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ON CONTINUITY OF EXTREMAL DISTANCE AND ITS APPLICATIONS TO CONFORMAL MAPPINGS

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§ 1. Introduction.

1. In the present paper we shall first discuss the continuity of the extremal distance between two sets of the boundary of a plane region with respect to its exhaustion. Here the continuity of the extremal length means the convergence of a sequence of extremal lengths of curve families to that of a well-defined family. The continuity of a sequence of increasing curve families was shown by the author [14] and Ziemer [17]. In the problem for extremal distance, the corresponding is regarded as a decreasing sequence. The problem was dealt with by Wolontis [16] for two compact sets in a region, later by Strebel [11] for two compact sets of boundary components of a Riemann surface and recently by the author [15] for two boundary parts on a boundary component of a plane region. Another generalization for curves was given by Marden and Rodin [6].

In this paper we shall define a new boundary part of a plane region Ω which is considered as a sort of element in the research of ideal boundary by means of a filter. Let $\{\Omega_n\}$ be an exhaustion of Ω towards two disjoint boundary parts. Then we shall show the continuity, if the extremal distance of the two sets of the relative boundary components of Ω_1 is positive. The proofs due to Wolontis and Strebel are based on the semicontinuity of the distance measured by an admissible metric and the second proof in [15] on a conformal representation. In the present proof we shall make use of an auxiliary metric so that the former method may be available.

After the continuity is established, cannonical conformal mappings as in [14] and [15] can be easily constructed, which will be discussed in § 4. The images of boundary parts will be investigated by making use of the method of extremal metrics. The continuity of the extremal distance will be effective in such conformal mappings.

In the final section we shall give an extension of the notion of prime ends, first introduced by Carathéodory [2], to an arbitrary plane region in such a way that it gives a compactification in a suitable topology. Every conformal mapping of the region is extended to the compactification and it introduces a well-defined boundary correspondence.

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§ 2. Preliminaries.

2. *Definition of boundary part.* Let Ω be a plane region which is not the Riemann sphere. We may assume that Ω does not contain the point at infinity throughout this paper. We shall mean a *boundary part* by a sequence of open subset $\{\Delta_\nu\}$ of Ω satisfying:

I) the relative boundary of Δ_ν consists of a finite number of Jordan curves either closed or open, both of whose end arcs tend to a common boundary component in the latter case,

II) the open set $\Omega-\bar{\Delta}_\nu$ is connected, where $\bar{\Delta}_\nu$ denotes the closure of Δ_ν with respect to the relative topology in Ω,

III) $\Delta_\nu \supset \Delta_{\nu+1}$, and

IV) $\cap \Delta_\nu = \phi$.

Two boundary parts $\{\Delta_\nu\}$ and $\{\Delta'_\nu\}$ are said to be *equivalent,* if every Δ_n contains a Δ'_m and vice versa. The sequence $\{\Delta_\nu\}$ is termed a *defining sequence* of the boundary part. A boundary part $A'=\{\Delta'_\nu\}$ is said to *lie on* $A=\{\Delta_\nu\}$, if the first half of the conditions of the above equivalence is satisfied.

A sequence $\{\Omega_\nu\}$ with $\Omega_\nu=\Omega-\bar{\Delta}_\nu$ is called an *exhaustion of Ω towards* $A=\{\Delta_\nu\}$. We say that two boundary parts $A=\{\Delta^A_\nu\}$ and $B=\{\Delta^B_\nu\}$ are *disjoint,* if $\bar{\Delta}^A_n \cap \bar{\Delta}^B_n=\phi$ for some n. Then we can construct an exhaustion $\{\Omega_\nu\}$ of Ω *towards A and B* with $\Omega_\nu=\Omega-\bar{\Delta}^A_{n+\nu}\cup\bar{\Delta}^B_{n+\nu}$.

For a boundary part $A=\{\Delta_n\}$, the set $I(A)=\cap \mathrm{Cl}(\Delta_n)$ is called the *realization* of A, where Cl(*) denotes the closure on the Riemann sphere.

3. *Prime ends.* We mention prime ends of a simply connected region which are needed to establish an important lemma in the proof of our continuity theorem of extremal distance. Let Δ be a bounded simply connected region. A *prime end* is a boundary part $\xi=\{\Delta_n\}$ with the following properties: the relative boundary of Δ_n, denoted by q_n, is a Jordan arc with end points on the boundary of Δ in the Riemann sphere, no two of q_n have any point, including their ends points in common and the diameter of q_n tends to zero. Following Collingwood and Lohwater [3], we call the sequence $\{q_n\}$ a chain. It is known that every prime end can be defined by a chain consisting of concentric circular arcs [3].

4. Let U be the unit disc $|z|<1$ and let c_0 be a Jordan curve $z=z_0(t)$, $0<t<1$, in U such that $|z_0(t)|\to 1$ as $t\to 0$ and $t\to 1$ respectively. The c_0 divides U into two simply connected subregions, say V_1 and V_2. Let z_1 be a fixed point in V_1 and let c be an arbitrary curve in U joining z_1 and a point of V_2. Then we state

LEMMA 1. *There exist at most two points ζ_j on the circle $|z|=1$ and a sequence of discs $|z-\zeta_j|<r^{(n)}_j$ with $r^{(n)}_j\to 0$ such that the first intersection of c with c_0 lies on a fixed compact subarc c_n of c_0, unless it runs through the two discs for fixed n.*

The author expresses his warmest thanks to Mr. M. Tsuzuki for his suggestions in the proof of this lemma.

