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ON THE CHARACTERISTIC OF AN ALGEBROID FUNCTION

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Let $f(z)$ be an n-valued transcendential algebroid function in $|z|<\infty$ defined by an irreducible equation

$$F(z, f)\equiv A_n(z)f^n+A_{n-1}(z)f^{n-1}+\cdots+A_0(z)=0,$$

where the coefficients $A_0, \cdots, A_n$ are entire functions without any common zeros. We set

$$A(z)=\max(|A_0|, \cdots, |A_n|).$$

Let $\mu(r, A)$ be defined by

$$\mu(r, A)=\frac{1}{2n\pi}\int_0^{2\pi}\log A(re^{i\theta})d\theta.$$

Recently Ozawa [1] obtained

LEMMA. *Suppose that there is at least one index j satisfying*

$$m\left(r, \frac{1}{A_j}\right)\leqq cm(r, A), \qquad c<1,$$

then

$$(1-c)m(r, A)\leqq n\mu(r, A)\leqq m(r, A).$$

In connection with this lemma he proposed the following problem.

Are there any algebroid functions satisfying

$$(1) \qquad \varliminf_{r\to\infty}\frac{n\mu(r, A)}{m(r, A)}=0?$$

In this note using Ozawa's method we construct a two-valued algebroid function satisfying (1).

In the first place we consider

$$h(x)=\frac{(\log x)^{\rho}}{x},$$

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where $\rho>0$. $h(x)$ is a strictly decreasing function in $x>x_0>e$. Let r_1 be a real number such that

$$r_1>x_0>e, \qquad (\log r_1)^{\rho}>2.$$

We suppose that the real numbers $r_1<r_2<\cdots<r_n$ have been defined. Then we choose r_{n+1} such that

$$h(r_{n+1})=\frac{1}{n^{\rho}r_n}. \tag{2}$$

By this process we get an increasing sequence $\{r_n\}$ $(n=1, 2, \cdots)$, satisfying (2). We set

$$N_1=[1\cdot\log r_1],$$

where $[x]$ denotes the greatest integer not larger than x. Suppose that the numbers $N_1<N_2<\cdots<N_n$ have already been defind and let

$$S_1=1, \qquad S_{n+1}=\sum_{\nu=1}^{n} N_{\nu} \qquad (n\geqq 1).$$

Then we define

$$N_{n+1}=[(n+1)S_{n+1}\log r_{n+1}]. \tag{3}$$

Thus we have an increasing sequence $\{N_n\}$ $(n=1, 2, \cdots)$. Now for a positive number λ

$$\frac{N_n}{r_n^{\lambda}}\Big/\frac{N_{n+1}}{r_{n+1}^{\lambda}}=\left(\frac{r_{n+1}}{r_n}\right)^{\lambda}\frac{nS_n\log r_n}{(n+1)S_{n+1}\log r_{n+1}}(1+o(1))$$

$$=\frac{n^{\lambda\rho}(\log r_{n+1})^{\lambda\rho}}{n\log r_{n+1}}(1+o(1)) \qquad (n\to\infty).$$

Therefore the series

$$\sum_{n=1}^{\infty}\frac{N_n}{(3r_n/2)^{\lambda}}=\left(\frac{2}{3}\right)^{\lambda}\sum_{n=1}^{\infty}\frac{N_n}{r_n^{\lambda}}$$

is convergent if $\lambda>1/\rho$ and divergent if $\lambda<1/\rho$. For $\rho>1$ let $g(z)$ be

$$\prod_{n=1}^{\infty}\left(1+\frac{2}{3}\frac{z}{r_n}\right)^{N_n}. \tag{4}$$

By the above result $g(z)$ has the order $1/\rho$. For the zeros of $g(z)$ we get

$$\frac{n(r_n, 0)\log r_n}{n(2r_n, 0)}=\frac{n(r_n, 0)\log r_n}{n(2r_n, 0)-n(r_n, 0)+n(r_n, 0)}$$

$$=\frac{1}{n}(1+o(1)) \qquad (n\to\infty),$$

and by Shea's result [2, p. 226] we obtain

$$(5)\qquad \lim_{n\to\infty}\frac{N(r_n, 0, g)}{m(r_n, g)}=0, \qquad \lim_{n\to\infty}\frac{m(r_n, 1/g)}{m(r_n, g)}=1.$$

Now we set

$$(6)\qquad g_1(z)=\sum_{n=1}^{\infty}\left(1+\frac{z}{3r_n/2-2/3r_n}\right)^{N_n}.$$

