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DEFICIENCIES OF AN ALGEBROID FUNCTION

BY MITSURU OZAWA

1. Let $f(z)$ be an n-valued transcendental algebroid function in $|z|<\infty$ defined by an irreducible equation

$$F(z,f)\equiv A_n(z)f^n+A_{n-1}(z)f^{n-1}+\cdots+A_0(z)=0,$$

where the coefficients $A_0,\cdots,A_n$ are entire functions without any common zeros.

It is well known that if a transcendental meromorphic function f has two full deficient values a_1, a_2, that is, $\delta(a_j,f)=1$, $j=1,2$, then f is either of positive integral order or of infinite order.

In this note we shall be concerned with an extension of the above fact to algebroid functions. The problem is the following: Does $\delta(a_k,f)=1$ for $k=1,\cdots,n+1$ imply the positive integrity of order of f? We could not answer to this question in any way.

In general, $n+1$, the number of full deficient values, cannot be improved. This is evident for the one-valued case. We now construct a two-valued transcendental algebroid function having a further property. Let g be an entire function of finite order λ satisfying $\Delta(0,g)=1$ with the Valiron deficiency Δ. The existence of such functions for any given order λ was shown in [3]. Let $f(z)$ be a two-valued entire algebroid function, which is defined by

$$F(z,f)\equiv f^2+gf-1=0.$$

Evidently $\delta(0,f)=\delta(\infty,f)=1$. Further we have $F(z,1)=g$, $F(z,-1)=-g$. Hence by Valiron's theorem, which is listed as Lemma 1 below,

$$\varliminf_{r\to\infty}\frac{N(r;\pm1,f)}{T(r,f)}=\varliminf_{r\to\infty}\frac{N(r;0,g)}{m(r,g)}=1-\Delta(0,g)=0.$$

This shows that $\Delta(\pm1,f)=1$. If λ is finite but is not any positive integer, then f does not have any full Nevanlinna deficient value, which is shown by Theorem 1 below.

2. Lemmas.

LEMMA 1. Valiron [4]. *Let* $A(z)$ *be* $\max(|A_0|,\cdots,|A_n|)$. *Let* $\mu(r,A)$ *be*

$$\frac{1}{2n\pi}\int_0^{2\pi}\log A(re^{i\theta})d\theta.$$

Then

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$$|T(r,f)-\mu(r,A)|=O(1).$$

LEMMA 2. Selberg [2]. *There are at most $2n$ values a_k for which*

$$\delta(a_k,f)=1.$$

LEMMA 3. *Suppose that*

$$B_\nu(z)=\sum_{j=0}^{n} a_\nu^j A_j(z), \qquad \nu=1,\cdots,n+1,$$

then

$$\mu(r,B)=\mu(r,A)+O(1),$$
$$B=\max(|B_1|,\cdots,|B_{n+1}|).$$

Proof. By the given equations

$$|B_\nu|\leqq\sum_{j=0}^{n}|a_\nu|^j|A_j|\leqq c_\nu\max(|A_j|)=c_\nu A$$

and hence

$$B\leqq cA, \qquad c=\max(c_\nu).$$

This implies that

$$\mu(r,B)\leqq\mu(r,A)+O(1).$$

By solving the given equations we have

$$A_j=\sum_{\nu=1}^{n+1}\alpha_{j\nu}B_\nu,$$

which implies similarly

$$\mu(r,A)\leqq\mu(r,B)+O(1).$$

LEMMA 4.

$$\{1-\delta(0,F(z,a))\}\varliminf_{r\to\infty}\frac{m(r,F(z,a))}{n\mu(r,A)}\leqq 1-\delta(a,f).$$

Proof.

$$\varlimsup_{r\to\infty}\frac{N(r;0,F(z,a))}{m(r,F(z,a))}\varliminf_{r\to\infty}\frac{m(r,F(z,a))}{n\mu(r,A)}$$
$$\leqq\varlimsup_{r\to\infty}\frac{N(r;0,F(z,a))}{n\mu(r,A)}=\varlimsup_{r\to\infty}\frac{N(r;a,f)}{\mu(r,A)}.$$

This implies the desired result.

LEMMA 5. *Suppose that there is at least one index j satisfying*

$$m\left(r,\frac{1}{A_j}\right)\leqq cm(r,A), \qquad c<1,$$

then

$$(1-c)m(r, A) \leqq n\mu(r, A) \leqq m(r, A).$$

Proof. Evidently we have $m(r, A)-m(r, 1/A)=n\mu(r, A)$. Let E be the set on $|z|=r$ in which $A<1$. Then

$$m\left(r, \frac{1}{A}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{A}d\theta=\frac{1}{2\pi}\int_E \log\frac{1}{A}d\theta$$
$$\leqq \frac{1}{2\pi}\int_E \log\frac{1}{|A_\mu|}d\theta \leqq m\left(r, \frac{1}{A_\mu}\right)$$

for every μ. Hence we have

$$m\left(r, \frac{1}{A}\right) \leqq \min_{0\leqq\mu\leqq n} m\left(r, \frac{1}{A_\mu}\right) \leqq m\left(r, \frac{1}{A_j}\right) \leqq cm(r, A),$$

which implies

$$(1-c)m(r, A) \leqq n\mu(r, A).$$

The second half is evident.