Proof. We set

$$C(z_0, 0)=\bigcap_{\varepsilon>0}\mathrm{Cl}(\{z_0(t)|t<\varepsilon\})$$

and

$$C(z_0, 1)=\bigcap_{\varepsilon>0}\mathrm{Cl}(\{z_0(t)|t>1-\varepsilon\})$$

which are the cluster sets of $z_0(t)$ at 0 and 1 and write C_0 and C_1 for them, respectively. We first see that there exist one or two prime ends of V_1 with non-void intersection of their realizations with the union $C_0\cup C_1$. In fact, at least one of the end points of q_n of a chain of such a prime end ξ lies on c_0 except for a finite number of n. Indeed, if both the end points lie on the circle $|z|=1$ for infinitely many n, we have $I(\xi)\cap(C_0\cup C_1)=\phi$, which is a contradiction. Thus we may assume that one of the end points is on c_0 for all n. Then there are possible two cases where both the end points lie on c_0 for infinitely many n or only one of them on the unit circle for almost all n. If one of the end points is on c_0 and the other is on the circle, then all the q_m with $m>n$ are the same in their end points. For, if not, both the end points are on the subarc of c_0 on $\bar{\varDelta}_n$, which contradicts the fact that $I(\xi)\cap(C_0\cap C_1)\neq\phi$. Hence, in the first case, we have a unique prime end $\xi=\{\varDelta_n\}$, since $V_1-\bar{\varDelta}_n$ is relatively compact and there are no such prime ends. In the second case we may assume that one of the end points of q_n is on c_0 and the other is on the circle for all n. The sequence of the end points of q_n on c_0 has accumulation points on one of C_0 and C_1, say C_0. Then $V_1-\bar{\varDelta}_n$ has another prime end $\xi'=\{\varDelta_n'\}$ such that $I(\xi')\cap C_1\neq\phi$ and each simply connected subregion $V_1-(\bar{\varDelta}_n\cup\bar{\varDelta}_n')$ has no desired prime ends, since the intersection of its relative boundary in U with c_0 is on a compact subarc.

As is remarked in No. 3, we can take as chains of these prime ends a sequence of concentric circular arcs with centers at ζ_j on the unit circle and with radii $r_j^{(n)}$ tending to zero. These points are so-called principal points [3] which are the desired. To see this, it is sufficient to show it in the case where there exist two prime ends, since otherwise the proof is easier. If the point z_1 is contained in the subregion $V_1-(\bar{\varDelta}_n\cup\bar{\varDelta}_n')$ for sufficiently large n so that $\bar{\varDelta}_n\cap\bar{\varDelta}_n'=\phi$, then the first intersection of c with the relative boundary of this region is either on $q_n\cup q_n'$ or on $c_n=\mathrm{Cl}(c_0-(\varDelta_n\cup\varDelta_n'))$, which implies the assertion.

§ 3. Extremal distance and its continuity.

5. *Extremal length.* Let Γ be a family of locally rectifiable curves in Ω. We denote by $P(\Gamma)$ the family of Borel measurable metrics $\rho(z)|dz|$ defined in Ω and satisfying

$$\int_c\rho|dz|\geqq 1$$

for all $c\in\Gamma$, which is called the class of *admissible* metrics for Γ. The *module* of the curve family is defined by

$$\operatorname{mod} \Gamma = \inf_{\rho \in P(\Gamma)} \|\rho\|^2 = \inf_{\rho \in P(\Gamma)} \iint_\Omega \rho^2 \, dxdy$$

and the *extremal length* of Γ, denoted by $\lambda(\Gamma)$, is its reciprocal. The extremal length of the family of curves joining two sets is called the *extremal distance* between them.

We prepare a lemma for the later use.

LEMMA 2. (Hersch [5]). *Let $\{\Gamma_n\}$ be at most a countable number of curve families. Then*

$$\operatorname{mod} (\cup \Gamma_n) \leqq \sum \operatorname{mod} \Gamma_n.$$

6. The closure of the intersection of $P(\Gamma)$ with the space of L_2-metrics is called the class of *L_2-admissible* metrics denoted by $P^*(\Gamma)$. There exists always a unique metric ρ_0 in $P^*(\Gamma)$ such that $\|\rho_0\|^2 = \operatorname{mod} \Gamma$, if $P^*(\Gamma) \neq \phi$ [10]. The metric ρ_0 is termed the *extremal* metric for Γ. The following inequality is known [13]:

$$\|\rho - \rho_0\|^2 \leqq \|\rho\|^2 - \|\rho_0\|^2. \tag{1}$$

We quote

LEMMA 3. (Suita [14] and Ziemer [17]). *Let $\{\Gamma_n\}_{n=1}^\infty$ be a sequence of curve families such that $\Gamma_n \subset \Gamma_{n+1}$ for all n. Setting $\Gamma_0 = \cup \Gamma_n$, we have*

$$\lambda(\Gamma_0) = \lim_{n \to \infty} \lambda(\Gamma_n).$$

Furthermore, the extremal metrics for Γ_n tend strongly to the extremal metric for Γ_0, if $\lambda(\Gamma_0) > 0$.

A curve family with vanishing module is called an *exceptional family.* By Lemma 2 the union of at most a countable number of exceptional families is also exceptional. The extremal length of a curve family remains unchanged, if an exceptional family is added or subtracted. A proposition about a curve family is said to be true *for almost all curves,* if it holds except for an exceptional family.

7. *Continuity theorem.* Let A and B be two disjoint boundary parts with respective defining sequences $\{\Delta_n^A\}$ and $\{\Delta_n^B\}$ such that $\bar{\Delta}_1^A \cap \bar{\Delta}_1^B = \phi$ and let $\{\Omega_n\}$ be an exhaustion of Ω towards A and B. We denote by A_n and B_n the sets of components of the relative boundary of Δ_n which are those of Δ_n^A and Δ_n^B respectively. Let Γ_n be the family of curves joining A_n and B_n in Ω. We define the curve family Γ_0 *joining* A and B by the family of curves running through Ω, intersecting all the members of Δ_n^A and Δ_n^B and clustering at no point of Ω. This family was called the clustering curve family in [15]. We state

THEOREM 1. *If $\lambda(\Gamma_1) > 0$, we have*

$$\lambda(\Gamma_0) = \lim_{n \to \infty} \lambda(\Gamma_n).$$

Moreover the extremal metrics ρ_n for Γ_n tend strongly to the extremal ρ_0 for Γ_0.

The condition $\lambda(\Gamma_1)>0$ is indispensable, which was shown in [15]. After an auxiliary metric is constructed so that Strebel's proof [11] may be available, we shall need his proof and restate it here for completeness.

8. *Proof.* Let $\{D_n\}$ be a normal exhaustion of Ω, that is an exhaustion of Ω towards the whole boundary. We may assume that D_ν is disjoint from $\bar{\Delta}_1^A \cup \bar{\Delta}_1^B$ with ν fixed. We first show the continuity for the subfamilies of curves intersecting D_ν. The relative boundary of Ω_n consists of a finite number of open or closed Jordan curves and each of the former is a curve whose end arcs tend to a common boundary component α by definition. Suppose α is a point. We may assume that α is the point at the origin. Since the extremal distance of the α from D_ν is infinite in the Riemann sphere, we can construct a metric ρ_ε^* such that $\|\rho_\varepsilon^*\|^2<\varepsilon$ and $\int_c \rho_\varepsilon^*|dz|\geqq 1$, where c is an arbitrary curve joining D_ν and the disc $|z|\leqq\delta$ for sufficiently small δ within Ω. Next if α is not a point, by Riemann's mapping theorem to the component of α^c containing Ω, we may assume that α is the unit circle $|z|=1$ which is its outer boundary. Then, by the same reason, there exists an admissible metric ρ_ε^* for the family of curves joining D_ν and the one or two discs $|z-\zeta_j|\leqq r_j^{(n)}$ in Lemma 1 for so large n that $\|\rho_\varepsilon^*\|^2<\varepsilon$, where each relative boundary component of Δ_n tending to α is taken as c_0 in the lemma.

