $\mathcal{T}_2^1(M)$,

$$
\begin{aligned}
(3.5)\qquad & S^0{}_{ji}{}^h=0, && S^0{}_{ji}{}^{\bar{h}}=0, && S^0{}_{ji}{}^{\bar{\bar{h}}}=S_{ji}{}^h,\\
& S^{\mathrm{I}}{}_{ji}{}^h=0, && S^{\mathrm{I}}{}_{ji}{}^{\bar{h}}=S_{ji}{}^h, && S^{\mathrm{I}}{}_{ji}{}^{\bar{\bar{h}}}=\mathcal{L}_V S_{ji}{}^h,\\
& S^{\mathrm{II}}{}_{ji}{}^h=S_{ji}{}^h, && S^{\mathrm{II}}{}_{ji}{}^{\bar{h}}=2\mathcal{L}_V S_{ji}{}^h, && S^{\mathrm{II}}{}_{ji}{}^{\bar{\bar{h}}}=\mathcal{L}_V{}^2 S_{ji}{}^h,
\end{aligned}
$$

$S_{ji}{}^h$ being the components of S.

The linear isomorphism B defined in § 2 is the differential mapping γ_V' of the cross-section mapping γ_V: $M\to\gamma_V(M)$. Then we denote sometimes by $\gamma_V'X$ the vector field BX, X being an arbitrary element of $\mathcal{T}_0^1(M)$. Given an element ω of $\mathcal{T}_1^0(M)$, we denote by $\gamma_V'\omega$ the image of ω by the dual mapping of B^{-1} (=the restriction of π_2 to $\gamma_V(M)$). The mapping γ_V' is extended as a linear mapping γ_V': $\mathcal{T}(M)\to\mathcal{T}(\gamma_V(M))$ by

$$\gamma_V'(P\otimes Q)=(\gamma_V'P)\otimes(\gamma_V'Q),$$

P and Q being arbitrary tensor fields in M.

Now we will define the operation $\sharp$ in $\mathcal{T}(T_2(M))$ as follows. For an element $\tilde{X}$ of $\mathcal{T}_0^1(T_2(M))$, we put

$$\tilde{X}^\sharp=\tilde{X}^i B_i.$$

Let $\tilde{\omega}$ be a tensor field of type (0. 1) defined along $\gamma_V(M)$. Then putting along $\gamma_V(M)$

$$\tilde{\omega}^\sharp(BX)=\tilde{\omega}(BX)$$

for $X\in\mathcal{T}_0^1(M)$, we can define an element $\tilde{\omega}^\sharp$ of $\mathcal{T}_1^0(\gamma_V(M))$ which is called the 1-form induced in $\gamma_V(M)$ from $\tilde{\omega}$. Let $\tilde{h}$ be a tensor field of type (0, 2) defined along $\gamma_V(M)$. Then putting along $\gamma_V(M)$

$$\tilde{h}^\sharp(BX, BY)=\tilde{h}(BX, BY)$$

for $X, Y\in\mathcal{T}_0^1(M)$, we can define an element $\tilde{h}^\sharp$ of $\mathcal{T}_2^0(\gamma_V(M))$ which is called the tensor field induced in $\gamma_V(M)$ from $\tilde{h}$. Let $\tilde{F}$ be a tensor field of type (1, 1) defined along $\gamma_V(M)$ such that, for any vector field $\tilde{A}$ tangent to $\gamma_V(M)$, $\tilde{F}\tilde{A}$ is also tangent to $\gamma_V(M)$. Then putting

$$\tilde{F}^\sharp(BX)=\tilde{F}(BX)$$

for $X\in\mathcal{T}_0^1(M)$, we can define an element $\tilde{F}^\sharp$ of $\mathcal{T}_1^1(\gamma_V(M))$ which is called the tensor field induced in $\gamma_V(M)$ from $\tilde{F}$. Let $\tilde{S}$ be a tensor field of type (1, 2) defined along $\gamma_V(M)$ such that for any vector field $\tilde{A}, \tilde{B}$ tangent to $\gamma_V(M)$, $\tilde{S}(\tilde{A}, B)$ is tangent to

