KŌDAI MATH. SEM. REP.
21 (1969), 281—289

ON GENERALIZED $|V, \lambda|$ SUMMABILITY FACTORS OF INFINITE SERIES

BY R. G. VARSHNEY

1. Let $\sum a_n$ be a given infinite series with partial sum s_n and let $\lambda=\{\lambda_n\}$ be a monotone non-decreasing sequence of natural numbers with $\lambda_{n+1}-\lambda_n\leqq 1$ and $\lambda_1=1$.

The sequence-to-sequence transformation

$$V_n(\lambda)=\frac{1}{\lambda_n}\sum_{\nu=n-\lambda_n+1}^{n} s_\nu$$

defines generalized de la Vallée Poussin means of the sequence $\{s_n\}$ generated by the sequence $\{\lambda_n\}$.

The series $\sum a_n$ is said to be summable $|V, \lambda|$, if the sequence $\{V_n(\lambda)\}$ is of bounded variation, that is to say

$$\sum_{n=1}^{\infty} |V_{n+1}(\lambda)-V_n(\lambda)|<\infty \qquad [1].$$

The series $\sum a_n$ will be said to be summable $|V, \lambda|_k$, $k\geqq 1$, if the series

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} |V_{n+1}(\lambda)-\lambda_n(\lambda)|^k<\infty.$$

For $\lambda_n=n$ it reduces to $|C, 1|_k$ and for $k=1$ it is the same as summability $|V, \lambda|$. If

$$\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu}=O(\log n), \qquad n\to\infty$$

then $\sum a_n$ is said to be strongly bounded by logarthmic means with index 1 or simply bounded $[R, \log n, 1]$.

A sequence $\{\varepsilon_n\}$ is said to be convex when

$$\Delta^2\varepsilon_n\geqq 0, \qquad n=1, 2, 3, \cdots,$$

with the notation

$$\Delta\varepsilon_n=\varepsilon_n-\varepsilon_{n+1}, \qquad \Delta^2\varepsilon_n=\Delta(\Delta\varepsilon_n).$$

2. Concerning $|C, 1|$ summability factors of infinite series.

Prasad and Bhatt proved the following theorem:

THEOREM A [7]. *If $\{\varepsilon_n\}$ is a convex sequence such that $\sum n^{-1}\varepsilon_n<\infty$, and*

Received January 13, 1969.

$$\sum_{\nu=1}^{n} |s_\nu - s| = O\{n(\log n)^k\}, \qquad k \geqq 0$$

as $n \to \infty$, *then the series* $\sum \{\log(n+1)\}^{-k} \varepsilon_n a_n$ *is summable* $|C, 1|$.

In 1962 Pati proved another theorem of $|C, 1|$ summability factors of infinite series. His theorem is as following:

THEOREM B [6]. *Let* $\{\varepsilon_n\}$ *be a convex sequence such that* $\sum n^{-1}\varepsilon_n < \infty$. *If* $\sum a_n$ *is bounded* $[R, \log n, 1]$, *then* $\sum a_n \varepsilon_n$ *is summable* $|C, 1|$.

Very recently the author has generalized the above theorem of Pati by proving the following:

THEOREM C [9]. *Let* $\{\varepsilon_n\}$ *be a convex sequence such that* $\sum \lambda_n^{-1}\varepsilon_n < \infty$. *If*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\lambda_\nu} = O(\mu_n)$$

where $\mu_n = \sum_{\nu=1}^{n} \lambda_\nu^{-1}$, *then* $\sum a_n \varepsilon_n$ *is summable* $|V, \lambda|$.

Mazhar[1)] established the following theorem which generalizes Theorem B.

THEOREM D [3]. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum n^{-1}\varepsilon_n < \infty$, *and*

$$\sum_{\nu=1}^{n} \frac{|s_\nu|^k}{\nu} = O(\log n), \qquad (k \geqq 1)$$

then $\sum a_n \varepsilon_n$ *is summable* $|C, 1|_k$.