Since there are at most a countable number of such boundary components over all the Ω_n, by arranging them, we can construct a sequence of metrics $\rho_{\varepsilon/2^{k+1}}^*(k\geqq 1)$ which are admissible for the families of curves joining D_ν and E_k, where E_k is the inverse image of the subset of the closed discs constructed above in the image region by the Riemann's theorem. We set $\rho_\varepsilon'(z)=\sup_{k\geqq 1}\{\rho_{\varepsilon/2^{k+1}}^*(z)\}$ which is clearly admissible for the union of these curve families. We have $\|\rho_\varepsilon'\|^2\leqq\varepsilon/2$.

We construct another auxiliary metric ρ_ε'' which is continuous, positive and such that $\|\rho_\varepsilon''\|^2<\varepsilon/2$. Indeed, setting $\rho_\varepsilon''=\kappa\min(|z-a|^{-2}, |z-b|^{-2})$ with different a and b, we get a desired metric for sufficiently small κ.

9. We shall see how Strebel's proof [11] is applied. Let $\Gamma_{n\nu}$ be the subfamily of Γ_n which consists of curves of Γ_n intersecting D_ν. It is easily verified that $\operatorname{mod}\Gamma_{n\nu}$ is decreasing with n and $\operatorname{mod}\Gamma_{0\nu}\leqq\lim\operatorname{mod}\Gamma_{n\nu}$. To show the opposite inequality, setting $\rho_\varepsilon=\max(\rho, \rho_\varepsilon', \rho_\varepsilon'')$ with $\|\rho\|^2<\infty$ and $\rho\in P(\Gamma_{0\nu})$ and

$$L_n=\inf_{c\in\Gamma_{n\nu}}\int_c \rho_\varepsilon|dz|,$$

we show

$$\lim_{n\to\infty} L_n\geqq 1. \tag{2}$$

Contrary to the assertion, suppose $\lim L_n=l<1$. Then there exists a sequence of curves $c_n\in\Gamma_{n\nu}$ such that

$$\int_{c_n}\rho_\varepsilon|dz|<l'=\frac{l+1}{2}<1,$$

since L_n is increasing. Let m be arbitrarily fixed. The c_n with its initial point on A_n intersects A_m and B_m for $n \geqq m$ and its last and first intersections with them are on a compact subset K_m of them in Ω respectively. In fact, if a noncompact component of A_m and B_m terminates at a component α which is a point, its intersection with K_m is the complement of the interior of an E_k with respect to it. If it terminates at a continuum α, its intersection is the conformally equivalent arc of the arc in Lemma 1, by applying the lemma to the conformally equivalent unit disc of the component of α^c containing Ω.

Thus, by diagonal process, we may assume that the two sequences of the above intersections are convergent to the points a_m and b_m on A_m and B_m respectively. Since $\|\rho_\varepsilon\|^2 < \infty$, there exist two sequences of discs $|z-a_m| \leqq r_m$ and $|z-b_m| \leqq r_m$ contained in Ω and such that

$$(3) \qquad \int_{|z-a_m|=r_m} \rho_\varepsilon |dz| + \int_{|z-b_m|=r_m} \rho_\varepsilon |dz| < \frac{\eta}{2^{m+1}} \quad (m=1, 2, \cdots).$$

We denote by $c_n^{(m)}$ and $c_n^{(-m)}$ the subarcs of c_n between its last intersections with A_m and A_{m+1} and first intersections with B_m and B_{m+1} respectively. Let $c_n^{(0)}$ denote its subarc between its last and first intersections with A_1 and B_1. Setting

$$\lim_{n\to\infty} \int_{c_n^{(m)}} \rho_\varepsilon |dz| = l_m, \qquad m=0, \pm 1, \cdots,$$

we have easily

$$\sum_{m=-\infty}^{\infty} l_m \leqq l'.$$

Hence we can take arcs $c_{nm}^{(m)}$ such that

$$\int_{c_{nm}^{(m)}} \rho_\varepsilon |dz| < l_m + \frac{\eta}{4^{|m|+2}}$$

and intersecting both the circles around a_m and b_m for sufficiently large n. Connecting these arcs by the circular arcs passing through $\Omega - \bar{\Omega}_n$, we obtain a curve $\hat{c}$ such that

$$(5) \qquad \int_{\hat{c}} \rho_\varepsilon |dz| < l' + \eta.$$

The curve $\hat{c}$ belongs to $\Gamma_{0\nu}$. It is sufficient to show that $\hat{c}$ has no cluster points in Ω. Indeed, if any, there exists a relatively compact subregion containing the closure of a neighborhood of a cluster point. Then it intersects both the boundaries of these sets infinitely often and since ρ_ε has a positive lower bound in any compact subregion by virtue of ρ_ε'', we have $\int_{\hat{c}} \rho_\varepsilon |dz| = \infty$, which contradicts (5). Since η is arbitrary in (5), we get a contradiction to the admissibility of ρ_ε, which implies (2).

Then by (2) $(1+\eta)\rho_\varepsilon$ is admissible for $\Gamma_{n\nu}$ for a sufficiently large n and we have

$$\lim_{n\to\infty} \operatorname{mod} \Gamma_{n\nu} \leqq \|\rho_\varepsilon\|^2.$$

Moreover $\|\rho_\varepsilon\|^2 \leqq \|\rho\|^2+\varepsilon$ holds by definition and since the ε is arbitrary, we have $\operatorname{mod} \Gamma_{0\nu} \geqq \lim \operatorname{mod} \Gamma_{n\nu}$ and

$$\operatorname{mod} \Gamma_{0\nu} = \lim_{n\to\infty} \operatorname{mod} \Gamma_{n\nu}. \tag{6}$$

10. To show that $\lambda(\Gamma_n)\to\lambda(\Gamma_0)$, we have at first

$$\lim_{n\to\infty} \operatorname{mod} \Gamma_n \geqq \operatorname{mod} \Gamma_0,$$

since $P(\Gamma_n)\subset P(\Gamma_0)$. Next let $\Lambda_{n\nu}$ be the subfamily of Γ_n consisting of curves which never intersect D_ν. Then we have $\Gamma_n=\Gamma_{n\nu}\cup\Lambda_{n\nu}$ and from Lemma 2

$$\operatorname{mod} \Gamma_n \leqq \operatorname{mod} \Gamma_{n\nu} + \operatorname{mod} \Lambda_{n\nu}.$$