3. In connection with Lemma 5 there happens another problem. Are there any algebroid functions satisfying

$$\varliminf_{r\to\infty}\frac{n\mu(r, A)}{m(r, A)}=0?$$

Now we construct a two-valued algebroid function satisfying

$$\varliminf_{r\to\infty}\frac{2\mu(r, A)}{m(r, A)}<\varepsilon$$

for a given positive number ε.

Let a_{jk} be

$$2^{10^j/\rho}+\frac{k}{2^{10^j}}, \qquad \begin{matrix} j=0, 1, \cdots, \\ k=0, 1, \cdots, 2^{10^j}-1, \end{matrix}$$

with a fixed ρ satisfying $0<\rho<9/10$. Let $g(z)$ be

$$\prod_{j=0}^{\infty}\prod_{k=0}^{2^{10^j}-1}\left(1+\frac{z}{a_{jk}}\right)\equiv\prod_{j=0}^{\infty}h_j(z).$$

Let $n(r)$ be the number of zeros of $g(z)$ in $|z|\leqq r$. Then

$$n(r)=\begin{cases} 0, & \text{if } 0\leqq r<2^{1/\rho}, \\ 2+2^{10}+\cdots+2^{10^j}, & \text{if } 2^{10^j/\rho}+1-\dfrac{1}{2^{10^j}}\leqq r<2^{10^{j+1}/\rho}, \\ 2+2^{10}+\cdots+2^{10^j}+k+1, & \text{if } 2^{10^{j+1}/\rho}+\dfrac{k}{2^{10^{j+1}}}\leqq r<2^{10^{j+1}/\rho}+\dfrac{k+1}{2^{10^{j+1}}}. \end{cases}$$

$n(r)$ satisfies the following inequalities

$$2^{10^j-1}<n(r)<2^{10^j+1}$$

for $r_j < r \leqq r_{j+1}$, $r_j = 2^{10^j/\rho}$. Hence

$$\varlimsup_{r\to\infty} \frac{\log n(r)}{\log r} \leqq \varlimsup_{j\to\infty} \frac{\log 2^{10^j} + \log 2}{\log 2^{10^j/\rho}} = \varlimsup_{j\to\infty} \frac{10^j + 1}{10^j/\rho} = \rho.$$

Further

$$\varlimsup_{r\to\infty} \frac{\log n(r)}{\log r} \geqq \varlimsup_{m\to\infty} \frac{\log n(r_m+1)}{\log (r_m+1)} \geqq \varlimsup_{m\to\infty} \frac{10^m \log 2}{(1/\rho)10^m \log 2} = \rho.$$

Therefore the order of $n(r)$ is equal to ρ.

Next we prove

$$\varliminf_{r\to\infty} \frac{n(r)\log r}{n(2r)} = 0.$$

Take the sequence $\{r_m\}$. Then

$$\frac{n(r_m)\log r_m}{n(2r_m)} \leqq \frac{2^{10^{m-1}+1} 10^m (1/\rho) \log 2}{2^{10^m}} \to 0$$

as $m \to \infty$. This implies the desired fact.

This condition implies the following fact by Shea's reasoning [3].

$$\varDelta(0, g) = \varlimsup_{r\to\infty} \frac{m(r, 1/g)}{m(r, g)} = \varlimsup_{m\to\infty} \frac{m(r_m, 1/g)}{(r_m, g)} = 1.$$

Still we need further properties of $g(z)$. Let $g_\nu(z)$ be $g(z)/h_\nu(z)$. Then

$$\min_{|z|=r_\nu} |g_\nu(z)| \geqq c > 0$$

with an absolute constant c. In order to prove this we should firstly remark that the minimum of $|g_\nu(z)|$ on $|z|=r_\nu$ is attained at $z=-r_\nu$. Evidently on $|z|=r_\nu$

$$\prod_{l=0}^{\nu-1} |h_l(z)| \geqq \prod_{l=0}^{\nu-1} |h_l(-r_\nu)| > 1.$$

Now we shall consider

$$X = \prod_{l=\nu+1}^{\infty} |h_l(-r_\nu)|.$$

Taking its logarithm we have

$$\log X = \sum_{l=\nu+1}^{\infty} \log |h_l(-r_\nu)|.$$

On the other hand

$$|h_l(-r_\nu)| = \prod_{k=0}^{2^{10^l}-1} \left|1 - \frac{r_\nu}{r_l + k/2^{10^l}}\right| \geqq \left|1 - \frac{r_\nu}{r_l}\right|^{2^{10^l}}$$