These theorems were subsequently generalized by the author in the following form.

THEOREM E [11]. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum \lambda_n^{-1}\varepsilon_n < \infty$, *and*

$$\sum_{\nu=1}^{n} \frac{|s_\nu|^k}{\lambda_\nu} = O(\mu_n), \qquad (k \geqq 1)$$

where $\mu_n = \sum_{\nu=1}^{n} \lambda_\nu^{-1}$, *then* $\sum a_n \varepsilon_n$ *is summable* $|V, \lambda|_k$.

Niranjan Singh has established the following theorem which generalizes Theorem A and Theorem B.

THEOREM F [5]. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum n^{-1}\varepsilon_n < \infty$, *and*

$$\sum_{\nu=1}^{n} \frac{|s_\nu|}{\nu} = O(\log n\gamma_n), \qquad n \to \infty$$

where $\{\gamma_n\}$ *is a positive non-decreasing sequence such that*

$$n\gamma_n \log n \Delta\left(\frac{1}{\gamma_n}\right) = O(1), \qquad n \to \infty$$

then $\sum a_n \varepsilon_n / \gamma_n$ *is summable* $|C, 1|$.

1) The same theorem has also been obtained by Misra [4].

In his thesis for Ph. D. Umar has extended Theorem F as following:

THEOREM G [8]. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum n^{-1}\varepsilon_n<\infty$ *and*

$$\sum_{\nu=1}^{n}\frac{|s_\nu|^k}{\nu}=O(\log n\gamma_n)$$

where $\{\gamma_n\}$ *is a positive non-decreasing sequence such that* $\{1/\beta_n\}$ *is a convex sequence and*

$$n\gamma_n \log n\Delta\frac{1}{\gamma_n}=O(1), \qquad n\to\infty$$

then $\sum a_n\varepsilon_n/\gamma_n$ *is summable* $|C,1|_k$, $k\geqq 1$.

The author has proved a theorem for $|V,\lambda|$ summability factors which extends theorem F. His theorem is as following:

THEOREM H [10]. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum \lambda_n^{-1}\varepsilon_n<\infty$, *and*

$$\sum_{\nu=1}^{n}\frac{|s_\nu|}{\lambda_\nu}=O(\mu_n\gamma_n)$$

where $\mu_n=\sum_{\nu=1}^{n}\lambda_\nu^{-1}$ *and* $\{\gamma_n\}$ *is a positive non-decreasing sequence such that*

$$\lambda_n\mu_n\gamma_n\Delta\left(\frac{1}{\gamma_n}\right)=O(1), \qquad as\ n\to\infty$$

then $\sum a_n\varepsilon_n/\gamma_n$ *is summable* $|V,\lambda|$.

The object of this note is to extend our theorem H to summability $|V,\lambda|_k$.

3. In what follows we establish the following theorem which includes, as special cases, all the previous theorems stated above.

THEOREM. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum \lambda_n^{-1}\varepsilon_n<\infty$, *and*

$$\sum_{\nu=1}^{n}\frac{|s_\nu|^k}{\lambda_\nu}=O(\mu_n\gamma_n) \qquad (k\geqq 1)$$

where $\mu_n=\sum_{\nu=1}^{n}\lambda_\nu^{-1}$ *and* $\{\gamma_n\}$ *is a positive non-decreasing sequence such that*

$$\lambda_n\gamma_n\mu_n\Delta\left(\frac{1}{\gamma_n}\right)=O(1), \qquad n\to\infty$$

then $\sum a_n\varepsilon_n/\gamma_n$ *is summable* $|V,\lambda|_k$.

4. We need the following Lemma for the proof of the theorem.

LEMMA. *If* $\{\varepsilon_n\}$ *is a convex sequence such that* $\sum \lambda_n^{-1}\varepsilon_n<\infty$, *then*

(i) $\{\varepsilon_n\}$ *is a non-negative decreasing sequence and* $\varepsilon_n\mu_n=o(1)$ *as* $n\to\infty$;

(ii) $$m\mu_m\Delta\varepsilon_m=O(1);$$

(iii) $$\sum_{n=1}^{m}\mu_n\Delta\varepsilon_n=O(1);$$

and

(iv) $$\sum_{n=1}^{m} n\mu_n \Delta^2 \varepsilon_n = O(1)$$

as $m \to \infty$.