There exists a metric $\mu\in P(\Gamma_1)$ such that $\|\mu\|^2<\infty$ by the assumption $\lambda(\Gamma_1)>0$. Then the metric

$$\mu_\nu = \begin{cases} \mu & \text{on } \Omega - D_\nu, \\ 0 & \text{elsewhere} \end{cases}$$

is admissible for $\Lambda_{n\nu}$, $n=1, 2, \cdots$ and such that $\|\mu_\nu\|^2\to 0$ as $\nu\to\infty$. Hence there exists an N such that $\operatorname{mod} \Lambda_{n\nu}<\varepsilon$ for $\nu\geqq N$. Letting $n\to\infty$, we have from (6)

$$\lim_{n\to\infty} \operatorname{mod} \Gamma_n \leqq \operatorname{mod} \Gamma_{0\nu}+\varepsilon,$$

for $\nu\geqq N$, and by letting $\nu\to\infty$, from Lemma 3

$$\lim_{n\to\infty} \operatorname{mod} \Gamma_n \leqq \operatorname{mod} \Gamma_0+\varepsilon.$$

Since ε is arbitrary, we get the assertion.

As for the convergence of ρ_n, we have from (1)

$$\|\rho_m-\rho_n\|^2 \leqq \|\rho_m\|^2 - \|\rho_n\|^2$$

for $n\geqq m$, which implies the strong convergence of ρ_n to a metric ρ_0. Since $\cup P(\Gamma_n)\subset P(\Gamma_0)$, $\rho_0\in P^*(\Gamma_0)$, whence ρ_0 is extremal for Γ_0.

§ 4. Conformal representation.

11. After the continuity of extremal distance is established, we can give a conformal representation of Ω onto a slit rectangle by a function related to the extremal metric for Γ_0 of special A and B, Let A and B be two disjoint boundary parts with $\bar{\Delta}_1^A \cap \bar{\Delta}_1^B = \phi$ which are on a common boundary component α of Ω. Suppose that Ω can be divided into two subregions by a Jordan curve in such a way that A and B are regarded as the boundary parts of different parts. Each Δ_n^A consists of a finite number of components Δ_{nj}^A whose relative boundary is a single Jordan curve. Then we can select a sequence $\{\Delta_{nj_n}^A\}_{n=1}^{\infty}$ which makes a boundary part. Such a boundary part is said to be *elementary*. Then A and B are decomposed into elementary boundary parts.

Let $\{\Omega_n\}$ be an exhaustion of Ω towards A and B and let A_n and B_n be the relative boundaries of Δ_n^A and Δ_n^B respectively. Let α_n denote the boundary component of Ω_n containing them. The component of α_n^c containing Ω_n is mapped onto the unit disc by Riemann's mapping theorem. Then the images of A_{nj} and B_{nk} are open arcs, denoted by A'_{nj} and B'_{nk}, on the unit circle, where A_{nj} and B_{nk} are the relative boundaries of components Δ_{nj}^A and Δ_{nk}^B respectively. When the circle is positively oriented with respect to the disc, we may assume that A_{nj} and B_{nk} are arranged in such a way that $A'_{n1}, A'_{n2}, \cdots, B'_{n1}, B'_{n2}, \cdots$ lie in this orientation.

Next we can construct a defining sequence $\{\Delta_m^\alpha\}$ of α, each of whose relative boundaries intersects every A_{nj} and B_{nk} just twice. We denote by $\Delta_{mj}^{A_{nj}}$ the non-compact subregion of Δ_m^α bounded by the subarcs of A_{nj}, A_{nj+1} and the relative boundary of Δ_m^α. The $\{\Delta_{mj}^{A_{nj}}\}$ defines an elementary boundary part A_{nj}^* which is called the boundary part *determined by these two end arcs*. We denote by B_{nk}^*, C_n^* and D_n^* the boundary part between B_{nk} and B_{nk+1}, the last element of A_{nj} and the first of B_{nk} and the last of B_{nk} and the first of A_{nj} similarly defined respectively. Then we can define an exhaustion $\{\Omega_{nm}\}$ of Ω_n towards the sets $A_{nj}^*, B_{nk}^*, C_n^*, D_n^*$ and the other boundary components than α. We designate the relative boundaries of the m-th members of the defining sequences of the first four as $A_{nj}^{*(m)}$, $B_{nk}^{*(m)}$, $C_n^{*(m)}$ and $D_n^{*(m)}$. It should be remarked that these sets may be void except for C_n^* and D_n^*.

12. Since the Ω_{nm} is regularly imbedded, by a standard method [1], we can construct a unique function u_{nm} harmonic in Ω_{nm} and such that

i) $u_{nm}=0$ on the intersection of A_{nj} with $\bar{\Omega}_{nm}$, denoted by $A_{nj}^{(m)}$,

ii) $u_{nm}=1$ on the intersection of B_{nk} with $\bar{\Omega}_{nm}$, denoted by $B_{nk}^{(m)}$, and

iii) $du_{nm}^*=0$ along the relative boundary of Ω_{nm} in Ω_n.

Set $\psi_{nm}=u_{nm}+iv_{nm}$, where v_{nm} is the conjugate harmonic function of u_{nm} which is single-valued from iii), normalized so that $v_{nm}=0$ on $C_n^{*(m)}$. We have $v_{nm}=h_{nm}$ on $D_n^{*(m)}$. The function ψ_{nm} maps Ω_{nm} onto an incised horizontal slit rectangle in such a way that

i) $\psi_{nm}(\alpha_{nm})$ is the boundary of a horizontally incised rectangle, whose closure is such that $0 \leqq \operatorname{Re} \psi_{nm} \leqq 1$ and $0 \leqq \operatorname{Im} \psi_{nm} \leqq h_{nm}$, where α_{nm} is the boundary component containing $A_{nj}^{(m)}$,

ii) $\psi_{nm}(A_{nj}^{(m)})$ and $\psi_{nm}(B_{nk}^{(m)})$ are arcs on the imaginary axis and the line $\operatorname{Re} \psi_{nm}=1$ which are non-overlapping and whose sums over j and k are equal to h_{nm} respectively,

iii) $\psi_{nm}(A_{nj}^{*(m)})$ and $\psi_{nm}(B_{nk}^{*(m)})$ are horizontal incisions emanating from the boundary points of adjacent vertical arcs,

iv) $\phi_{nm}(C_n^{*(m)})$ and $\phi_{nm}(D_n^{*(m)})$ are horizontal sides on the real axis and the line $\operatorname{Im}\phi_{nm}=h_{nm}$ respectively,

v) the images of the boundary components other than α_{nm} under ϕ_{nm} are a finite number of horizontal slits, and

vi) the metric $\rho_{nm}=|\phi'_{nm}|$ is extremal for the family Γ_{nm} of curves joining two points of $A_{nj}^{(m)}$'s and $B_{nk}^{(m)}$'s and $\operatorname{mod}\Gamma_{nm}=h_{nm}$.