for $l \geqq \nu+1$. Further for $l \geqq \nu+1$

$$\frac{r_\nu}{r_l} \leqq \frac{r_\nu}{r_{\nu+1}} < \frac{1}{2}.$$

Hence, by $\log(1-x) \geqq -2x$ for $0<x<1/2$,

$$\log X \geqq \sum_{l=\nu+1}^{\infty} 2^{10l} \log\left(1-\frac{r_\nu}{r_l}\right)$$

$$\geqq -2\sum_{l=\nu+1}^{\infty} 2^{10l}\frac{r_\nu}{r_l} \equiv -2U,$$

$$U \leqq \sum_{l=\nu+1}^{\infty} \frac{1}{2^{((1/\rho)(9/10)-1)10l}} \leqq \sum_{l=\nu+1}^{\infty} \frac{1}{((1/\rho)(9/10)-1)10^l}$$

$$\leqq \frac{100\rho}{(9-10\rho)9}.$$

Therefore

$$\log X \geqq -2\frac{100\rho}{(9-10\rho)9},$$

$$X \geqq \exp\left(-2\frac{100\rho}{(9-10\rho)9}\right) \equiv c>0.$$

This shows that

$$\min_{|z|=r_\nu} |g_\nu(z)| \geqq c>0.$$

Now we prove

$$\frac{2}{\pi}\int_{\pi-\delta/2}^{\pi} \log^+ \frac{1}{|h_\nu(r_\nu e^{i\theta})|} d\theta \leqq D(\delta) m(r_\nu, g),$$

where $D(\delta)$ is a constant being independent of ν and satisfies $D(\delta)\to 0$ when $\delta\to 0$. In order to prove this inequality we need

$$-\int_0^{\delta/2} \log|1-e^{-i\theta}+\tilde{A}_k e^{-i\theta}|\, d\theta$$

$$\leqq -\frac{\delta}{4}\log(2-2\tilde{A}_k)+\frac{1}{2}\int_0^{\delta/2}\log\frac{1}{1-\cos\theta}d\theta, \qquad \tilde{A}_k=\frac{k}{r_\nu^{\rho+1}+k}.$$

This is not so difficult to prove. Indeed

$$-\int_0^{\delta/2} \log|1-e^{-i\theta}+\tilde{A}_k e^{-i\theta}|\, d\theta$$

$$=-\frac{1}{2}\int_0^{\delta/2}\log\{(2-2\tilde{A}_k)(1-\cos\theta)+\tilde{A}_k^2\}d\theta$$

$$=-\frac{\delta}{4}\log(2-2\tilde{A}_k)-\frac{1}{2}\int_0^{\delta/2}\log\left(1-\cos\theta+\frac{\tilde{A}_k^2}{2-2\tilde{A}_k}\right)d\theta$$

$$\leqq -\frac{\delta}{4}\log(2-2\tilde{A}_k)+\frac{1}{2}\int_0^{\delta/2}\log\frac{1}{1-\cos\theta}d\theta.$$

Further we need

$$\int_0^{\delta/2} \log \frac{1}{1-\cos\theta}\,d\theta \leqq -\frac{\delta}{2}\log 2+\delta\log\pi+\delta-\delta\log\frac{\delta}{2}.$$

This inequality is easily obtained by $1-\cos\theta=2\sin^2(\theta/2)$ and by $\sin(\theta/2)\geqq\theta/\pi$.

Under these preparations we proceed to the original inequality.

$$\frac{2}{\pi}\int_{\pi-\delta/2}^{\pi} \log^+ \frac{1}{|h_\nu(r_\nu e^{i\theta})|}\,d\theta=\sum_{k=0}^{2^{10^\nu}-1}\frac{2}{\pi}\int_{\pi-\delta/2}^{\pi}\log^+\frac{1}{|1-r_\nu e^{i\theta}/(r_\nu+k)|}\,d\theta$$

$$=-\sum_{k=0}^{2^{10^\nu}-1}\frac{2}{\pi}\int_0^{\delta/2}\log|1-e^{-i\theta}+\tilde{A}_k e^{-i\theta}|\,d\theta$$

$$\leqq-\frac{\delta}{\pi}\sum_{k=0}^{2^{10^\nu}-1}\log(2-2\tilde{A}_k)+\frac{2^{10^\nu}}{\pi}\int_0^{\delta/2}\log\frac{1}{1-\cos\theta}\,d\theta$$

$$\leqq-\frac{\delta}{\pi}2^{10^\nu}\log\frac{3}{2}+\frac{2^{10^\nu}}{\pi}\left(-\frac{\delta}{2}\log 2+\delta+\delta\log\pi-\delta\log\frac{\delta}{2}\right).$$

Here by Shea's result [3]

$$2^{10^\nu}=r_\nu^{\rho}\leqq n(2r_\nu)$$
$$\leqq N(2er_\nu)\leqq\frac{1}{K}m(r_\nu, g)$$

with a positive constant K being independent of ν. Therefore we have the desired result. Here

$$D(\delta)=\frac{1}{K\pi}\left(-\delta\log\frac{3}{2}-\frac{\delta}{2}\log 2+\delta+\delta\log\pi-\delta\log\frac{\delta}{2}\right).$$

Hence $D(\delta)$ does not depend upon ν and $D(\delta)\to 0$ when $\delta\to 0$.