This Lemma is a special case of certain more general results due to Mazhar [2].

5. Proof of the theorem. Let

$$T_n = V_{n+1}(\lambda; \varepsilon_n) - V_n(\lambda; \varepsilon_n)$$

where $V_n(\lambda; \varepsilon_n)$ is the n-th de la Vallée Poussin means of the series $\sum a_n \varepsilon_n / \gamma_n$. Then to prove the theorem it is sufficient to show that

$$\sum_{n=1}^{\infty} \lambda_n^{k-1} |T_n|^k < \infty.$$

Let $\sum'$ be the summation over all n satisfying $\lambda_{n+1} = \lambda_n$; and $\sum''$ the summation over all n where $\lambda_{n+1} > \lambda_n$. We have

$$T_n = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{\nu = n - \lambda_n + 2}^{n+1} [(\lambda_{n+1} - \lambda_n)(\nu - n - 1) + \lambda_n] \frac{a_\nu \varepsilon_\nu}{\gamma_\nu}$$

when $\lambda_{n+1} = \lambda_n$, then we have

$$\begin{aligned}
T_n &= \frac{1}{\lambda_{n+1}} \sum_{\nu = n - \lambda_n + 2}^{n+1} \frac{a_\nu \varepsilon_\nu}{\gamma_\nu} \\
&= \frac{1}{\lambda_{n+1}} \left[\sum_{\nu = n - \lambda_n + 2}^{n} s_\nu \Delta\left(\frac{\varepsilon_\nu}{\gamma_\nu}\right) + \frac{s_{n+1}\varepsilon_{n+1}}{\gamma_{n+1}} - \frac{s_{n-\lambda_n+1}\varepsilon_{n-\lambda_n+2}}{\gamma_{n-\lambda_n+2}} \right] \\
&= \frac{1}{\lambda_n} \sum_{\nu = n - \lambda_n + 2}^{n} s_\nu \frac{\Delta \varepsilon_\nu}{\gamma_\nu} + \frac{1}{\lambda_n} \sum_{\nu = n - \lambda_n + 2}^{n} s_\nu \varepsilon_{\nu+1} \Delta\left(\frac{1}{\gamma_\nu}\right) + \frac{s_{n+1}\varepsilon_{n+1}}{\lambda_{n+1}\gamma_{n+1}} - \frac{s_{n-\lambda_n+1}\varepsilon_{n-\lambda_n+2}}{\lambda_{n-\lambda_n+1}\gamma_{n-\lambda_n+2}} \\
&= L_1^{(n)} + L_2^{(n)} + L_3^{(n)} + L_4^{(n)}, \qquad \text{say.}
\end{aligned}$$

By Minkowski's inequality it is therefore, sufficient to prove that

$$\sum{}' \lambda_n^{k-1} |L_r^{(n)}|^k < \infty \qquad \text{for } r = 1, 2, 3, 4.$$