The properties from i) to v) are obvious since Ω_{nm} is a regularly imbedded subregion of Ω. We remark that the ϕ_{nm} can be directly constructed by duplication [13].

The last property vi) is verified as follows: We have $\|\rho\|^2\geqq h_{nm}$ for $\rho\in P(\Gamma_{nm})$ by Schwarz's inequality and since $\rho_{nm}=|\phi'_{nm}|$ belongs to $P(\Gamma_{nm})$ with $\|\rho_{nm}\|^2=h_{nm}$, it is thus extremal.

13. The sequence Γ_{nm} is increasing and we set $\Gamma_n=\cup_m\Gamma_{nm}$. Note that a curve of Γ_n joins two points of A_n and B_n within Ω. We have by Lemma 3

$$\operatorname{mod}\Gamma_n=\lim_{m\to\infty}\operatorname{mod}\Gamma_{nm},$$

and the metrics ρ_{nm} tend to the extremal metric ρ_n strongly, if $\lambda(\Gamma_n)>0$. We can deduce, from this convergence, that there exists a univalent function ϕ_n such that $\rho_n=|\phi'_n|$ and $\|\phi'_{nm}-\phi'_n\|^2_{\Omega_{nm}}\to 0$ [15]. Clearly $\operatorname{Re}\phi_n=0$ on A_n and $\operatorname{Re}\phi_n=1$ on B_n, since a half neighborhood of every point of them is regularly imbedded.

Finally letting $n\to\infty$, we have $\lambda(\Gamma_n)\to\lambda(\Gamma_0)$ from Theorem 1 and the metrics ρ_n tend to its extremal metric ρ_0 strongly where Γ_0 is the family of curves joining A and B. Suppose further that $\lambda(\Gamma_0)<\infty$, that is $h_0=\operatorname{mod}\Gamma_0>0$. Then ϕ_n tends to a univalent function ϕ_0 in such a way that $\|\phi'_n-\phi'_0\|^2_{\Omega_n}\to 0$ as $n\to\infty$. Then we state

THEOREM 2. *Suppose $\lambda(\Gamma_1)>0$ and $\lambda(\Gamma_0)<\infty$. The function $\phi_0(z)$, above constructed, is univalent and possesses the following properties:*

i) *$I(\phi_0(\alpha))$ is the boundary of horizontally incised rectangle whose closure is such that* $0\leqq\operatorname{Re}\phi_0\leqq 1$ *and* $0\leqq\operatorname{Im}\phi_0\leqq h_0$,

ii) *$I(\phi_0(A))$ (resp. $\phi_0(B)$) contains the vertical side* $[0, ih_0]$ *(resp.* $[1, 1+ih_0]$*) and is disjoint from the open opposite side including its possible incisions,*

iii) *$I(\phi_0(\xi))$, $\xi\in A$(resp. $\xi\in B$), is either a closed arc (possibly a point) on the vertical side* $[0, ih_0]$ *(resp.* $[1, 1+ih_0]$*) with possible horizontal incisions emanating from it or a segment (possibly a point) on an incision, where ξ denotes an elementary boundary part,*

iv) *$I(\phi_0(\partial\Omega-\alpha))$ is a minimal set of horizontal slits,*

v) *the area of $I(\phi_0(\partial\Omega))$ vanishes, and*

vi) *the metric $\rho_0=|\phi'_0|$ is extremal for Γ_0 and* $\operatorname{mod}\Gamma_0=h_0$.

14. *Proof.* We have already proved vi). To prove i), iv) and v), we shall make use of Reich's method [8].

Let c be a curve of Γ_0 with its parametrization $z=z(t)$, $0<t<1$, which intersects all the members of a defining sequence of A as $t\to 0$. Then we show

$$\lim_{t\to 1} \operatorname{Re} \psi_0(z(t)) - \lim_{t\to 0} \operatorname{Re} \psi_0(z(t)) = 1 \tag{7}$$

for almost all curves $c\in\Gamma_0$. In fact, the function ψ_n in No. **13** is such that $\operatorname{Re}\psi_n=0$ and $\operatorname{Re}\psi_n=1$ on A_n and B_n respectively. Set

$$\mu_n=\begin{cases} |\operatorname{grad} \operatorname{Re}(\psi_0-\psi_n)| & \text{in } \Omega_n, \\ 0 & \text{elsewhere,} \end{cases}$$

then we have by a lemma given in [6] or directly as in [13]

$$\left|\int_{c\cap\Omega_n} d \operatorname{Re}\psi_0 - d \operatorname{Re}\psi_n\right| \to 0$$

for almost all $c\in\Gamma_0$ as $n\to\infty$, since $\|\mu_n\|^2\to 0$, which implies (7).

Let Λ denote the exceptional family of curves along which the equality (7) does not hold. Set $w=\psi_0$, $\Delta=\psi_0(\Omega)$ and denote by Γ_0' and Λ' the families of the image curves of the members of Γ_0 and Λ under ψ_0, respectively. Then $\operatorname{mod}\Gamma_0' = \operatorname{mod}(\Gamma_0'-\Lambda')=h_0$, since Λ' is also exceptional. Next consider two rectangles R_ε: $0<\operatorname{Re} w<\varepsilon$, $0<\operatorname{Im} w<h_0$ and R_ε': $1-\varepsilon<\operatorname{Re} w<1$, $0<\operatorname{Im} w<h_0$ and the family, denoted by Γ_ε', of curves joining two points of vertical sides within $\Delta_\varepsilon=\Delta\cup R_\varepsilon\cup R_\varepsilon'$. Then we have $\Gamma_\varepsilon'\supset\Gamma_0'-\Lambda'$ and $\operatorname{mod}\Gamma_\varepsilon'\geqq h_0$. On the other hand the metric $\rho\equiv 1$ is admissible, which thus extremal and $\operatorname{mod}\Gamma_\varepsilon'=h_0$. Then by a criterion of minimality given in [13] (cf. [15]) the inner boundary components of Δ_ε form a minimal set of horizontal slits. Thus we have proved the properties other than ii) and iii).