$\gamma_V(M)$. Then putting

$$\tilde{S}^{\#}(BX, BY)=\tilde{S}(BX, BY)$$

for $X, Y \in \mathcal{T}_0^1(M)$, we can define an element $\tilde{S}^{\#}$ of $\mathcal{T}_2^1(\gamma_V(M))$, which is called the tensor field induced in $\gamma_V(M)$ from $\tilde{S}$.

We have from (3. 1),

PROPOSITION 3. 1. *Let X be an element of $\mathcal{T}_0^1(M)$. Then X^{II} is tangent to $\gamma_V(M)$ if and only if $\mathcal{L}_V X=0$. In this case $X^{II\#}=\gamma_V' X$ holds. For any element X of $\mathcal{T}_0^1(M)$, $X^{0\#}=0$ and $X^{I\#}=0$ hold.*

We have from (3. 2),

PROPOSITION 3. 2. *For any element ω of $\mathcal{T}_1^0(M)$,*

$$\omega^{II\#}=\gamma_V'(\mathcal{L}_V{}^2\omega), \qquad \omega^{I\#}=\gamma_V'(\mathcal{L}_V\omega) \qquad \text{and} \qquad \omega^{0\#}=\gamma_V'\omega$$

hold.

We have from (3. 3)

PROPOSITION 3. 3. *For any element h of $\mathcal{T}_2^0(M)$,*

$$h^{II\#}=\gamma_V'(\mathcal{L}_V{}^2 h), \qquad h^{I\#}=\gamma_V'(\mathcal{L}_V h) \qquad \text{and} \qquad h^{0\#}=\gamma_V' h$$

hold, and hence $h^{0\#}(BX, BY)=h(X, Y)^0$.

PROPOSITION 3. 4. *Let g be a Riemannian metric in M. Then $g^{0\#}$ is a Riemannian metric in $\gamma_V(M)$ and γ_V is isometry, i.e. $g^{0\#}=\gamma_V' g$.*

Suppose that the vector field V in M satisfies the condition $\mathcal{L}_V g=cg$, g being a Riemannian metric in M and c a constant, that is, V is an infinitesimal homothetic transformation with respect to g. Then we have from Proposition 3. 3 the relation $g^{II\#}=cg^{I\#}=c^2 g^{0\#}$.

If for each point σ of $\gamma_V(M)$ the tangent space $T_\sigma(\gamma_V(M))$ is invariant by the action of a tensor field $\tilde{F}$ defined along $\gamma_V(M)$, then the cross-section $\gamma_V(M)$ is said to be invariant by $\tilde{F}$. For any $F \in \mathcal{T}_1^1(M)$, we have from (3. 4)

$$F^0(BX)=DFX, \qquad F^I(BX)=C(FX)+D((\mathcal{L}_V F)X),$$

$$F^{II}(BX)=B(FX)+2C((\mathcal{L}_V F)X)+D((\mathcal{L}_V{}^2 F)X)$$

for $X \in \mathcal{T}_0^1(M)$. Thus we have

PROPOSITION 3. 5. *Let F be an element of $\mathcal{T}_1^1(M)$. The cross-section $\gamma_V(M)$ is invariant by F^{II} if and only if $\mathcal{L}_V F=0$. In this case, $F^{II\#}=\gamma_V' F$ holds. The lifts F^0 and F^I do not leave $\gamma_V(M)$ invariant, unless $F=0$.*

PROPOSITION 3. 6. *If F is an almost complex structure such that $\mathcal{L}_V F=0$, then $F^{II\#}$ is an almost complex structure in $\gamma_V(M)$ and $F^{II\#}=\gamma_V' F$ holds.*