$$\begin{aligned}
\sum{}' \lambda_n^{k-1} |L_1^{(n)}|^k &= \sum{}' \frac{1}{\lambda_n} \left| \sum_{\nu = n - \lambda_n + 2}^{n} s_\nu \frac{\Delta \varepsilon_\nu}{\gamma_\nu} \right|^k \\
&\leqq \sum{}' \frac{1}{\lambda_n} \left\{ \sum_{\nu = n - \lambda_n + 2}^{n} |s_\nu| \frac{\Delta \varepsilon_\nu}{\gamma_\nu} \right\}^k \\
&\leqq \sum{}' \frac{1}{\lambda_n} \left\{ \sum_{\nu = n - \lambda_n + 2}^{n} |s_\nu|^k \frac{\Delta \varepsilon_\nu}{\gamma_\nu} \right\} \left\{ \sum_{\nu = n - \lambda_n + 2}^{n} \frac{\Delta \varepsilon_\nu}{\gamma_\nu} \right\}^{k/k'} \\
&= O(1) \sum{}' \frac{1}{\lambda_n} \sum_{\nu = n - \lambda_n + 2}^{n} |s_\nu|^k \frac{\Delta \varepsilon_\nu}{\gamma_\nu} \\
&= O(1) \sum_{\nu=1}^{\infty} |s_\nu|^k \frac{\Delta \varepsilon_\nu}{\gamma_\nu} \sum_{n=\nu}^{\nu + \lambda_\nu - 1} \frac{1}{\lambda_n} \\
&= O(1) \sum_{\nu=1}^{\infty} |s_\nu|^k \frac{\Delta \varepsilon_\nu}{\gamma_\nu}.
\end{aligned}$$

Now,

$$\sum_{\nu=1}^{m+1} \frac{|s_\nu|^k}{\lambda_\nu} \cdot \frac{\lambda_\nu \Delta\varepsilon_\nu}{\lambda_\nu} = \sum_{\nu=1}^{m} \left(\sum_{r=1}^{\nu} \frac{|s_r|^k}{\lambda_r} \right) \Delta\left(\frac{\lambda_\nu}{\gamma_\nu} \Delta\varepsilon_\nu \right) + \frac{\lambda_{m+1}\Delta\varepsilon_{m+1}}{\gamma_{m+1}} \sum_{\nu=1}^{m+1} \frac{|s_\nu|^k}{\lambda_\nu}$$

$$= O(1)\left[\sum_{\nu=1}^{m} \mu_\nu \gamma_\nu \frac{\lambda_\nu}{\gamma_\nu} \Delta^2\varepsilon_\nu + \sum_{\nu=1}^{m} \lambda_\nu \mu_\nu \gamma_\nu \Delta\varepsilon_{\nu+1} \Delta\left(\frac{1}{\gamma_\nu} \right) + \frac{\lambda_{m+1}\Delta\varepsilon_{m+1}}{\gamma_{m+1}} \mu_{m+1}\gamma_{m+1} \right]$$

$$= O(1)\left[\sum_{\nu=1}^{m} \nu\mu_\nu \Delta^2\varepsilon_\nu + \sum_{\nu=1}^{m} \Delta\varepsilon_\nu + (m+1)\mu_{m+1}\Delta\varepsilon_{m+1} \right]$$

$$= O(1)$$

as $m \to \infty$, by virtue of the Lemma. Hence

$$\sum{}' \lambda_n^{k-1} |L_1^{(n)}|^k = O(1).$$

$$\sum{}' \lambda_n^{k-1} |L_2^{(n)}|^k = \sum{}' \frac{1}{\lambda_n} \left| \sum_{\nu=n-\lambda_n+2}^{n} s_\nu \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \right|^k$$

$$\leqq \sum{}' \frac{1}{\lambda_n} \left\{ \sum_{\nu=n-\lambda_n+2}^{n} |s_\nu|^k \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \right\} \left\{ \sum_{\nu=n-\lambda_n+2}^{n} \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \right\}^{k/k'}$$

$$= O(1)\left[\sum{}' \frac{1}{\lambda_n} \sum_{\nu=n-\lambda_n+2}^{n} |s_\nu|^k \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \right]$$

$$= O(1)\left[\sum_{\nu=1}^{\infty} |s_\nu|^k \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \sum_{n=\nu}^{\nu+\lambda_\nu-1} \frac{1}{\lambda_n} \right]$$

$$= O(1)\left[\sum_{\nu=1}^{\infty} |s_\nu|^k \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \right]$$

$$= O(1)\left[\sum_{\nu=1}^{\infty} \frac{|s_\nu|^k}{\lambda_\nu} \cdot \frac{\varepsilon_\nu}{\mu_\nu \gamma_\nu} \lambda_\nu \mu_\nu \gamma_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) \right]$$

$$= O(1)\left[\sum_{\nu=1}^{\infty} \frac{|s_\nu|^k}{\lambda_\nu} \cdot \frac{\varepsilon_\nu}{\mu_\nu \gamma_\nu} \right].$$