In order to prove ii), suppose at first that, for example, $I(\psi_0(A))$ does not contain the side $[0, ih_0]$. Then there exists an open segment J contained in the vertical side $[0, ih_0]$ and disjoint from $I(\psi_0(A))$. As before, the module of the subfamily Γ' of Γ_0 consisting of the curves satisfying (7) is equal to h_0. Consider another module problem for the family Γ'' of curves joining the vertical side $[0, ih_0]$ less J and the opposite vertical side in the rectangle: $0<\operatorname{Re} w<1$, $0<\operatorname{Im} w<h_0$. Clearly $\Gamma''\supset\Gamma'$ and $\operatorname{mod}\Gamma''\geqq\operatorname{mod}\Gamma'$. On the other hand, from the result in No. **12** and the unicity of the extremal metric, we get $\operatorname{mod}\Gamma''<\operatorname{mod}\Gamma'$, which is a contradiction.

Next suppose that $I(\psi_0(A))$ contains an interior point of the side $[1, 1+ih_0]$ or a point of an incision emanating from it. Let ω be that point with $0<\operatorname{Im}\omega<h_0$ and let J_1 and J_2 be two closed subsets of $I(\psi_0(B))$ divided by the line $\operatorname{Im} w=\operatorname{Im}\omega$. Then one of J_k is contained in a single $I(\psi_0(\xi_0))$, $\xi_0\in A$. If it were not, we would construct a Jordan curve with ends in $\psi_0(B)$ dividing a boundary part of ψ_0(A) containing ω from its boundary parts on $[0, ih_0]$, which contradicts the assumption in No. **11**.

Let J' denote the projection of the J_k into the line $\operatorname{Re} w=1$. Then there exists a disc $|w-w_0|<r$, contained in Δ whose projection to the line $\mathrm{Re}\, w=1$ is contained in the segment J'. By the same reason as above, a curve c^* joining the disc and the $\phi_0(\xi_0)$ enjoys

$$\mathrm{Re}\, w \to 0 \tag{8}$$

as w tends to $\phi_0(\xi_0)$ along almost all c^*. Let Λ^* be the exceptional family for the property (8). Then from Strebel's inequality given in [15] for the disc and the segment we get

$$\mathrm{mod}\, \Lambda^* \geqq \int_{\mathrm{Im}\, w_0 - r}^{\mathrm{Im}\, w_0 + r} \frac{dy}{l(y)} > 0$$

where $l(y)$ is the length of the horizontal segment with $\mathrm{Im}\, w=y$ $(|y-\mathrm{Im}\, w_0|<r)$ between them, which is a contradiction to the property (8). Thus the property ii) has been completed. The property iii) immediately follows from the connectedness of $\phi_0(\xi)$.

15. *Remarks.* The realizations $I(\phi_0(\xi))$ and $I(\phi_0(\xi'))$ with $\xi, \xi' \subset A \cup B$ may have their intersection containing a continuum (not a point), if they are distinct. Note that these two elementary boundary parts are disjoint by definition. There exists an elementary part ξ disjoint from A and B such that $I(\phi_0(A)) \cap I(\phi_0(\xi))$ contains a segment on the vertical side $[0, ih_0]$. The latter example was given in [15] and examples of the former type can be constructed by making use of a characterization theorem in the next No. 16 which will be omitted.

16. *Extremal property of ϕ_0.* Let $\mathfrak{F}$ be the family of univalent function $f(z)$ in Ω such that $0<\mathrm{Re}\, f<1$, $\mathrm{Inf}\, \mathrm{Im}\, f=0$, and $\mathrm{Re}\, f(z) \to 0$ and $\mathrm{Re}\, f(z) \to 1$ as z tends to A and B along almost all curves joining A and B respectively. Setting $H(f)=\sup_{z\in\Omega} \mathrm{Im}\, f$, we have similarly as in [15]

THEOREM 3. *Suppose that the extremal distance between A and B in Ω is positive and finite. Then the function ϕ_0, if it exists, is the unique function which minimizes the quantity $H(f)$ in $\mathfrak{F}$.*

Proof. We have from (1) for ϕ_0 and f

$$\|\, |f'|-|\phi_0'|\, \|^2 \leqq \|f'\|^2 - \|\phi_0'\|^2 \leqq H(f)-h_0,$$

with $\rho=|f'|$ and $\rho_0=|\phi_0'|$. It is obvious, from Theorem 2 and its proof, that ϕ_0 belongs to $\mathfrak{F}$.

An example in which the assumption of the theorem does not guarantee the construction of ϕ_0 was given in [15].

The image is called a *minimal horizontal slit rectangle* with respect to A and B. We cite its characterization without proof (cf. [15]).

COROLLARY 1. *Let Ω be a region whose outer boundary is the periphery of the* *rectangle* $0<\mathrm{Re}\, z<1$, $0<\mathrm{Im}\, z<h$ *with possible horizontal incisions emanating from its vertical sides and let* A *and* B *be two disjoint boundary parts on the outer boundary component* α *whose realizations contain the sides* $[0, ih]$ *and* $[1, 1+ih]$ *respectively. Suppose that the extremal distance of the relative boundaries of the first members of their defining sequences is positive. Then any two of the following three conditions imply the minimality of* Ω *with respect to* A *and* B:

i) $\partial\Omega-\alpha$ *is a minimal set of horizontal slits,*

ii) $\mathrm{Re}\, z\to 0$ *and* $\mathrm{Re}\, z\to 1$ *as* z *tends to* A *and* B *along almost all curves joining* A *and* B *respectively, and*

iii) *the extremal distance of* A *and* B *is equal to* $1/h$.

Conversely a minimal slit rectangle with respect to A *and* B *possesses all the above properties.*

§ 5. Boundary elements.

17. *Introduction.* In this section we shall generalize the notion of prime ends for an arbitrary region, which introduces a well-defined boundary correspondence. Several approaches to the theory of prime ends in terms of extremal length and its extension were given by Schlesinger [9], Gál [4] and Ohtsuka [7]. Here we shall define a boundary element which corresponds to either a point or an incision in a radial slit disc mapping with a finite radius.