$g_1(z)$ has the same order as $g(z)$. Setting $3r_n/2=a_n$ we have for $z=r_ne^{i\theta}$

$$\begin{aligned}\left|\frac{g(z)}{g_1(z)}\right| &= \prod_{\nu=1}^{\infty}\left|\frac{1+z/a_\nu}{1+z/(a_\nu-a_\nu^{-1})}\right|^{N_\nu}\\ &=\prod_{\nu=1}^{\infty}\left|\frac{1+z/a_\nu}{1+z/a_\nu-1/a_\nu^2}\right|^{N_\nu}\prod_{\nu=1}^{\infty}\left(1-\frac{1}{a_\nu^2}\right)^{N_\nu}\\ &=C_1\prod_{\nu=1}^{\infty}\frac{1}{|1-1/a_\nu(a_\nu+z)|^{N_\nu}},\end{aligned}$$

where

$$C_1=\prod_{\nu=1}^{\infty}\left(1-\frac{1}{a_\nu^2}\right)^{N_\nu}$$

is a positive constant. Further

$$\left|1-\frac{1}{a_\nu(a_\nu+z)}\right|^{N_\nu}\leqq\left(1-\frac{1}{a_\nu(a_\nu+r_n)}\right)^{N_\nu}\leqq\left(1-\frac{1}{a_\nu^2 r_n}\right)^{N_\nu},$$

$$\left|1-\frac{1}{a_\nu(a_\nu+z)}\right|^{N_\nu}\geqq\left|1-\frac{1}{a_\nu|a_\nu-r_n|}\right|^{N_\nu}\geqq\left(1-\frac{1}{a_\nu}\right)^{N_\nu}.$$

Thus

$$C_2=\left\{\prod_{\nu=1}^{\infty}\left(1-\frac{1}{a_\nu}\right)^{N_\nu}\right\}^{-1}\geqq\prod_{\nu=1}^{\infty}\frac{1}{|1-1/a_\nu(a_\nu+z)|^{N_\nu}}\geqq\left\{\prod_{\nu=1}^{\infty}\left(1-\frac{1}{a_\nu^2 r_n}\right)^{N_\nu}\right\}^{-1}.$$

Hence C_2 is a positive constant and the right hand side converges to 1 as $n\to\infty$. Hence we can find n_0 such that for $n\geqq n_0$

$$\infty>C_1\cdot C_2\geqq\left|\frac{g(r_ne^{i\theta})}{g_1(r_ne^{i\theta})}\right|\geqq\frac{C_1}{2}>0.$$

Then we obtain

$$A(r_ne^{i\theta})=\max\left(|g(r_ne^{i\theta})|, |g_1(r_ne^{i\theta})|\right)\leqq K|g(r_ne^{i\theta})|,$$

where $K(>1)$ is a positive constant. By this estimate we have

$$(7)\qquad m(r_n, A)\leqq m(r_n, g)+K$$

and

$$
\begin{aligned}
m\left(r_n, \frac{1}{A}\right) &\geqq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{K|g(r_n e^{i\theta})|}\, d\theta \\
&\geqq m\left(r_n, \frac{1}{g}\right) - \log K - \log 2K.
\end{aligned} \tag{8}
$$

Finally consider the equation

$$
g_1(z) f^2 + g(z) f + g(z) = 0. \tag{9}
$$

For this two-valued algebroid function f, whose order is $1/\rho$ $(1<\rho)$,

$$
2\mu(r_n, A) = m(r_n, A) - m\left(r_n, \frac{1}{A}\right).
$$

By (7) and (8)

$$
\begin{aligned}
\frac{2\mu(r_n, A)}{m(r_n, A)} &\leqq 1 - \frac{m(r_n, 1/g) - 2\log 2K}{m(r_n, g) + K} \\
&= 1 - \frac{m(r_n, 1/g)}{m(r_n, g)}(1 + o(1)) \qquad (n \to \infty).
\end{aligned}
$$

Thus by (5)

$$
\varliminf_{r\to\infty} \frac{2\mu(r, A)}{m(r, A)} \leqq \lim_{n\to\infty} \frac{2\mu(r_n, A)}{m(r_n, A)} = 0.
$$

This is the desired result.

Remark. If we take $r_n^2 = r_{n+1}$ and $N_{n+1} = S_{n+1}(\log r_{n+1})^2$ for (2) and (3) respectively, $g(z)$ and $g_1(z)$ defined by (4), (6), with these r_n, N_n, have the same order 0. Then we have

$$
\lim_{n\to\infty} \frac{n(r_n, 0, g)\log r_n}{n(2r_n, 0, g)} = 0 \qquad \text{and} \quad (5).
$$

Moreover the above arguments remain for those $g(z)$ and $g_1(z)$. Hence if we use those $g(z)$, $g_1(z)$ in (9), we get a two-valued algebroid function of the order zero, which satisfies (2).

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References

[1] Ozawa, M., Deficiencies of an algebroid function. Kōdai Math. Sem. Rep. **21** (1969), 262–276.

[2] Shea, D. F., On the Valiron deficiencies of meromorphic functions of finite order. Trans. Amer. Math. Soc. **124** (1966), 201–227.

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