Now,

$$\sum_{\nu=1}^{m+1} \frac{|s_\nu|^k}{\lambda_\nu} \cdot \frac{\varepsilon_\nu}{\mu_\nu \gamma_\nu} = \sum_{\nu=1}^{m} \left(\sum_{r=1}^{\nu} \frac{|s_r|^k}{\lambda_r} \right) \Delta\left(\frac{\varepsilon_\nu}{\mu_\nu \gamma_\nu} \right) + \frac{\varepsilon_{m+1}}{\mu_{m+1}\gamma_{m+1}} \sum_{\nu=1}^{m+1} \frac{|s_\nu|^k}{\lambda_\nu}$$

$$= O(1)\left[\sum_{\nu=1}^{m} \mu_\nu \gamma_\nu \Delta\left(\frac{\varepsilon_\nu}{\mu_\nu \gamma_\nu} \right) + \frac{\varepsilon_{m+1}}{\mu_{m+1}\gamma_{m+1}} \mu_{m+1}\gamma_{m+1} \right]$$

$$= O(1)\left[\sum_{\nu=1}^{m} \mu_\nu \gamma_\nu \frac{\Delta\varepsilon_\nu}{\mu_\nu \gamma_\nu} + \sum_{\nu=1}^{m} \mu_\nu \gamma_\nu \varepsilon_{\nu+1} \Delta\left(\frac{1}{\mu_\nu \gamma_\nu} \right) \right] + O(1)$$

$$= O(1)\left[\sum_{\nu=1}^{m} \mu_\nu \gamma_\nu \varepsilon_\nu \frac{1}{\mu_\nu} \Delta\left(\frac{1}{\gamma_\nu} \right) + \sum_{\nu=1}^{m} \mu_\nu \gamma_\nu \varepsilon_\nu \frac{1}{\gamma_{\nu+1}} \Delta\left(\frac{1}{\mu_\nu} \right) \right]$$

$$= O(1)\left[\sum_{\nu=1}^{m} \frac{\varepsilon_\nu}{\lambda_\nu \mu_\nu} \lambda_\nu \mu_\nu \gamma_\nu \Delta\left(\frac{1}{\gamma_\nu} \right) + \sum_{\nu=1}^{m} \varepsilon_\nu \mu_\nu \Delta\left(\frac{1}{\mu_\nu} \right) \right]$$

$$= O(1)\left[\sum_{\nu=1}^{m} \frac{\varepsilon_\nu}{\lambda_\nu \mu_\nu} + \sum_{\nu=1}^{m} \varepsilon_\nu \mu_\nu \Delta\left(\frac{1}{\mu_\nu} \right) \right]$$

by hypotheses and by virtue of the following

$$\sum_{\nu=1}^{m} \varepsilon_\nu \mu_\nu \Delta\left(\frac{1}{\mu_\nu}\right) = \sum_{\nu=1}^{m} \varepsilon_\nu \Delta\left(\frac{1}{\mu_\nu}\right) \sum_{n=1}^{\nu} \frac{1}{\lambda_n}$$

$$= \sum_{n=1}^{m} \frac{1}{\lambda_n} \sum_{\lambda=n}^{m} \varepsilon_\nu \Delta\left(\frac{1}{\mu_\nu}\right)$$

$$\leqq \sum_{n=1}^{m} \frac{\varepsilon_n}{\lambda_n} \sum_{\nu=n}^{m} \Delta\left(\frac{1}{\mu_\nu}\right)$$

$$= O(1)\left[\sum_{n=1}^{m} \frac{\varepsilon_n}{\lambda_n}\right]$$

$$= O(1)$$

as $m \to \infty$. Therefore,

$$\sum{}' \lambda_n^{k-1} |L_2^{(n)}|^k < \infty,$$

$$\sum{}' \lambda_n^{k-1} |L_3^{(n)}|^k + \sum{}' \lambda_n^{k-1} |L_4^{(n)}|^k = O(1) \sum_{n=1}^{\infty} \frac{|s_n|^k}{\lambda_n} \cdot \frac{\varepsilon_n}{\gamma_n}.$$