Next we prove

$$\frac{2}{\pi}\int_0^{\delta/2}\log|g(re^{i\theta})|\,d\theta\leqq\sin\frac{\delta}{2}\,m(r, g).$$

By the well known representation

$$\frac{1}{\pi}\int_0^{\delta/2}\log|g(re^{i\theta})|\,d\theta=\frac{1}{\pi}\int_0^{\infty}N(t)\frac{r\sin(\delta/2)}{t^2+2tr\cos(\delta/2)+r^2}\,dt$$

$$\leqq\sin\frac{\delta}{2}\,\frac{1}{\pi}\int_0^{\infty}N(t)\frac{r}{t}\,\frac{t}{t^2+r^2}\,dt$$

$$=\sin\frac{\delta}{2}\,\frac{1}{\pi}\int_0^{\pi/2}\log|g(re^{i\theta})|\,d\theta\leqq\frac{1}{2}m(r, g)\sin\frac{\delta}{2}.$$

Finally we consider

$$g(z)f^2+g(ze^{i\delta})f+g(ze^{i\delta})=0.$$

Here δ is a positive number which is sufficiently small. For the two-valued algebroid function f, whose order is ρ,

$$2\mu(r, A)=m(r, A)-m\left(r, \frac{1}{A}\right)$$

and

$$m(r, g)\leqq m(r, A)\leqq m(r, g)+\frac{2}{\pi}\int_0^{\delta/2}\log|g(re^{i\theta})|\,d\theta,$$

$$m\left(r, \frac{1}{g}\right)-\frac{2}{\pi}\int_{\pi-\delta/2}^{\pi}\log^+\frac{1}{|g(re^{i\theta})|}\,d\theta=m\left(r, \frac{1}{A}\right)\leqq m\left(r, \frac{1}{g}\right).$$

Hence

$$\frac{2\mu(r, A)}{m(r, A)}\leqq 1-\frac{m(r, 1/g)}{m(r, g)}$$
$$+\frac{2}{m(r, g)}\left(\frac{1}{\pi}\int_0^{\delta/2}\log|g(re^{i\theta})|\,d\theta+\frac{1}{\pi}\int_{\pi-\delta/2}^{\pi}\log^+\left|\frac{1}{g(re^{i\theta})}\right|d\theta\right).$$

Thus

$$\varliminf_{r\to\infty}\frac{2\mu(r, A)}{m(r, A)}\leqq\varliminf_{m\to\infty}\frac{2\mu(r_m, A)}{m(r_m, A)}$$
$$\leqq D(\delta)+\sin\frac{\delta}{2}.$$

Now we can choose δ so that

$$D(\delta)+\sin\frac{\delta}{2}<\varepsilon.$$

This implies the desired result.

The following fact is suggestive: Assume

$$\min_{0\leqq j\leqq n}\varDelta(0, A_j)=0.$$

Then

$$m(r, A)\sim n\mu(r, A).$$

The proof of this fact is almost immediate. Since

$$\varDelta(0, A_j)=\varlimsup_{r\to\infty}\frac{m(r, 1/A_j)}{m(r, A_j)}\geqq\varlimsup_{r\to\infty}\frac{m(r, 1/A_j)}{m(r, A)},$$

there is an index j_0 such that $m(r, 1/A_{j_0})=o(m(r, A))$. This implies that $m(r, 1/A)=o(m(r, A))$ and hence $m(r, A)\sim n\mu(r, A)$.

This fact enables us to say that $m(r, A)\sim n\mu(r, A)$ if a coefficient A_j satisfies

$$\varlimsup_{r\to\infty}\frac{m(r, A_j)}{(\log r)^2}<\infty.$$

Indeed in this case

$$m(r, A_j)\sim\log\max_{|z|=r}|A_j(z)|\sim N\left(r, \frac{1}{A_j}\right)$$

is well known. Hence $\varDelta(0, A_j)=0$. This leads to the desired fact.

Especially every entire algebroid function satisfies

$$|m(r, A)-n\mu(r, A)|=O(1).$$

4. In what follows we denote the following condition by (A): There is at least one index j satisfying

$$m\left(r, \frac{1}{A_j}\right)\leqq cm(r, A), \qquad c<1.$$

THEOREM 1. *Suppose that there is an index* ν *satisfying*

$$\sum_{j\neq\nu} m(r, A_j)\leqq dm(r, A_\nu), \qquad d<1$$

and that f *satisfies the condition* (A). *If there is a non-zero finite value* a *with* $\delta(a,f)=1$, *then the order of* f *is a positive integer, unless it is* ∞.