If a Riemannian metric g in M satisfies the condition

$$g(FX, FY)=g(X, Y) \qquad \text{for any} \quad X, Y \in \mathcal{T}_0^1(M),$$

then (g, F) is called an almost Hermitian structure in M. If $\mathcal{L}_V F=0$ holds, then we get along $\gamma_V(M)$

$$g^{0\#}(F^{II\#}BX, F^{II\#}BY)=(\gamma_V' g)((\gamma_V' F)X, (\gamma_V' F)Y)$$
$$=(g(FX, FY))^0$$

because of Proposition 3. 3 and 3. 5. Thus we have

PROPOSITION 3. 7. *Suppose that there is given an almost Hermitian structure (g, F) in M. If $\mathcal{L}_V F=0$, then $(g^{0\#}, F^{II\#})$ is an almost Hermitian structure in $\gamma_V(M)$.*

For any $S \in \mathcal{T}_2^1(M)$, we have from (3. 5)

$$S^0(BX, BY)=D(S(X, Y)),$$
$$S^I(BX, BY)=C(S(X, Y))+D((\mathcal{L}_V S)(X, Y)), \tag{3. 7}$$
$$S^{II}(BX, BY)=B(S(X, Y))+2C((\mathcal{L}_V S)(X, Y))+D((\mathcal{L}_V{}^2 S)(X, Y))$$

for any $X, Y \in \mathcal{T}_0^1(M)$. Thus we get

PROPOSITION 3. 8. *Let S be an element of $\mathcal{T}_2^1(M)$. The vector fields $S^{II}(BX, BY)$ is tangent to $\gamma_V(M)$ for arbitrary elements X, Y of $\mathcal{T}_0^1(M)$, if and only if $\mathcal{L}_V S=0$, and in this case $S^{II\#}=\gamma_V' S$ holds. The vector fields $S^0(BX, BY)$ and $S^I(BX, BY)$ are not tangent to $\gamma_V(M)$, unless $S=0$.*

If an element F of $\mathcal{T}_1^1(M)$ satisfies $\mathcal{L}_V F=0$, then its Nijenhuis tensor satisfies $\mathcal{L}_V N_F=0$. By virtue of (1. 8), Proposition 3. 5 and 3. 8, we have

$$N_{F^{II}}{}^{\#}=N_{F^{II\#}}=\gamma_V' N_F$$

in the case that $\mathcal{L}_V F=0$. Thus we have

PROPOSITION 3. 9. *Let F be an element of $\mathcal{T}^1_1(M)$ such that $\mathcal{L}_V F=0$. Then the vector field $N_{F^{II}}(BX, BY)$ is tangent to $\gamma_V(M)$ for arbitrary elements X, Y of $\mathcal{T}^1_0(M)$, and $N_{F^{II}}{}^{\#}=N_{F^{II\#}}=\gamma_V{}'N_F$ hold. Especially $N_{F^{II\#}}$ vanishes identically in $\gamma_V(M)$ if and only if $N_F=0$.*

Consequently taking account of Proposition in § 1 and Proposition 3. 5, we get

PROPOSITION 3. 10. *If a complex structure F satisfies the condition $\mathcal{L}_V F=0$, then $F^{II\#}$ is a complex structure in $\gamma_V(M)$.*

§ 4. Prolongations of affine connections in cross-sections.

First of all, we recall some formulas on Lie derivations (cf. [4]). Let there be given an affine connection ∇ with coefficients Γ^h_{ji}. For vector fields X with local expression $X=X^i\partial_i$ and V, we have formulas as

$$\mathcal{L}_V(\nabla_j X^h)-\nabla_j(\mathcal{L}_V X^h)=(\mathcal{L}_V\Gamma^h_{ji})X^i, \tag{4. 1}$$

$$\nabla_k(\mathcal{L}_V\Gamma^h_{ji})-\nabla_j(\mathcal{L}_V\Gamma^h_{ki})=\mathcal{L}_V R_{kji}{}^h, \tag{4. 2}$$