Now,

$$\sum_{n=1}^{m+1} \frac{|s_n|^k}{\lambda_n} \cdot \frac{\varepsilon_n}{\gamma_n} = \sum_{n=1}^{m} \left(\sum_{\nu=1}^{n} \frac{|s_\nu|^k}{\lambda_\nu}\right) \Delta\left(\frac{\varepsilon_n}{\gamma_n}\right) + \frac{\varepsilon_{m+1}}{\gamma_{m+1}} \sum_{n=1}^{m+1} \frac{|s_\nu|^k}{\lambda_\nu}$$

$$= O(1)\left[\sum_{n=1}^{m} \mu_n \gamma_n \frac{\Delta \varepsilon_n}{\gamma_n} + \sum_{n=1}^{m} \mu_n \gamma_n \varepsilon_{n+1} \Delta\left(\frac{1}{\gamma_n}\right) + \varepsilon_{m+1} \mu_{m+1}\right]$$

$$= O(1)\left[\sum_{n=1}^{m} \mu_n \Delta \varepsilon_n + \sum_{n=1}^{m} \frac{\varepsilon_n}{\lambda_n} \lambda_n \mu_n \gamma_n \Delta\left(\frac{1}{\gamma_n}\right) + \varepsilon_{m+1} \mu_{m+1}\right]$$

$$= O(1)$$

as $m \to \infty$, by hypotheses and by virtue of the Lemma. Hence,

$$\sum{}' \lambda_n^{k-1} |L_3^{(n)}|^k + \sum{}' \lambda_n^{k-1} |L_4^{(n)}|^k = O(1).$$

Therefore,

$$\sum{}' \lambda_n^{k-1} |T_n|^k = O(1).$$

When $\lambda_{n+1} > \lambda_n$, then we have

$$|T_n| = \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{\nu=n-\lambda_n+2}^{n+1} (\lambda_n + \nu - n - 1) \frac{a_\nu \varepsilon_\nu}{\gamma_\nu} \right|$$

$$= \frac{1}{\lambda_n \lambda_{n+1}} \left| \sum_{\nu=n-\lambda_n+2}^{n} s_\nu \Delta\left\{(\lambda_n + \nu - n - 1) \frac{\varepsilon_\nu}{\gamma_\nu}\right\} + \frac{s_{n+1} \lambda_n \varepsilon_{n+1}}{\gamma_{n+1}} - \frac{s_{n-\lambda_n+1} \varepsilon_{n-\lambda_n+2}}{\gamma_{n-\lambda_n+2}} \right|$$

$$\leqq \frac{1}{\lambda_n^2} \sum_{\nu=n-\lambda_n+2}^{n} |s_\nu| \left| \Delta\left\{(\lambda_n + \nu - n - 1) \frac{\varepsilon_\nu}{\gamma_\nu}\right\} \right| + \frac{|s_{n+1}| \varepsilon_{n+1}}{\lambda_{n+1} \gamma_{n+1}} - \frac{|s_{n-\lambda_n+1}| \varepsilon_{n-\lambda_n+1}}{\lambda_n \lambda_{n-\lambda_n+1} \gamma_{n-\lambda_n+1}}$$

$$= M_1^{(n)} + M_2^{(n)} + M_3^{(n)}, \qquad \text{say.}$$

By Minkowski's inequality it is therefore, sufficient to prove that

$$\sum{}'' \lambda_n^{k-1} |M_k^{(n)}|^k < \infty, \qquad \text{for} \quad r=1,2,3.$$