Proof. By

$$a^\nu A_\nu=F(z, a)-\sum_{j\neq\nu} a^j A_j,$$

$$m(r, A_\nu)\leqq m(r, F(z, a))+\sum_{j\neq\nu} m(r, A_j)+O(1),$$

and hence

$$(1-d)m(r, A_\nu)\leqq m(r, F(z, a))+O(1).$$

By Lemma 5

$$(1-c)m(r, A_\nu)\leqq(1-c)m(r, A)\leqq n\mu(r, A)$$

$$\leqq\sum_{j=0}^{n} m(r, A_j)+O(1)\leqq(1+d)m(r, A_\nu)+O(1).$$

Hence

$$n\mu(r, A)\leqq\frac{1+d}{1-d}m(r, F(z, a))+O(1).$$

By Lemma 4 we have $\delta(0, F(z, a))=1$. Since $F(z, a)$ is entire, its order must be a positive integer, unless it is ∞. Since

$$m(r, F(z, a))\leqq\sum_{j=0}^{n} m(r, A_j)+O(1)$$

$$\leqq(1+d)m(r, A_\nu)+O(1)\leqq\frac{1+d}{1-c}n\mu(r, A)+O(1),$$

both orders of $F(z, a)$ and f are coincident with each other, which gives the desired result.

THEOREM 2. *Suppose that there are two indices* μ_0, ν_0 $(n>\mu_0>\mu_0>0)$ *such that*

$$\sum_{j\neq\mu_0,\nu_0} m(r, A_j)\leqq\alpha n\mu(r, A), \qquad \alpha<\frac{1}{3},$$

and that f satisfies the condition (A). *Further suppose that there is a non-zero finite value b such that $m(r, F(z, b))=o(\mu(r, A))$. If there are $\mu_0-\nu_0+1$ non-zero finite values a_j such that $\delta(a_j, f)=1$, then the order of f is a positive integer, unless it is ∞.*

Proof. Firstly by Lemma 5 $(1-c)(m(r, A)\leqq n\mu(r, A)\leqq m(r, A)$. Since

$$b^{\mu_0}A_{\mu_0}+b^{\nu_0}A_{\nu_0}=F(z, b)-\sum_{j\neq\mu_0,\nu_0} b^j A_j\equiv g(z),$$

$$\begin{aligned} m(r, g)&\leqq m(r, F(z, b))+\sum_{j\neq\mu_0,\nu_0} m(r, A_j)+O(1) \\ &\leqq(n\alpha+\varepsilon)\mu(r, A). \end{aligned}$$

Among the given $\{a_j\}$ there is at least one a_k such that

$$a_k^{\mu_0-\nu_0}\neq b^{\mu_0-\nu_0}, \qquad \delta(a_k, f)=1.$$

Hence we have

$$a_k^{\mu_0}A_{\mu_0}+a_k^{\nu_0}A_{\nu_0}=F(z, a_k)-\sum_{j\neq\mu_0,\nu_0} a_k^j A_j\equiv g_1(z).$$

By solving the equation we have

$$A_{\mu_0}=c_0 g+c_1 g_1,$$

$$A_{\nu_0}=d_0 g+d_1 g_1.$$

Let A^* be $\max(|A_{\mu_0}|, |A_{\nu_0}|)$. Then

$$\begin{aligned} n\mu(r, A)\leqq m(r, A)&\leqq m(r, A^*)+\sum_{j\neq\mu_0,\nu_0} m(r, A_j) \\ &=m(r, A^*)+n\alpha\mu(r, A) \end{aligned}$$

and

$$\begin{aligned} m(r, A^*)&\leqq m(r, g)+m(r, g_1)+O(1) \\ &\leqq m(r, g_1)+(n\alpha+\varepsilon)\mu(r, A)+O(1). \end{aligned}$$

Further

$$\begin{aligned} m(r, g_1)&\leqq m(r, F(z, a_k))+\sum_{j\neq\mu_0,\nu_0} m(r, A_j)+O(1) \\ &\leqq m(r, F(z, a_k))+(n\alpha+\varepsilon)\mu(r, A)+O(1). \end{aligned}$$

Hence

$$\varliminf_{r\to\infty}\frac{m(r, F(z, a_k))}{n\mu(r, A)}\geqq 1-3\alpha>0.$$

Then by Lemma 4 we have $\delta(0, F(z, a_k))=1$, which implies the positive integrity of order of $F(z, a_k)$, unless it is ∞. Since

$$\begin{aligned} m(r, F(z, a_k))&\leqq m(r, A)+O(1) \\ &\leqq\frac{1}{1-c}n\mu(r, A)+O(1), \end{aligned}$$

we have the desired result.

By a slight modification of proof we have the following

THEOREM 3. *Suppose that there are two indices* μ_0, ν_0 $(n>\mu_0>\nu_0>0)$ *such that*

$$\sum_{j\neq\mu_0,\nu_0} m(r, A_j)\leqq n\alpha\mu(r, A)$$

and that there is a non-zero finite value b such that

$$m(r, F(z, b))\leqq n\beta\mu(r, A)$$

with $0\leqq 3\alpha+\beta<1$. *Further suppose that f satisfies the conditions* (A). *If there are* $\mu_0-\nu_0+1$ *non-zero finite values* $\{a_j\}$ *such that* $\delta(a_j, f)=1$, *then the order of f is a positive integer, unless it is* ∞.