where $R_{kji}{}^h$ denotes the components of the curvature tensor R of ∇. Hence we have

$$\mathcal{L}_V{}^2(\nabla_j X^h)-\nabla_j(\mathcal{L}_V{}^2X^h)=(\mathcal{L}_V{}^2\Gamma^h_{ji})X^i+2(\mathcal{L}_V\Gamma^h_{ji})(\mathcal{L}_V X^i), \tag{4. 3}$$

$$\nabla_k(\mathcal{L}_V{}^2\Gamma^h_{ji})-\nabla_j(\mathcal{L}_V{}^2\Gamma^h_{ki})+2(\mathcal{L}_V\Gamma^h_{kt})(\mathcal{L}_V\Gamma^t_{ji})-2(\mathcal{L}_V\Gamma^h_{jt})(\mathcal{L}_V\Gamma^t_{ki})=\mathcal{L}_V{}^2R_{kji}{}^h. \tag{4. 4}$$

Taking account of (1. 9) and (2. 9), we have along $\gamma_V(M)$

$$\begin{aligned}\nabla^{II}{}_{Y^{II}}X^{II}&=B(\nabla_Y X)+2C(\mathcal{L}_V(\nabla_Y X))+D(\mathcal{L}_V{}^2(\nabla_Y X))\\&=(\nabla_Y X^h)^0 B_h+2(\mathcal{L}_V\nabla_Y X^h)^0 C_{\bar{h}}+(\mathcal{L}_V{}^2\nabla_Y X^h)^0 D_{\bar{\bar{h}}}\end{aligned} \tag{4. 5}$$

for $X, Y\in\mathcal{T}^1_0(M)$, where $X=X^i\partial_i$ is the local expression of X. On the other hand, taking account of (1. 4) and (2. 10), we have along $\gamma_V(M)$

$$\nabla^{II}{}_{Y^{II}}X^{II}=\nabla^{II}{}_{Y^{II}}((X^h)^0 B_h+2(\mathcal{L}_V X^h)^0 C_{\bar{h}}+(\mathcal{L}_V{}^2X^h)^0 D_{\bar{\bar{h}}})$$

i.e.

$$\nabla^{II}_{Y^{II}}X^{II}=(X^h)^0\nabla^{II}_{Y^{II}}B_h+2(\mathcal{L}_VX^h)^0\nabla^{II}_{Y^{II}}C_{\bar{h}}+(\mathcal{L}_V{}^2X^h)^0\nabla^{II}_{Y^{II}}D_{\bar{\bar{h}}}$$
$$+(Y^i\partial_iX^h)^0B_h+2(Y^i\partial_i(\mathcal{L}_VX^h))^0C_{\bar{h}}+(Y^i\partial_i(\mathcal{L}_V{}^2X^h))^0D_{\bar{\bar{h}}},$$

where $Y=Y^i\partial_i$ is the local representation of Y. For an arbitrary point σ of $\gamma_V(M)$, there exists a vector field Y in M with initial conditions $Y=\partial_j$, $\mathcal{L}_VY=0$, $\mathcal{L}_V{}^2Y=0$ at $p=\pi_2(\sigma)$. Then at σ, $Y^{II}=BY=B\partial_j=B_j$, and the value of $\nabla^{II}_{Y^{II}}X^{II}$ at σ is $\nabla^{II}_{B_j}X^{II}$. Comparing the two equations (4. 5) and (4. 6), we have at $\sigma\in\gamma_V(M)$,