Since

$$\left| \Delta\left\{(\lambda_n+\nu-n-1)\frac{\varepsilon_\nu}{\gamma_\nu}\right\}\right| \leqq \lambda_\nu \Delta\left(\frac{\varepsilon_\nu}{\gamma_\nu}\right)+\frac{\varepsilon_\nu}{\gamma_\nu},$$

we have

$$\sum{}'' \lambda_n^{k-1} |M_1^{(n)}|^k = O(1)\left[\sum{}'' \frac{1}{\lambda_n^{k-1}}\left\{\sum_{\nu=n-\lambda_n+2}^{n} |s_\nu| \lambda_\nu \Delta\left(\frac{\varepsilon_\nu}{\gamma_\nu}\right)\right\}^k + \sum{}'' \frac{1}{\lambda_n^{k+1}}\left\{\sum_{\nu=n-\lambda_n+2}^{n} |s_\nu| \frac{\varepsilon_\nu}{\gamma_\nu}\right\}^k\right]$$

$$= \sum{}_1'' + \sum{}_2'', \qquad \text{say.}$$

$$\sum{}_{11}'' = \sum{}'' \frac{1}{\lambda_n^{k+1}}\left\{\sum_{\nu=n-\lambda_n+2}^{n} |s_\nu| \lambda_\nu \frac{\Delta\varepsilon_\nu}{\gamma_\nu}\right\}^k$$

$$= O(1) \sum{}'' \frac{1}{\lambda_n^{k+1}}\left\{\sum_{\nu=n-\lambda_n+2}^{n} |s_\nu|^k \lambda_\nu \frac{\Delta\varepsilon_\nu}{\gamma_\nu}\right\}\left\{\sum_{\nu=n-\lambda_n+2}^{n} \lambda_\nu \frac{\Delta\varepsilon_\nu}{\gamma_\nu}\right\}^{k/k'}$$

$$= O(1) \sum \frac{1}{\lambda_n^2} \sum_{\nu=n-\lambda_n+2}^{n} |s_\nu|^k \lambda_\nu \frac{\Delta\varepsilon_\nu}{\gamma_\nu}$$

$$= O(1) \sum_{\nu=1}^{\infty} |s_\nu|^k \lambda_\nu \frac{\Delta\varepsilon_\nu}{\gamma_\nu} \sum_{n\geqq k}{}'' \frac{1}{\lambda_n^2}$$

$$= O(1) \sum_{\nu=1}^{\infty} |s_\nu|^k \frac{\Delta\varepsilon_\nu}{\gamma_\nu}$$

$$= O(1)$$

as proved earlier.

$$\sum{}_{12}'' = \sum{}'' \frac{1}{\lambda_n^{k+1}}\left\{\sum_{\nu=n-\lambda_n+2}^{n} |s_\nu| \lambda_\nu \frac{\Delta\varepsilon_\nu}{\gamma_\nu}\right\}^k$$

$$= O(1) \sum{}'' \frac{1}{\lambda_n^{k+1}}\left[\sum_{\nu=n-\lambda_n+2}^{n} |s_\nu|^k \lambda_\nu \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu}\right)\right]\left[\sum_{\nu=n-\lambda_n+2}^{n} \lambda_\nu \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu}\right)\right]^{k/k'}$$

$$= O(1) \sum{}'' \frac{1}{\lambda_n^2} \sum_{\nu=n-\lambda_n+2}^{n} |s_\nu|^k \lambda_\nu \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu}\right)$$

$$= O(1) \sum_{\nu=1}^{\infty} |s_\nu|^k \lambda_\nu \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu}\right) \sum_{n\geqq \nu}{}'' \frac{1}{\lambda_n^2}$$

$$= O(1) \sum_{\nu=1}^{\infty} |s_\nu|^k \varepsilon_\nu \Delta\left(\frac{1}{\gamma_\nu}\right)$$

$$= O(1)$$

as already proved.