DEFINITION. Let $\mathfrak{p}$ be a 3-vector $(a, b, 1)$ satisfying

$$m(r, aA_{\mu_0}+bA_{\nu_0}+A_{\varepsilon_0})=o(\mu(r, A)).$$

Then this vector is called *exceptional* for $(A_{\mu_0}, A_{\nu_0}, A_{\varepsilon_0})$.

THEOREM 4. *Suppose that there are three indices* $\mu_0, \nu_0, \varepsilon_0$ $(n>\mu_0>\nu_0>\varepsilon_0>0)$ *such that*

$$\sum_{j\neq\mu_0,\nu_0,\varepsilon_0} m(r, A_j)=o(\mu(r, A))$$

and that there are two exceptional vectors $\mathfrak{p}_1, \mathfrak{p}_2$, *for* $(A_{\mu_0}, A_{\nu_0}, A_{\varepsilon_0})$. *Further suppose that there is a non-zero finite value a satisfying* $\delta(a, f)=1$ *and* $(a^{\mu_0-\varepsilon_0}, a^{\nu_0-\varepsilon_0}, 1)\notin\lambda\mathfrak{p}_1+\mu\mathfrak{p}_2$, $\lambda+\mu=1$ *and that f satisfies the condition* (A). *Then the order of f is a positive integer, unless it is* ∞.

Proof. Let $\mathfrak{p}_1, \mathfrak{p}_2$ be $(\alpha, \beta, 1)$, $(\gamma, \delta, 1)$ respectively. Then

$$\begin{vmatrix} \alpha & \beta & 1 \\ \gamma & \delta & 1 \\ a^{\mu_0-\varepsilon_0} & a^{\nu_0-\varepsilon_0} & 1 \end{vmatrix}\neq 0,$$

since $(a^{\mu_0-\varepsilon_0}, a^{\varepsilon_0-\nu_0}, 1)\notin\lambda\mathfrak{p}_1+\mu\mathfrak{p}_2$, $\lambda+\mu=1$. Hence we can solve the equations:

$$\begin{aligned} \alpha A_{\mu_0}+\ \beta A_{\nu_0}+\ \ A_{\varepsilon_0}&=f_1, \\ \gamma A_{\mu_0}+\ \delta A_{\nu_0}+\ \ A_{\varepsilon_0}&=f_2, \\ a^{\mu_0}A_{\mu_0}+a^{\nu_0}A_{\nu_0}+a^{\varepsilon_0}A_{\varepsilon_0}&=F(z, a)-\sum_{j\neq\mu_0,\nu_0,\varepsilon_0} a^j A_j=f_3. \end{aligned}$$

Then we have

$$|A_l|\leqq\sum_{j=1}^{3}|c_{lj}|\,|f_j|, \qquad l=\mu_0, \nu_0, \varepsilon_0.$$

This implies

$$\frac{n}{3}\mu(r,A)-o(\mu(r,A))\leqq\frac{1}{3}\left(\sum_{j=0}^{n}m(r,A_j)-\sum_{j\neq\mu_0,\nu_0,\varepsilon_0}m(r,A_j)\right)$$
$$=\frac{1}{3}\sum_{j=\mu_0,\nu_0,\varepsilon_0}m(r,A_j)\leqq m\left(r,\max_{l=\mu_0,\nu_0,\varepsilon_0}|A_l|\right)$$
$$\leqq\sum_{l=1}^{3}m(r,f_l)+O(1)=m(r,f_3)+o(\mu(r,A))$$
$$\leqq m(r,F(z,a))+\sum_{j\neq\mu_0,\nu_0,\varepsilon_0}m(r,A_j)+O(1)$$
$$=m(r,F(z,a))+o(\mu(r,A)).$$

Thus we have

$$\varliminf_{r\to\infty}\frac{m(r,F(z,a))}{\mu(r,A)}\geqq\frac{n}{3}.$$

Then Lemma 4 implies $\delta(0,F(z,a))=1$. Thus the order of $F(z,a)$ is a positive integer, unless it is ∞. Then Lemma 5 implies the desired result.