$$(X^h)^0\nabla^{II}_{B_j}B_h+2(\mathcal{L}_VX^h)^0\nabla^{II}_{B_j}C_{\bar{h}}+(\mathcal{L}_V{}^2X^h)^0\nabla^{II}_{B_j}D_{\bar{\bar{h}}}$$
$$=(\nabla_jX^h-\partial_jX^h)^0B_h+2\{\mathcal{L}_V(\nabla_jX^h)-\nabla_j(\mathcal{L}_VX^h)\}^0C_{\bar{h}}$$
$$+\{\mathcal{L}_V{}^2(\nabla_jX^h)-\nabla_j(\mathcal{L}_V{}^2X^h)\}^0D_{\bar{\bar{h}}}$$
$$=(\Gamma^h_{ji}X^i)^0B_h+2\{\mathcal{L}_V(\nabla_jX^h)-\nabla_j(\mathcal{L}_VX^h)+\Gamma^h_{ji}\mathcal{L}_VX^i\}^0C_{\bar{h}}$$
$$+\{\mathcal{L}_V{}^2(\nabla_jX^h)-\nabla_j(\mathcal{L}_V{}^2X^h)+\Gamma^h_{ji}\mathcal{L}_V{}^2X^i\}^0D_{\bar{\bar{h}}}$$

which implies by virtue of (4. 1) and (4. 3),

$$(X^h)^0\nabla^{II}_{B_j}B_h+2(\mathcal{L}_VX^h)^0\nabla^{II}_{B_j}C_{\bar{h}}+(\mathcal{L}_V{}^2X^h)^0\nabla^{II}_{B_j}D_{\bar{\bar{h}}}$$
$$(4.6)\qquad =(\Gamma^h_{ji}X^i)^0B_h+2\{(\mathcal{L}_V\Gamma^h_{ji})X^i+\Gamma^h_{ji}(\mathcal{L}_VX^i)\}^0C_{\bar{h}}$$
$$+\{(\mathcal{L}_V{}^2\Gamma^h_{ji})X^i+2(\mathcal{L}_V\Gamma^h_{ji})(\mathcal{L}_VX^i)+\Gamma^h_{ji}(\mathcal{L}_V{}^2X^i)\}^0D_{\bar{\bar{h}}}.$$

Let a^h, $b^{\bar{h}}$ and $c^{\bar{\bar{h}}}$ be arbitrary real numbers. For any point σ of $\gamma_V(M)$, there exists a vector field X in M with local expression $X=X^i\partial_i$ such that it satisfies $X^h=a^h$, $\mathcal{L}_VX^h=b^{\bar{h}}$, $\mathcal{L}_V{}^2X^h=c^{\bar{\bar{h}}}$ at $p=\pi_2(\sigma)$. Thus (4. 6) gives

$$(4.7)\qquad \begin{aligned} \nabla^{II}_{B_j}B_i&=(\Gamma^h_{ji})^0B_h+2(\mathcal{L}_V\Gamma^h_{ji})^0C_{\bar{h}}+(\mathcal{L}_V{}^2\Gamma^h_{ji})^0D_{\bar{\bar{h}}},\\ \nabla^{II}_{B_j}C_{\bar{i}}&=\qquad\qquad\quad (\Gamma^h_{ji})^0C_{\bar{h}}\quad+(\mathcal{L}_V\Gamma^h_{ji})^0D_{\bar{\bar{h}}},\\ \nabla^{II}_{B_j}D_{\bar{\bar{i}}}&=\qquad\qquad\qquad\qquad\qquad\qquad (\Gamma^h_{ji})^0D_{\bar{\bar{h}}}.\end{aligned}$$

Putting

$$(4.8)\qquad \begin{aligned} '\nabla^{II}{}_jB_i&=\nabla^{II}_{B_j}B_i-(\Gamma^h_{ji})^0B_h,\\ '\nabla^{II}{}_jC_{\bar{i}}&=\nabla^{II}_{B_j}C_{\bar{i}}-(\Gamma^h_{ji})^0C_{\bar{h}},\\ '\nabla^{II}{}_jD_{\bar{\bar{i}}}&=\nabla^{II}_{B_j}D_{\bar{\bar{i}}}-(\Gamma^h_{ji})^0D_{\bar{\bar{h}}},\end{aligned}$$

