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COMPLEX SUBMANIFOLDS OF THE COMPLEX PROJECTIVE SPACE WITH SECOND FUNDAMENTAL FORM OF CONSTANT LENGTH

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1. Statement of result.

In a recent work [1] Chern, do Carmo and Kobayashi have established a pinching problem, with respect to the length of the second fundamental form, for compact minimal submanifolds of a sphere and have classified compact minimal submanifolds of a sphere whose lengths of the second fundamental form are certain constants.

In the present paper we shall give a complex analogue. Let $P_{n+p}(\boldsymbol{C})$ be the complex projective space of complex dimension $n+p$ with the Fubini-Study metric. Let M be an n-dimensional compact complex submanifold of $P_{n+p}(\boldsymbol{C})$ and let h be the second fundamental form. We denote by S the square of the length of h. Then we can see that

$$\int_M \left\{ \left(2-\frac{1}{2p}\right)S-\frac{n+2}{2}\right\} S\,dv \geqq 0,$$

where dv denotes the volume element of M. It follows that if

$$S \leqq \frac{n+2}{4-1/p} \qquad \text{everywhere on } M,$$

then either

(1) $$S=0 \quad \text{(i.e., } M \text{ is totally geodesic)}$$

or

(2) $$S=\frac{n+2}{4-1/p}.$$

The purpose of the present paper is to determine all compact complex submanifolds M of $P_{n+p}(\boldsymbol{C})$ satisfying

$$S=\frac{n+2}{4-1/p}.$$

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Our result is the following

THEOREM. *The complex quadric in $P_2(\boldsymbol{C})$ is the only compact complex submanifolds of dimension n in $P_{n+p}(\boldsymbol{C})$ satisfying*

$$S=\frac{n+2}{4-1/p}.$$

For notations and formulae we refer to [1].

2. Outline of the proof.

Since M is a minimal submanifold in $P_{n+p}(\boldsymbol{C})$ and since the curvature tensor of $P_{n+p}(\boldsymbol{C})$ is given by

$$K_{ABCD}=\frac{1}{4}(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}+J_{AC}J_{BD}-J_{AD}J_{BC}+2J_{AB}J_{CD}),$$

where J denotes the complex structure, we have, from (2. 23) in [1],

$$\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}=-\sum(h_{ik}^{\alpha}h_{kj}^{\beta}-h_{ik}^{\beta}h_{kj}^{\alpha})(h_{il}^{\alpha}h_{lj}^{\beta}-h_{il}^{\beta}h_{lj}^{\alpha})-\sum h_{ij}^{\alpha}h_{kl}^{\alpha}h_{ij}^{\beta}h_{kl}^{\beta}+\frac{n+2}{2}\sum h_{ij}^{\alpha}h_{ij}^{\alpha}.$$

Corresponding to (3. 10) in [1], we have

$$-\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}\leqq\left(2-\frac{1}{2p}\right)S^{2}-\frac{n+2}{2}S.$$

This, together with the fact that

$$\frac{1}{2}\Delta\left(\sum h_{ij}^{\alpha}h_{ij}^{\alpha}\right)=\sum h_{ijk}^{\alpha}h_{ijk}^{\alpha}+\sum h_{ij}^{\alpha}\Delta h_{ij}^{\alpha}$$

and the theorem of Green, implies

PROPOSITION. *Let M be an n-dimensional compact eomplex submanifold of $P_{n+p}(\boldsymbol{C})$. Then*

$$\int_{M}\left\{\left(2-\frac{1}{2p}\right)S-\frac{n+2}{2}\right\}S\,dv\geqq 0,$$

where dv denotes the volume element of M.

COROLLARY. *Let M be an n-dimensional compact complex submanifold of $P_{n+p}(\boldsymbol{C})$. If M is not totally geodesic and if*

$$S\leqq\frac{n+2}{4-1/p}\quad \textit{everywhere on } M,$$

then

$$S=\frac{n+2}{4-1/p}.$$

Let M be an n-dimensional compact complex submanifold of $P_{n+p}(\boldsymbol{C})$ satisfying

$$S=\frac{n+2}{4-1/p}.$$

Then the argument quite similar to that of §4 in [1] yields that $n=p=1$ and that $\Omega_2^1=(1/2)\omega^1\wedge\omega^2$. This implies that M is isometric with the complex quadric in $P_2(\boldsymbol{C})$.

Bibliography

[1] CHERN, S. S., M. DO CARMO, AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length. To appear in J. Differential Geometry.

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