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18. *Definitions.* Let Ω be a region which is not the Riemann sphere. Let K be a compact disc $|z-a|\leqq r$ contained in Ω. Consider an elementary boundary part $\xi=\{\Delta_n\}_{n=1}^{\infty}$ satisfying $\Delta_n\supset\bar{\Delta}_{n+1}$. The relative boundary of every Δ_n is either closed or open. In either case $V_n=\Delta_n-\bar{\Delta}_{n+1}$ is a region and we denote by $\hat{\Gamma}_n$ the family of curves running within V_n and intersecting all the curves joining the relative boundaries of Δ_n and Δ_{n+1}. Such a family is called a *dividing curve family.* An elementary boundary part ξ with the following properties is called a *boundary element*:

I) $\lambda(\hat{\Gamma}_n)<\infty$ i.e. mod $\hat{\Gamma}_n>0$, and

II) the extremal distance between K and ξ is infinite.

The condition II) is equivalent to the following

II*) $\lim \lambda(\Gamma_n)=\infty$, where Γ_n is the family of curves joining K and Δ_n, from Theorem 1.

19. *Radial slit disc mapping.* We review the properties of radial slit disc mappings. Let α be a boundary component of Ω. Suppose that the extremal distance of K and α, denoted by $d(K, \alpha)$ for simplicity, is finite. Then the radial slit disc mapping g_α with a finite radius $R=R(\alpha)$ normalized by $g_\alpha(a)=0$ and $g'_\alpha(a)=1$ is uniquely constructed, which has the following properties:

i) $I(g_\alpha(\alpha))$ is a circle $|w|=R$ with possible radial incisions emanating from it,

ii) $I(g_\alpha(\partial\Omega-\alpha))$ is a minimal set of radial slits,

iii) the area of $I(g_\alpha(\partial\Omega))$ vanishes, and

iv) $|g_\alpha(z)|\to R$ as z tends to α along almost all curves joining K and α.
The properties except for iv) are essentially due to Strebel [11] (cf. [8], [10]) and iv) was shown in [13] and [6]. We called $g_\alpha(\Omega)$ a minimal radial slit disc.

If $d(K, \alpha)=\infty$, the α is a boundary element by definition and a radial slit disc mapping with radius infinite can be constructed [11, 12], but the uniqueness is not yet established. We show

LEMMA 4. *Let Ω be a minimal radial slit disc with a finite radius R in the w-plane. The points $Re^{i\theta}$ from which no incisions emanate are accessible in Ω. They form a G_δ set on the circle $|w|=R$ with angular measure 2π. Moreover each $Re^{i\theta}$ is accessible in the intersection of Ω with a sector $|\arg w-\theta|<\varepsilon$ for an arbitrary positive ε.*

Proof. The set of incisions whose lengths are not less than $1/n$ is compact, whence so is its projection into the circle $|w|=R$, denoted by Θ_n. Since the angular measure of Θ_n vanishes from the property iii), so does the union $\cup_n \Theta_n$. Hence the set in the lemma is a G_δ set of angular measure 2π. Next let $Re^{i\theta}$ be a point in the lemma. Consider an exhaustion $\{\Omega_n\}$ of Ω towards α and set $V_n=\Omega_n-\bar{\Omega}_{n+1}$. Let A_n denote the set of boundary components of V_n other than its relative boundary. Each boundary component $\beta\in A_n$ is enclosed by a Jordan curve along which the oscillation of $\arg w$ is less than ε, since $I(\beta)$ is a radial segment. We denote by U_n^β its interior. $I(A_n)$ is covered by a finite number of $U_n^{\beta_j}$'s, since it is compact. Then the segments on the radius $[0, Re^{i\theta}]$ interior of all the $U_n^{\beta_j}$ are at most countable and replacing them by a union of subarcs of the boundary of $U_n^{\beta_j}$, we get a desired curve.

20. *Boundary correspondence*. If a boundary component α is a boundary element, then it contains no other boundary element. To see it we prove

THEOREM 4. *If a boundary element ξ exists on a boundary component α with $\xi\neq\alpha$, then the extremal distance between K and α is finite.*

Proof. Set $\xi=\{\Delta_n\}$. We may suppose that the relative boundary of every Δ_n is an open Jordan curve, since $\alpha\neq\xi$. The end arcs of the relative boundary of $V_n=\Delta_n-\bar{\Delta}_{n+1}$ determine two boundary parts ξ_n and ξ_n'. Take an auxiliary disc K': $|z-b|\leqq\delta$ contained in the region V_n. We show that the extremal distances $d(K', \xi_n)$ and $d(K', \xi_n')$ in Ω are both finite. In fact, if one of them, for example, $d(K', \xi_n)=\infty$, the subfamily $\hat{\Gamma}_n^{K'}$ of curves of $\hat{\Gamma}_n$ passing through K' is exceptional, since every curve of $\Gamma_n^{K'}$ contains a curve of the former module problem as a subset. The finiteness of the extremal distance is independent of such a reference set [13]. Enlarging K' so that it may remain simply connected in the region V_n and approach to both the relative boundary of Δ_n and Δ_{n+1}, we see that Γ_n is the union of such $\hat{\Gamma}_n^{K'}$. We have $\lambda(\hat{\Gamma}_n)=\infty$ from Lemma 3 which contradicts the condition I) of the definition of a boundary element. So the extremal distances $d(K', \xi_n)$ and $d(K', \xi_n')$ or equivalently $d(K, \xi_n)$ and $d(K, \xi_n')$ are finite. Since they are on α, we have $d(K, \alpha)<\infty$.

We now state

THEOREM 5. *If the extremal distance between K and a boundary component α is finite and if a boundary element ξ is on the α, then the image $I(g_\alpha(\xi))$ is either a point on the circle $|w|=R$ or a possible incision including its end point on the circle, where g_α is the radial slit disc mapping with radius R.*

Proof. $g_\alpha(K)$ contains a disc $|w|\leqq r'$, denoted by K'. $I(g_\alpha(\xi))$ is a continuum and so is its projection into the circle $|w|=R$. If it is an arc $\widehat{Re^{i\theta_1}Re^{i\theta_2}}$ (not a point), we get as in No. 13 by Strebel's inequality [10] a contradiction

$$d(K', \xi)=\left(\int_{\theta_1}^{\theta_2}\frac{d\theta}{l(\theta)}\right)^{-1}<\infty,$$

where $l(\theta)$ is the logarithmic length of the radial segment between $r'e^{i\theta}$ and the set $I(g_\alpha(\xi))$ for $\theta_1\leqq\theta\leqq\theta_2$.