we have

$$'\nabla^{\mathrm{II}}{}_j B_i = 2(\mathcal{L}_V \Gamma^h_{ji})^0 C_{\bar{h}} + (\mathcal{L}_V{}^2 \Gamma^h_{ji})^0 D_{\bar{\bar{h}}}, \tag{4.9}$$

$$'\nabla^{\mathrm{II}}{}_j C_i = (\mathcal{L}_V \Gamma^h_{ji})^0 D_{\bar{\bar{h}}}, \tag{4.10}$$

$$'\nabla^{\mathrm{II}}{}_j D_{\bar{\bar{i}}} = 0. \tag{4.11}$$

These are nothing but the structure equations for the cross-section $\gamma_V(M)$, which imply

PROPOSITION 4.1. *The cross-section $\gamma_V(M)$ is totally geodesic in $T_2(M)$ with respect to the connection ∇^{II}, if and only if the vector field V is infinitesimal affine transformation in M with respect to ∇ (i.e. $\mathcal{L}_V \Gamma^h_{ji} = 0$).*

Taking account of (4.8) we have

$$\nabla^{\mathrm{II}}{}_{B_j} BX = (\nabla_j X^h)^0 B_h + (X^h)^0 \, '\nabla_j{}^{\mathrm{II}} B_i \tag{4.12}$$

for $X \in \mathcal{T}^1_0(M)$. For $Y = Y^i \partial_i$, putting $\nabla_{BY}{}^{\mathrm{II}} B_h = Y^j \, '\nabla_j{}^{\mathrm{II}} B_h$, we have

$$\nabla^{\mathrm{II}}{}_{BY} BX = B(\nabla_Y X) + (X^h)^0 \, '\nabla_{BY}{}^{\mathrm{II}} B_h.$$

We can now defined an affine connection $\nabla^{\sharp}$ in $\gamma_V(M)$ by the equation

$$\nabla^{\sharp}{}_{BY} BX = B(\nabla_Y X) \tag{4.13}$$

for $X, Y \in \mathcal{T}^1_0(M)$ and call $\nabla^{\sharp}$ the affine connection induced in $\gamma_V(M)$ from ∇, or the induced affine connection of $\gamma_V(M)$. Now (4.12) is written as

$$\nabla^{\mathrm{II}}{}_{BY} BX = \nabla^{\sharp}{}_{BY} BX + (X^h)^0 \, '\nabla_{BY}{}^{\mathrm{II}} B_h \tag{4.14}$$

for $X = X^i \partial_i$, $Y \in \mathcal{T}^1_0(M)$.

Let there be given an element h of $\mathcal{T}^0_2(M)$. By virtue of Proposition 3.3, we have

$$BZ(h^{0\sharp}(BX, BY)) = (\nabla^{\sharp}{}_{BZ} h^{0\sharp})(BX, BY) + h^{0\sharp}(\nabla^{\sharp}{}_{BZ} BX, BY) + h^{0\sharp}(BX, \nabla^{\sharp}{}_{BZ} BY)$$

for $X, Y, Z \in \mathcal{T}^1_0(M)$. On the other hand, taking account of (2.11), we have

$$(BZ)(h^{0\#}(BX, BY))=(BZ)(h(X, Y))^0=(Z(h(X, Y)))^0$$
$$=((\nabla_Z h)(X, Y))^0+(h(\nabla_Z X, Y))^0+(h(X, \nabla_Z Y))^0$$
$$=(\nabla_Z h)^{0\#}(BX, BY)+h^{0\#}(B(\nabla_Z X), BY)+h^{0\#}(BX, B(\nabla_Z Y))$$

for $X, Y, Z \in \mathcal{T}_0^1(M)$. If we compare the two equations obtained above, taking account of (4. 13), we have

$$\nabla^{\#}_{BZ} h^{0\#}=(\nabla_Z h)^{0\#} \tag{4. 15}$$

for $Z \in \mathcal{T}_0^1(M)$.