THEOREM 5. *Let $\mathfrak{p}_j$, $j=1,\cdots,n$ be a system of linearly independent $(n+1)$-vectors $(\alpha_{jn},\alpha_{j,n-1},\cdots,1)$ satisfying*

$$m(r,\alpha_{jn}A_n+\alpha_{j,n-1}A_{n-1}+\cdots+A_0)=o(\mu(r,A)).$$

Suppose that there is a value a such that $\delta(a,f)=1$ and

$$(a^n,a^{n-1},\cdots,1)\notin\sum_{j=1}^{n}\lambda_j\mathfrak{p}_j,\qquad\sum_{j=1}^{n}\lambda_j=1.$$

Further suppose that f satisfies the condition (A). *Then the order of f is a positive integer, unless it is ∞.*

Proof. Let

$$\sum_{\mu=0}^{n}\alpha_{j\mu}A_\mu=f_j,\qquad j=1,\cdots,n,\qquad\alpha_{jo}=1$$

and

$$\sum_{\mu=0}^{n}a^\mu A_\mu=f_{n+1}.$$

Here

$$\begin{vmatrix}\alpha_{1n}&\alpha_{1,n-1}&\cdots&1\\ &\cdots\cdots\cdots&&\\ \alpha_{nn}&\alpha_{n,n-1}&\cdots&1\\ a^n&a^{n-1}&\cdots&1\end{vmatrix}\neq0$$

and

$$m(r,f_j)=o(\mu(r,A)),\qquad j=1,\cdots,n.$$

Therefore by solving the above equations

$$\begin{aligned} n\mu(r; A) &\leqq m(r, A) \\ &\leqq m(r, f_{n+1}) + \sum_{j=1}^{n} m(r, f_j) + O(1) \\ &= m(r, f_{n+1}) + o(\mu(r, A)). \end{aligned}$$

This shows that

$$\varliminf_{r\to\infty} \frac{m(r, f_{n+1})}{n\mu(r, A)} \geqq 1.$$

By Lemma 4 $\delta(0, F(z, a))=1$, which implies that $F(z, a)$ has either a positive integral order or an infinite order. By Lemma 5 we have the desired result.

THEOREM 6. *Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be n linearly independent $n+1$ vectors satisfying*

$$\mathfrak{p}_l = (\alpha_{ln}, \alpha_{l,n-1}, \cdots, \alpha_{l1}, 1),$$

$$m(r, \alpha_{ln}A_n + \cdots + A_0) \leqq n\beta_l \mu(r, A), \qquad \sum_{l=1}^{n} \beta_l < 1, \qquad \beta_l \geqq 0.$$

Suppose that f satisfies the condition (A) *and that there is a value a such that $\delta(a, f)=1$ and*

$$(a^n, a^{n-1}, \cdots, a, 1) \notin \sum_{j=1}^{n} \lambda_j \mathfrak{p}_j, \qquad \sum_{j=1}^{n} \lambda_j = 1.$$

Then the order of f is a positive integer, unless it is ∞.

Proof. Essentially the same method as in Theorem 5 does work. We have

$$\begin{aligned} n\mu(r, A) &= n\mu\left(r, \max_{1\leqq l\leqq n+1} |f_l|\right) + O(1) \\ &\leqq m(r, f_{n+1}) + \sum_{j=1}^{n} m(r, f_j) + O(1) \\ &\leqq m(r, f_{n+1}) + n\mu(r, A) \sum_{l=1}^{n} \beta_l + O(1). \end{aligned}$$

Hence

$$n\left(1 - \sum_{l=1}^{n} \beta_l\right)\mu(r, A) \leqq m(r, f_{n+1}) + O(1),$$

which implies the desired result.

5. We can make use of recent results in the theory of meromorphic functions [1].

THEOREM 7. *Let f be an algebroid function of order less than* 1/2. *Suppose* *that there is an index* ν *satisfying*

$$\sum_{j\neq\nu} m(r, A_j)=o(m(r, A_\nu)).$$

Then there are at most two deficient values 0 *and* ∞, *for which* $\delta(0,f)=1$ *and* $\delta(\infty,f)=1$ *if they are actually deficient values.*

Proof. In the first place we assume that $\nu\neq 0$, n. Then

$$m(r, A_\nu)-o(m(r, A_\nu))\leqq m(r, F(z, a))$$

holds for every finite $a\neq 0$. Further we have

$$n\mu(r, A)\leqq m(r, A)\leqq \sum_{j=0}^{n} m(r, A_j)=m(r, A_\nu)+o(m(r, A_\nu))$$

and hence

$$\varliminf_{r\to\infty}\frac{m(r, F(z, a))}{n\mu(r, A)}\geqq 1.$$

Therefore by Lemma 4 $1-\delta(0, F(z, a))\leqq 1-\delta(a,f)$. On the other hand we have $m(r, F(z, a))\leqq m(r, A)+O(1)$ and $m(r, A)=n\mu(r, A)$ in a set of r of positive upper density. Hence the order of $F(z, a)$ is less than 1/2 and this implies $\delta(0, F(z, a))=0$. Hence $\delta(a,f)=0$. It is very easy to show that $\delta(0,f)=\delta(\infty,f)=1$ in this case.

If $\nu=n$, then $\delta(0,f)=1$ and $\delta(a,f)=0$ for $a\neq 0$. If $\nu=0$, then $\delta(\infty,f)=1$ and $\delta(a,f)=0$ for $a\neq\infty$.