Hence,

$$\sum{}_1'' = O(1)[\sum{}_{11}'' + \sum{}_{12}''] = O(1),$$

$$\begin{aligned}\sum\nolimits_2'' &= O(1)\sum'' \frac{1}{\lambda_n^{k+1}}\left(\sum_{\nu=n-\lambda_n+2}^{n}|s_\nu|^k \frac{\varepsilon_\nu}{\gamma_\nu}\right)\left(\sum_{\nu=n-\lambda_n+2}^{n}\frac{\varepsilon_\nu}{\gamma_\nu}\right)^{k/k'}\\ &= O(1)\sum''\frac{1}{\lambda_n^{k+1}}\sum_{\nu=n-\lambda_n+2}^{n}|s_\nu|^k\frac{\varepsilon_\nu}{\gamma_\nu}\lambda_n^{k/k'}\frac{\varepsilon_{n-\lambda_n+2}^{k/k'}}{\gamma_{n-\lambda_n+2}^{k/k'}}\\ &= O(1)\sum''\frac{1}{\lambda_n^2}\sum_{\nu=n-\lambda_n+2}^{n}|s_\nu|^k\frac{\varepsilon_\nu}{\gamma_\nu}\\ &= O(1)\sum_{\nu=1}^{\infty}|s_\nu|^k\frac{\varepsilon_\nu}{\gamma_\nu}\sum_{n\geqq\nu}''\frac{1}{\lambda_n^2}\\ &= O(1)\sum_{\nu=1}^{\infty}\frac{|s_\nu|^k}{\lambda_\nu}\cdot\frac{\varepsilon_\nu}{\gamma_\nu}\\ &= O(1).\end{aligned}$$

Therefore,

$$\sum'' \lambda_n^{k-1}|M_1^{(n)}|^k = O(1).$$

Also,

$$\sum'' \lambda_n^{k-1}|M_2^{(n)}|^k = O(1)$$

as proved in the previous case for L_2.

$$\begin{aligned}\sum'' \lambda_n^{k-1}|M_2^{(n)}|^k &= O(1)\sum_{n=1}^{\infty}\frac{|s_n|^k\varepsilon_n}{\lambda_n\gamma_n}\\ &= O(1).\end{aligned}$$

This completes the proof of the theorem.

The author would like to express his warmest thanks to Dr. S. M. Mazhar for his kind encouragement and helpful suggestions.

References

[1] Leindler, L., On the absolute summability factors of Fourier series. Acta Sci. Math. Szeged **28** (1967), 323-336.

[2] Mazhar, S. M., On the summability factors of infinite series. Publicationes Mathematicae **13** (1966), 229-236.

[3] Mazhar, S. M., On $|C, 1|_k$ summability factors of infinite series. Acta Sci. Math. Szeged **27** (1966), 67-70.

[4] Misra, B. P., On the absolute Cesàro summability factors of infinite series. Rend. Circ. Matem. Palermo (2) **14** (1965), 189-193.

[5] Niranjan Singh, Ph. D. Thesis, Aligarh Muslim University (1967).

[6] Pati, T., Absolute Cesàro summability factors of infinite series. Math. Zeits. **78** (1962), 293-297.

[7] Prasad, B. N., and S. N. Bhatt, The summability factors of a Fourier series. Duke Math. Journ. **24** (1957), 103-120.

[8] Umar, S., Ph. D. Thesis, Aligarh Muslim University (1967).

[9] VARSHNEY, R. G. On $|V, \lambda|$ summability factors of infinite series, I. Under communication.

[10] VARSHNEY, R. G., On $|V, \lambda|$ summability factors of infinite series, II. Under communication.

[11] VARSHNEY, R. G., On $|V, \lambda|_k$ summability factors of infinite series. Under communication.

[12] ZYGMUND, A., Trigonometric Series, I (1935).

DEPARTMENT OF MATHEMATICS AND STATISTICS,
ALIGARH MUSLIM UNIVERSITY, ALIGARH, INDIA.