When an affine connection ∇ in M is torsionfree, ∇^{II} is torsionfree in $T_2(M)$ too (cf. (1. 10)). Hence the induced connection $\nabla^{\#}$ of $\gamma_V(M)$ is also torsionfree. Thus we obtain from (4. 15)

PROPOSITION 4. 2. *Let g be a Riemannian metric in M and ∇ the Riemannian connection determined by g in M. Then the connection $\nabla^{\#}$ induced in $\gamma_V(M)$ from ∇ is the Riemannian connection determined by the induced metric $g^{0\#}$ of $\gamma_V(M)$.*

Let there be given an element F of $\mathcal{T}_1^1(M)$ satisfying the condition $\mathcal{L}_V F=0$, then by virtue of (4. 13) and Proposition 3. 5, we have

$$\nabla_{BZ}{}^{\#} F^{II\#}=(\nabla_Z F)^{II} \tag{4. 16}$$

for $Z \in \mathcal{T}_0^1(M)$. Thus we have

PROPOSITION 4. 3. *Let F be an element of $\mathcal{T}_1^1(M)$ satisfying the condition $\mathcal{L}_V F=0$. If $\nabla F=0$ in M, then $\nabla^{\#} F^{II\#}=0$ in $\gamma_V(M)$.*

An almost Hermitian structure (g, F) in M is called Kählerian if $\nabla F=0$, ∇ being the Riemannian connection determined by g.

PROPOSITION 4. 4. *If (g, F) is a Kählerian structure satisfying the condition $\mathcal{L}_V F=0$, so is $(g^{0\#}, F^{II\#})$ in $\gamma_V(M)$.*

Operating $\nabla^{II}{}_{B_k}$ to the first equation (4. 7) and taking the skew symmetric part of the equation obtained with respect to the indices j and k, by virtue of (2. 11) and (4. 7) we have

$$\nabla^{II}{}_{B_k}\nabla^{II}{}_{B_j}B_i-\nabla^{II}{}_{B_j}\nabla^{II}{}_{B_k}B_i$$
$$=(R_{kji}{}^h)^0 B_h+2\{\nabla_k(\mathcal{L}_V\Gamma^h_{ji})-\nabla_j(\mathcal{L}_V\Gamma^h_{ki})\}^0 C_{\bar{h}}$$
$$+\{(\nabla_k\mathcal{L}_V{}^2\Gamma^h_{ji}-\nabla_j\mathcal{L}_V{}^2\Gamma^h_{ki})+2(\mathcal{L}_V\Gamma^h_{kt})(\mathcal{L}_V\Gamma^t_{ji})-2(\mathcal{L}_V\Gamma^h_{jt})(\mathcal{L}_V\Gamma^t_{ki})\}^0 D_{\bar{\bar{h}}}$$

which reduces to

$$R^{II}(B_k, B_j)B_i=(R_{kji}{}^h)^0 B_h+2(\mathcal{L}_V R_{kji}{}^h)^0 C_{\bar{h}}+(\mathcal{L}_V{}^2 R_{kji}{}^h)^0 D_{\bar{\bar{h}}} \tag{4. 17}$$

because of (1. 10), (4. 2) and (4. 4), where $R_{kji}{}^h$ denote the components of the curvature tensor R of the given affine condection ∇ and R^{II} the 2-nd lift of R to $T_2(M)$. As a direct consequence of (4. 17), we have

PROPOSITION 4. 5. *Let R and R^{II} be the curvature tensors of affine connections ∇ given in M and ∇^{II}, respectively. Then the curvature transformation $R^{II}(BX, BY)$ X and Y being arbitrary elements of $\mathcal{T}_0^1(M)$, leaves invariant the tangent space of the cross-section $\gamma_V(M)$ at each point of $\gamma_V(M)$, if and only if $\mathcal{L}_V R=0$. In this case $R^{II\#}=\gamma_V{}'R$ holds, where $R^{II\#}$ denotes the tensor field induced in $\gamma_V(M)$ from R.*

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