THEORFM 8. *Let f be an algebroid function of order less than* 1/2 *satisfying the following conditions*: 1) *There are two indices* μ_0, ν_0 *such that*

$$\sum_{j\neq\mu_0,\nu_0} m(r, A_j)=o(m(r, A)), \qquad n>\mu_0>\nu_0>0,$$

2) *for every* $b\neq 0, \infty$,

$$m(r, b^{\mu_0-\nu_0}A_{\mu_0}+A_{\nu_0})\sim\alpha_b m(r, A^*), \qquad A^*=\max(|A_{\mu_0}|, |A_{\nu_0}|), \qquad 0\leqq\alpha_b\leqq 1,$$

and 3) *there are at most two constants* b_1, b_2 *for which* $\alpha_{b_1}<1$, $\alpha_{b_2}<1$, $b_1^{\mu_0-\nu_0}\neq b_2^{\mu_0-\nu_0}$. *Then* $\delta(0,f)=1$, $\delta(\infty,f)=1$ *and*

$$\sum_{a\neq 0,\infty}\delta(a,f)\leqq\mu_0-\nu_0.$$

Proof. In the first place we prove $\alpha_{b_1}+\alpha_{b_2}\geqq 1$. b_1 and b_2 are not 0 and ∞ and satisfy $b_1^{\mu_0-\nu_0}\neq b_2^{\mu_0-\nu_0}$. Hence we can solve

$$b_1^{\mu_0}A_{\mu_0}+b_1^{\nu_0}A_{\nu_0}=F(z, b_1)-\sum_{j\neq\mu_0,\nu_0} b_1^j A_j\equiv g_1(z),$$

$$b_2^{\mu_0}A_{\mu_0}+b_2^{\nu_0}A_{\nu_0}=F(z, b_2)-\sum_{j\neq\mu_0,\nu_0} b_2^j A_j\equiv g_2(z).$$

Then

$$A_{\mu_0}=\gamma_1 g_1+\gamma_2 g_2, \qquad \gamma_1\gamma_2\delta_1\delta_2\neq 0.$$
$$A_{\nu_0}=\delta_1 g_1+\delta_2 g_2,$$

Hence

$$m(r, A^*)\leqq m(r, g_1)+m(r, g_2)+O(1).$$

This implies the desired result.

In general

$$n\mu(r, A)\leqq m(r, A)\leqq m(r, A^*)+\sum_{j\neq\mu_0,\nu_0} m(r, A_j)$$
$$=m(r, A^*)+o(m(r, A))\leqq m(r, A)+o(m(r, A)).$$

Further $n\mu(r, A)=m(r, A)$ in a set of r of positive upper density. Hence $n\mu(r, A)=m(r, A)\sim m(r, A^*)$ holds in a set of r of positive upper density.

By the assumption for every non-zero finite b,

$$\alpha_b m(r, A^*)\sim m(r, b^{\mu_0}A_{\mu_0}+b^{\nu_0}A_{\nu_0})$$
$$\leqq m(r, F(z, b))+\sum_{j\neq\mu_0,\nu_0} m(r, A_j)+O(1)$$
$$\leqq m(r, F(z, b))+o(m(r, A))$$
$$\leqq m(r, b^{\mu_0}A_{\mu_0}+b^{\nu_0}A_{\nu_0})+\sum_{j\neq\mu_0,\nu_0} m(r, A_j)+O(1)$$
$$\sim\alpha_b m(r, A^*)+o(m(r, A))\sim\alpha_b m(r, A^*).$$

Therefore

$$m(r, F(z, b))\sim\alpha_b m(r, A^*)\sim\alpha_b m(r, A)=\alpha_b n\mu(r, A)$$

in a set of r of positive upper density. Hence by Lemma 4

$$1-\delta(b, f)\geqq\alpha_b(1-\delta(0, F(z, b))).$$

On the other hand $\delta(0, F(z, b))=0$ which implies $\delta(b, f)\leqq 1-\alpha_b$. Further these are $\mu_0-\nu_0$ solutions of $z^{\mu_0-\nu_0}=b^{\mu_0-\nu_0}$. We denote them by $b\omega_j$, $\omega_j{}^{\mu_0-\nu_0}=1$. Hence

$$\sum_{j=1}^{\mu_0-\nu_0}\delta(b\omega_j, f)\leqq(1-\alpha_b)(\mu_0-\nu_0).$$

This holds only for b_1 and b_2 at most. Hence

$$\sum_{a\neq 0,\infty}\delta(a, f)\leqq\sum_{j=1}^{\mu_0-\nu_0}\delta(b_1\omega_j, f)+\sum_{j=1}^{\mu_0-\nu_0}\delta(b_2\omega_j, f)$$
$$\leqq(2-\alpha_{b_1}-\alpha_{b_2})(\mu_0-\nu_0)\leqq\mu_0-\nu_0.$$

If there is a non-zero finite b for which $\alpha_b=0$, there is no non-zero finite deficient value other than b. In this case we have

$$\sum_{a \neq 0, \infty} \delta(a, f) = \mu_0 - \nu_0.$$

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Department of Mathematics,
Tokyo Institute of Technology.