If no incision emanates from the projection of $I(g_\alpha(\xi))$, it is really a point. We show that if an incision τ emanates from the projection, $I(g_\alpha(\xi))$ coincides with τ. We parametrize the relative boundary of $g_\alpha(\Delta_n)$, denoted by γ_n, by $w_n(t)$ $(0<t<1)$ in such a way that $w_n(t)$ runs positively with respect to $g_\alpha(\Delta_n)$ as it moves from 0 to 1, where $\{\Delta_n\}$ denotes a defining sequence of ξ. We can deduce that the intersections of the projections of the cluster sets $C(w_n, 0)$ and $C(w_{n+1}, 0)$ and the cluster sets at $t=1$ are both void. Suppose, for example, that the intersection of the projections of the cluster sets at $t=0$ contains a point $Re^{i\theta}$. If no incision emanates from the $Re^{i\theta}$, we can construct, by Lemma 4, a defining sequence of an elementary boundary part $\xi^*=\{\Delta_\nu^*\}$ such that $I(\xi^*)=Re^{i\theta}$ and that the relative boundary of Δ_ν^* intersects both the Δ_n and Δ_{n+1}. Then every curve of $\hat{\Gamma}_n$ clusters at the point $Re^{i\theta}$ since it must pass through all the Δ_m^* and the metric

$$\mu_\varepsilon=\frac{\varepsilon}{|w-Re^{i\theta}|\ |\log|w-Re^{i\theta}|\ |}$$

is admissible for $\hat{\Gamma}_n$ [15]. We get $||\mu_\varepsilon||^2\to 0$ as $\varepsilon\to 0$, which contradicts the condition I).

Next, if an incision emanates from the point $Re^{i\theta}$, we take again, by Lemma 4, a defining sequence $\{\Delta_\nu^*\}$ such that its realization coincides with the incision and that the relative boundary of Δ_ν^* intersects both the Δ_n and Δ_{n+1}. Then every curve of $\hat{\Gamma}_n$ clusters either at the point $Re^{i\theta}$ or a point on the incision and in the disc $|w|<R$. Let K'' be a compact disc in the region $g_\alpha(\Delta_n-\bar{\Delta}_{n+1})$. Then both the families of curves of the former and the latter type are exceptional. Enlarging K'' as before, we get the same contradiction.

Therefore the projection of the cluster set $C(w_n, 0)$ is left to the incision τ and that of $C(w_n, 1)$ is right to it with respect to the outer normal. Then the set $\mathrm{Cl}(g_\alpha(\Delta_n))$ contains τ which implies the assertion.

21. Next we show, using the above conformal representation, THEOREM 6. *If two boundary elements ξ and ξ' are distinct, then they are disjoint.*

Proof. Set $\xi=\{\Delta_n\}$ and $\xi'=\{\Delta'_n\}$. If they are on distinct boundary components separately, there is nothing to prove. Suppose that they are on a boundary component α. Then from Theorem 4 we have $d(K, \alpha)<\infty$ and there exists the radial slit disc mapping g_α with finite radius R normalized at $a\in\Omega$. Contrary to the assertion, we have $\bar{\Delta}_n \cap \bar{\Delta}'_n \neq \phi$ for all n and $I(g_\alpha(\xi))=I(g_\alpha(\xi'))$. To see their equivalence, we denote by Θ_n (resp. Θ'_n) the circular arc between the left and right ends of the projections of the cluster sets of the relative boundary of Δ_n (resp. Δ'_n) at $t=0$ and $t=1$, respectively, where the relative boundaries are parametrized as before. Θ_{n+1} is contained in the open arc between the right and left ends of the projections of the cluster sets from Θ_n as is seen in the proof of the preceding theorem. Let n_0 be fixed. Then Θ'_n is contained in the open arc of Θ_{n_0} for a sufficiently large $n\geqq n_0$, since Θ_n and Θ'_n are shrinking to a point as $n\to\infty$. Therefore $g_\alpha(\Delta'_m)$ is contained in $g_\alpha(\Delta_{n_0})$ for a sufficiently large m and, hence, we get their equivalence.

We have immediately

COROLLARY 2. *If the extremal distance between a boundary component α and a compact disc K is finite, there exists a one to one correspondence between the boundary elements and the points $Re^{i\theta}$ under the radial slit mapping g_α with finite radius R.*

22. *A compactification.* We denote by $\mathcal{E}$ the set of all boundary elements. We set $\tilde{\Omega}=\Omega\cup\mathcal{E}$ and define a topology for it by giving a base for the topology. Let ξ be a boundary element with a defining sequence $\{\Delta_n\}$. We denote by $\tilde{\Delta}_n$ the union of Δ_n and all the boundary elements a member of each of whose defining sequences is contained in Δ_n. The base is given by the union of the original open sets in Ω and such a set $\tilde{\Delta}_n$ for every $\xi\in\tilde{\Omega}$. Using the Stoïlow compactification and conformal representations onto radial slit discs, we show

THEOREM 7. *The topological space $\tilde{\Omega}$ is a compact Hausdorff space.*

Proof. We first see the compactness. Let $\{O\}$ be a covering of $\tilde{\Omega}$ and let $\{O^\alpha\}$ be the class of its elements containing a boundary element on α, where α is a boundary component. Then there exist a finite number of O^α_j's such that $\cup_j O^\alpha_j$ contains all the boundary elements on α and a member of a defining sequence of α. Indeed, it is obvious if α is a boundary element. Otherwise, let g_α be a radial slit disc mapping with radius R. Since every O containing ξ contains a Δ_n of its defining sequence $\{\Delta_n\}$, the $g_\alpha(O)$ contains the intersection of $g_\alpha(\Omega)$ with a disc $|w-Re^{i\theta}|<\delta$ for a sufficiently small δ, where $Re^{i\theta}$ is the projection of $g_\alpha(\xi)$ into the circle $|w|=R$. The compactness of the circle implies the assertion. Using the Stoïlow compactification [1], we cover all the boundary components by a finite number of such unions. The complementary set of their union is compact on the Riemann sphere, whence $\tilde{\Omega}$ is compact.

The separation of distinct ξ and ξ' of $\mathscr{E}$ follows from Theorem 6.

23. *Remarks.* Every conformal mapping of Ω is topologically extended to the $\tilde{\Omega}$ and especially the radial slit disc mapping g_α with a finite radius maps distinct boundary elements on α onto those with disjoint realizations.

A boundary component is identified with a connected component of $\mathscr{E}$.

An elementary boundary part with extremal distance infinite from a compact disc in Ω is contained in a unique boundary element. It is easily verified by making use of radial slit disc mappings. But there exists a boundary part with a finite extremal distance which contains no boundary elements.

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